Locking-free shells with structure-preserving finite elements

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Project J 4824-N

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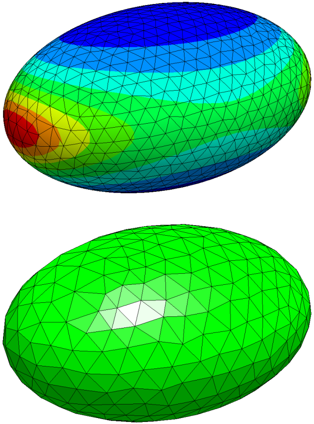
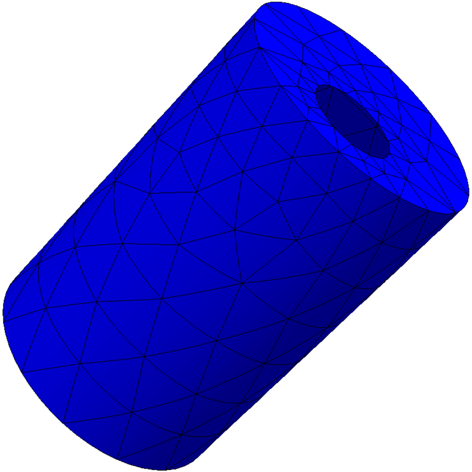
Approximate extrinsic curvature of non-smooth surfaces



Application to shells



Motivation

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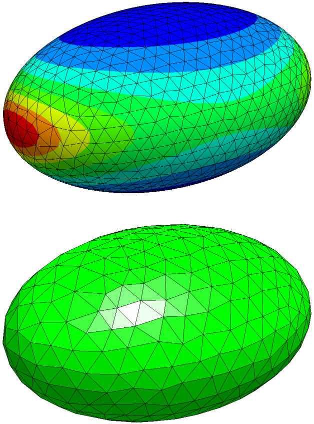

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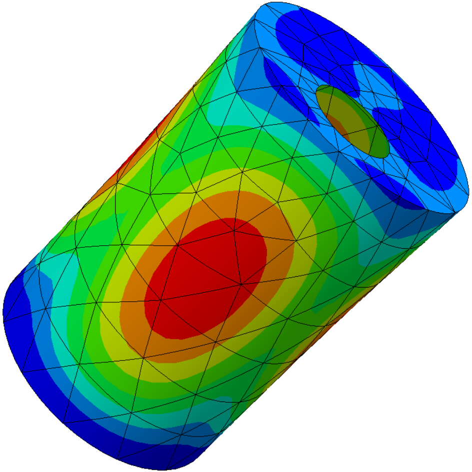

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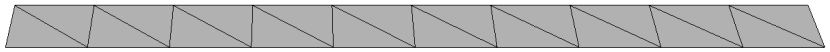

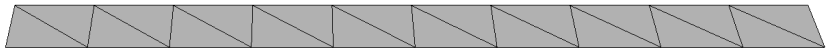

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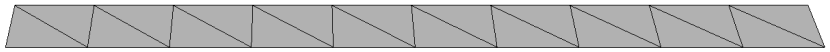

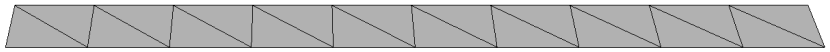

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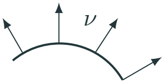
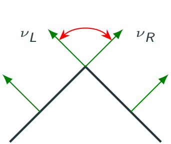
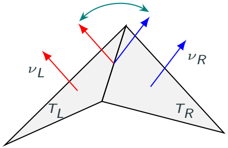


- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces

Weingarten tensor (shape operator)





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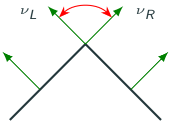
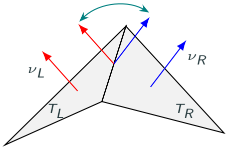
- Normal vector ν is piecewise constant and jumps

Weingarten tensor (shape operator)





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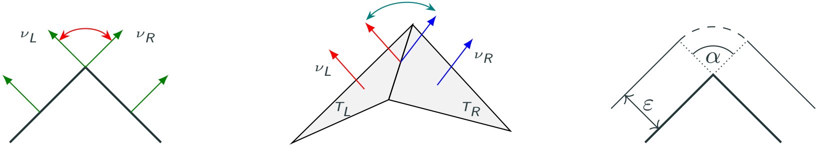

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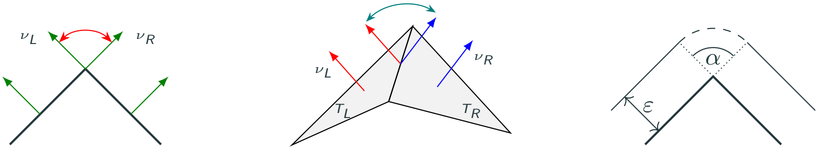
Steiner: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

Grinspun, Gingold, Reisman, Zorin Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

Weingarten tensor (shape operator)



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How to define a generalized Weingarten tensor object? Combine FEM & DDG!

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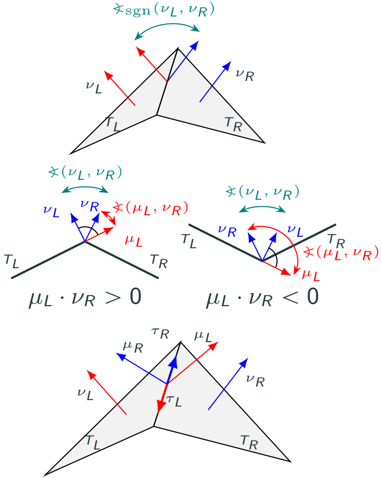
- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla \nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\widetilde{\nabla \nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$$

- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$



DDG & FEM: Generalized Weingarten tensor

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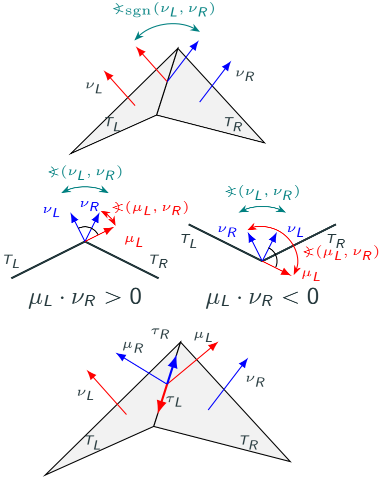
- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$
- Test function space (Hellan–Herrmann–Johnson space):

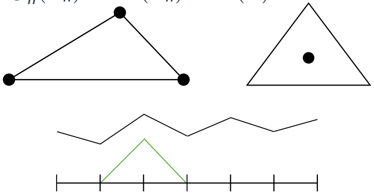
$$\Sigma = \{\boldsymbol{\sigma}\in L^2(\mathscr{T},\mathbb{R}^{3\times3}_{\mathrm{sym}}) : (\boldsymbol{\sigma}\nu)|_T = 0,\ (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

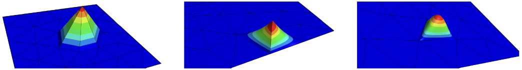


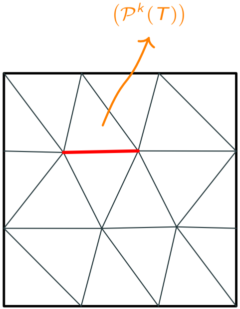
Scalar-valued spaces

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\},$$
$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega),$$

$$L^2(\Omega) = \{u : \Omega \to \mathbb{R} \mid u^2 \text{ integrable}\},$$
$$\mathcal{DG}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \subset L^2(\Omega)$$





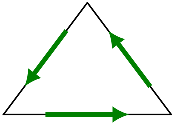


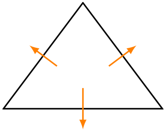
Examples: Density, pressure, temperature
Finite elements: Lagrange (H^1, continuous), discontinuous Galerkin (L^2, discontinuous)

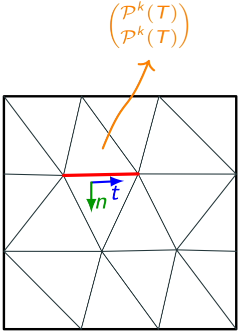
Vector-valued spaces

$$\begin{aligned} H(\mathrm{curl},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{curl}\,\sigma \in [L^2(\Omega)]^{2d-3}\}, \\ \mathcal{N}^k_{II,h} &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_t]\!]_F = 0\} \subset H(\mathrm{curl},\Omega), \end{aligned}$$

$$\begin{aligned} H(\mathrm{div},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}\sigma \in L^2(\Omega)\}, \\ \mathrm{BDM}^k_h &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div},\Omega) \end{aligned}$$





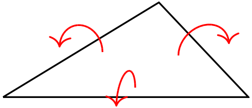


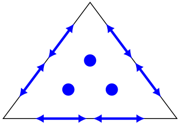
Examples: Deformation, velocity, momentum
Finite elements: Nédélec ($H(\mathrm{curl})$, tangential), Raviart–Thomas/BDM ($H(\mathrm{div})$, normal)

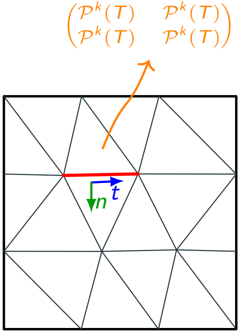
Tensor-valued spaces

$$H(\mathrm{divdiv},\Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathbf{div}\,\mathbf{div}\,\sigma \in H^{-1}(\Omega)\},$$
$$M_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![n^T\sigma n]\!]_F = 0\},$$

$$H(\mathrm{curl\,curl},\Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{curl\,curl}\,\sigma \in H^{-1}(\Omega)\},$$
$$\mathrm{Reg}_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^T\sigma t]\!]_F = 0\}$$





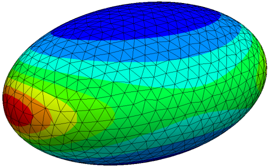


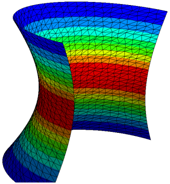
Examples: Strain, stress, metric
Finite elements: Regge ($H(\mathrm{curl\,curl})$, tt),
Hellan–Herrmann–Johnson ($H(\mathrm{divdiv})$, nn)

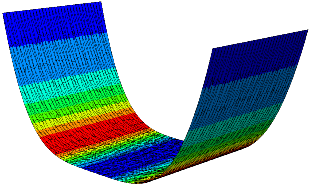
Lifting of distributional Weingarten tensor

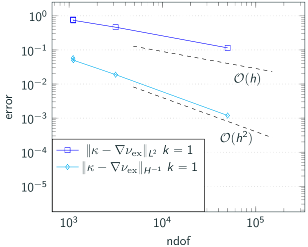
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

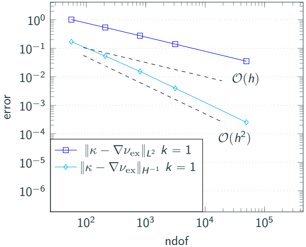
$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla \nu}(\boldsymbol{\sigma}).$$

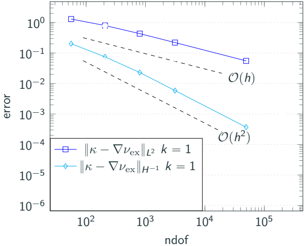












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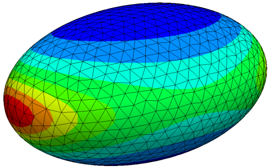

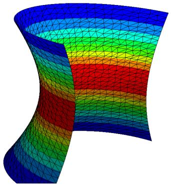

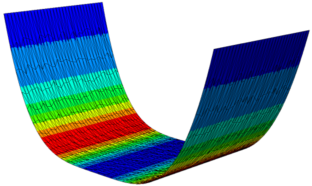

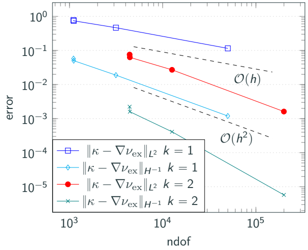

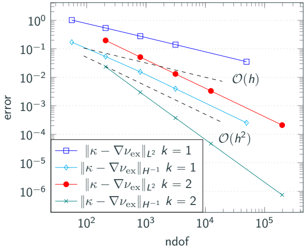

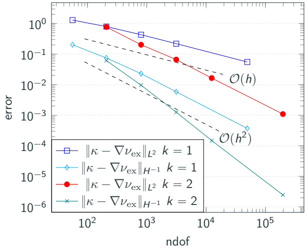

Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t)) + b(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t);\boldsymbol{\sigma},\dot{\Phi}(t))$ and $b(\Phi(t);\boldsymbol{\sigma},\dot{\Phi}(t))$ $\|\measuredangle_{\mathrm{sgn}}(\nu_L,\nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

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Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

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Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ always converges in H^{-1}!

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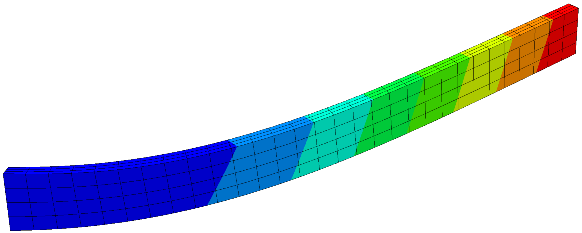
Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\Phi_h = \mathcal{I}_h^{\mathrm{Lag}^k}\bar{\Phi}_h$ for $k \geqslant 1$. Let $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ be the lifted Weingarten tensor. Then $\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$.

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

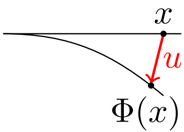
Shells

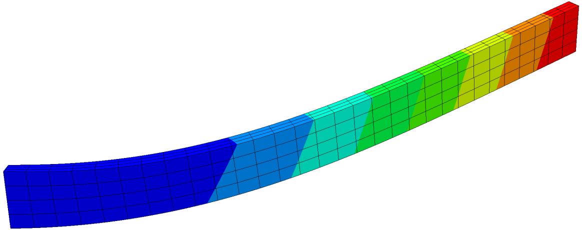
Deformation $\Phi : \Omega \to \mathbb{R}^3$



Deformation $\Phi : \Omega \to \mathbb{R}^3$

Displacement $u := \Phi - id$

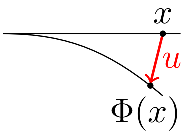


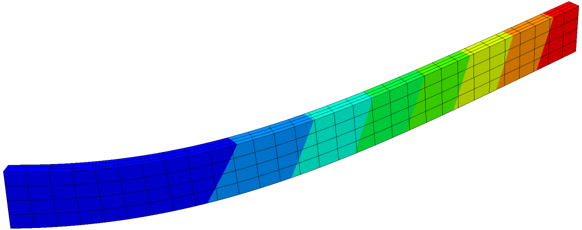


Deformation $\Phi : \Omega \to \mathbb{R}^3$

Displacement $u := \Phi - id$

Deformation gradient $\boldsymbol{F} := \nabla\Phi$

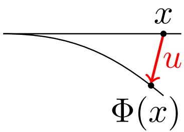


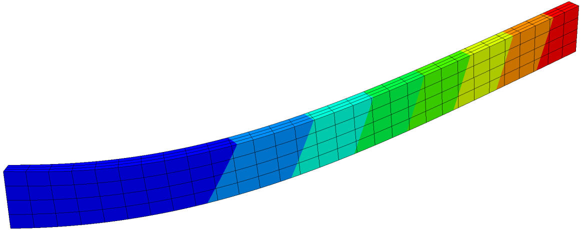


Deformation $\Phi : \Omega \to \mathbb{R}^3$

Displacement $u := \Phi - id$

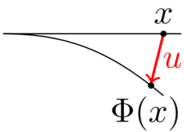
Deformation gradient $\boldsymbol{F} := \boldsymbol{I} + \nabla u$



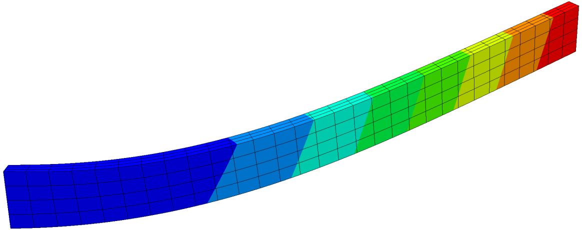


Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{I} + \nabla u + \nabla u^T + \nabla u^T \nabla u$

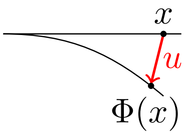


$$\Delta x^T \Delta x = \Delta X^T \boldsymbol{C} \Delta X$$



Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I} + \nabla u + \nabla u^T + \nabla u^T \nabla u$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$
Linearized strain	$\varepsilon := \mathrm{sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$



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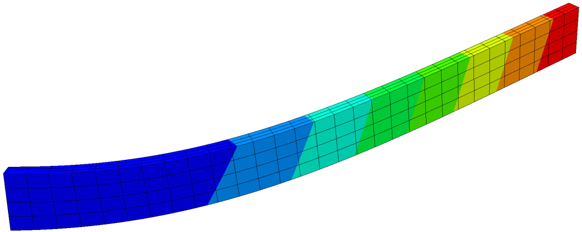

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
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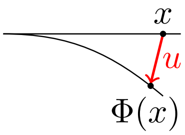
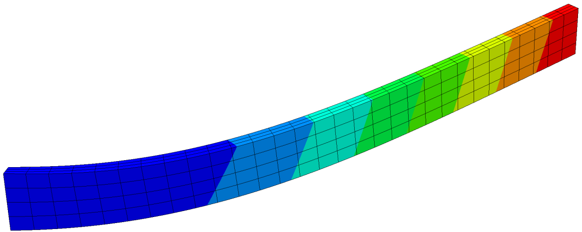
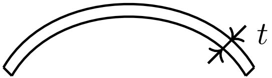


Elasticity

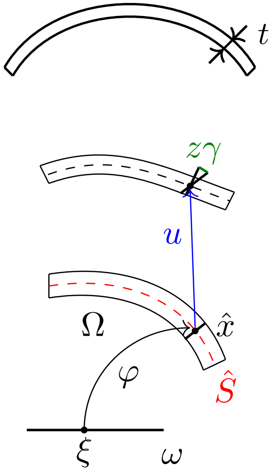
$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|_{\boldsymbol{M}}^2 - \langle f, u \rangle \to \min_{u \in V}!$$



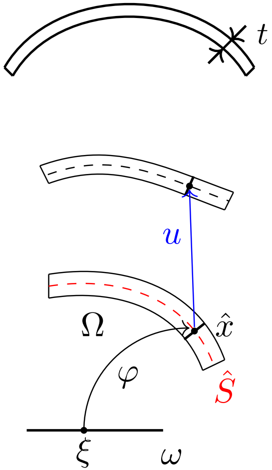
Thin-walled structures



- Reduce 3D elasticity to 2D shell model



- Reduce 3D elasticity to 2D shell model
- $\Omega = \{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in [-\frac{t}{2}, \frac{t}{2}]\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z \underbrace{(\nu + \gamma) \circ \phi}_{=\tilde{\nu}\circ\phi}(\hat{x})$
- Reissner-Mindlin/Naghdi shell



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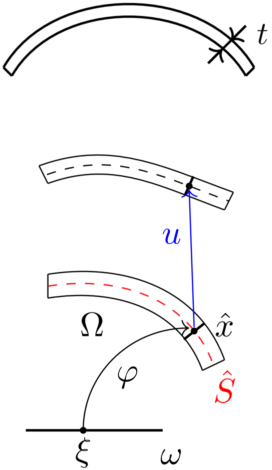
- Kirchhoff–Love/Koiter shell



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- Kirchhoff–Love/Koiter shell
- Insert Φ in 3D elasticity and integrate over thickness, neglect higher order terms $\mathcal{O}(t^4)$ (asymptotical analysis)

Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathbb{M}}$$

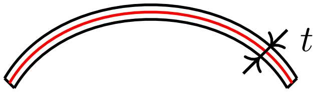
u . . . displacement of mid-surface

t . . . thickness

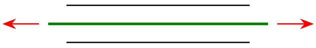
$\mathbb{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$


membrane energy

u ... displacement of mid-surface

t ... thickness

$\mathbb{M}$ ... material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

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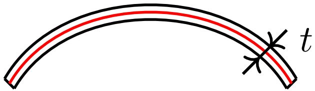
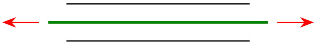


Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathbb{M}}$$



membrane energy

u . . . displacement of mid-surface

t . . . thickness

$\mathbb{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

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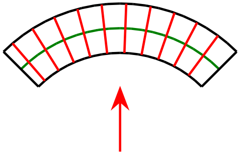
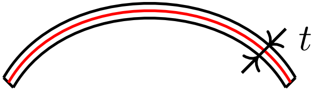


bending energy



Naghdi/Reissner–Mindlin shell

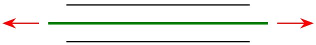
$$\mathcal{W}(u,\gamma) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla(\tilde{\nu}\circ\phi)) - \nabla\hat{\nu}\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2$$

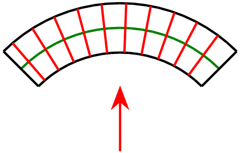
γ . . . shearing

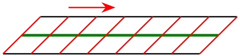
$\tilde{\nu} = \dfrac{\nu+\gamma}{\|\nu+\gamma\|}$ . . . director

G . . . shearing modulus

$\kappa = 5/6$ . . . shear correction factor







Lifted shape operator: $\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\Psi}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\Psi}) := \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f,u\rangle$$
$$+ \sum_{T\in\mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$- \sum_{E\in\mathscr{E}} \int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma}\in\Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}}$ → two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ &+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\end{aligned}$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

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$$+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx$$
$$+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$

Use hybridization to eliminate $\boldsymbol{\sigma}$ $\rightarrow$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

Shell problem

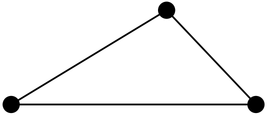
Find $u \in [V_h^k]^3$, $\boldsymbol{\sigma} \in M_h^{\mathrm{dc},k-1}$, and $\alpha \in \Gamma_h^{k-1}$ for

$$\mathcal{L}(u,\boldsymbol{\sigma},\alpha) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle$$
$$+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} + [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\alpha_{\hat{\mu}}\,ds$$

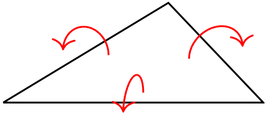
$$\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} = \tfrac{1}{2}\big((\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} + (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R}\big),\ [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} - (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R}$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

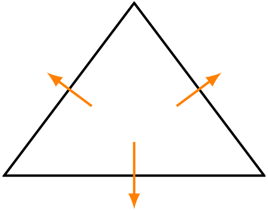
Shell element



Displacement u

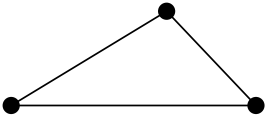


Moment $\boldsymbol{\sigma}$

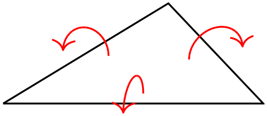


Hybridization

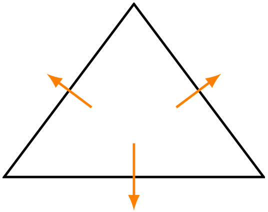
Shell element



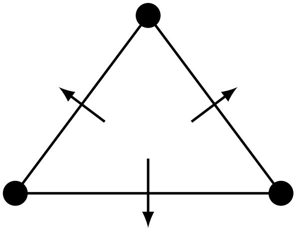
Displacement u



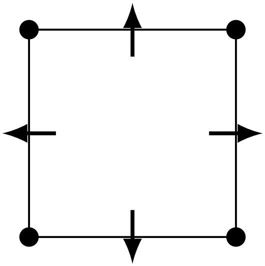
Moment $\boldsymbol{\sigma}$



Hybridization



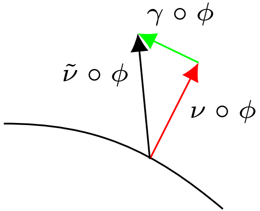
Morley



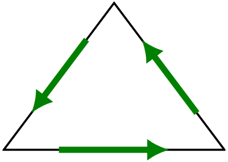
Quadrilateral (hybridized)

Extension to nonlinear Naghdi shells

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \frac{\nu\circ\phi+\gamma\circ\phi}{\|\nu\circ\phi+\gamma\circ\phi\|}$
- Free of shear locking



$$H(\mathrm{curl}) := \{u\in[L^2(\Omega)]^d \mid \mathrm{curl}\, u\in[L^2(\Omega)]^{2d-3}\}$$
$$\mathcal{N}_{II}^k := \{u\in[\mathcal{P}^k(\mathcal{T}_h)]^d \mid u_t \text{ is continuous over elements}\}$$



ECHTER, R. AND OESTERLE, B. AND BISCHOFF, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg.* (2013).

Extension to nonlinear Naghdi shells

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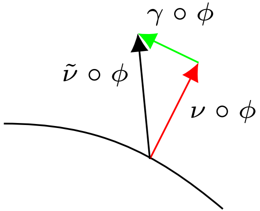
- $\tilde{\nu} \circ \phi = \nu \circ \phi + \gamma \circ \phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



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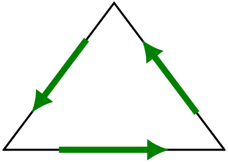

Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg.* (2013).

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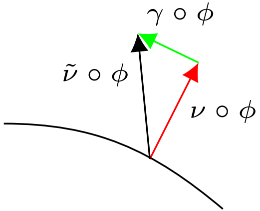
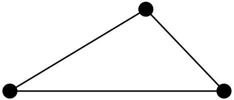
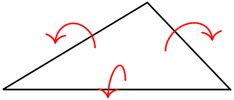


$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = {} & \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} \\
& + \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right) : \boldsymbol{\sigma}\,dx \\
& - \sum_{E\in\mathscr{E}_h}\int_E \left(\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R) - [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

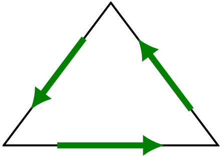
Shell element (Naghdi)



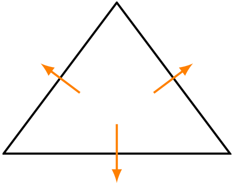
displacement u



bending moment tensor $\boldsymbol{\sigma}$

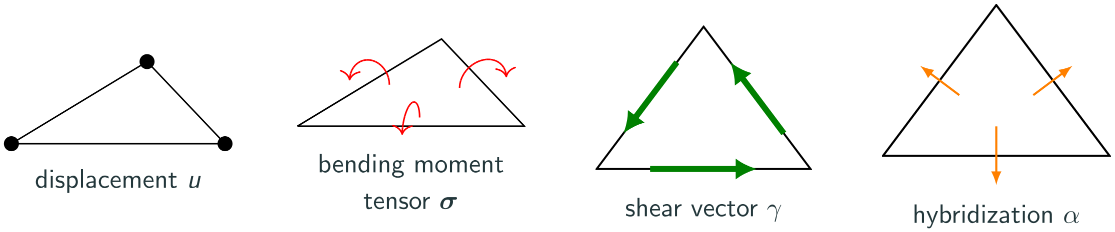


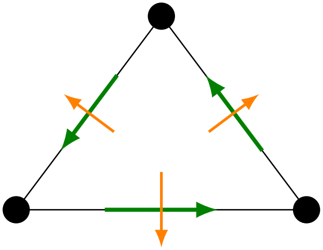
shear vector γ

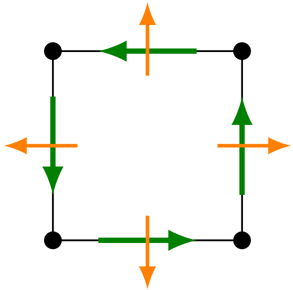


hybridization α

Shell element (Naghdi)







Linear Kirchhoff–Love shell and plate

TECHNISCHE UNIVERSITÄT WIEN

$$\mathcal{L}^{\mathrm{shell}}_{\mathrm{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\Big(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\, dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big)$$

$$\mathcal{L}^{\mathrm{plate}}_{\mathrm{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\Big(\int_T \nabla^2 w : \boldsymbol{\sigma}\, dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big)$$

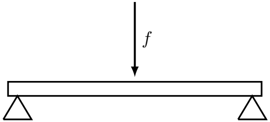
Linear Kirchhoff–Love shell and plate

$$\mathcal{L}^{\text{shell}}_{\text{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\text{cov}} u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

$$\mathcal{L}^{\text{plate}}_{\text{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

Biharmonic plate equation:

$$\mathbf{divdiv}(\mathbb{M}\nabla^2 w) = f \Leftrightarrow \begin{cases}\boldsymbol{\sigma} = \mathbb{M}\nabla^2 w,\\ \mathbf{divdiv}\boldsymbol{\sigma} = f,\end{cases}$$



M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. (1989)*

$$\mathcal{L}^{\mathrm{shell}}_{\mathrm{lin}}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}}$$
$$+ \sum_{T\in\mathscr{T}_h}\Big(\int_T (\boldsymbol{H}_{\hat{\nu}} - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}((\nabla u^\top \hat{\nu})_{\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big)$$

$$\mathcal{L}^{\mathrm{plate}}_{\mathrm{lin}}(w,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}}$$
$$+ \sum_{T\in\mathscr{T}_h}\Big(\int_T (\nabla^2 w - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}(\frac{\partial w}{\partial\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big)$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017)

Membrane locking

$$\mathcal{W}(u) = t\, E_{\mathrm{mem}}(u) + t^3\, E_{\mathrm{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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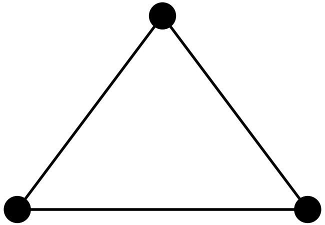
Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$

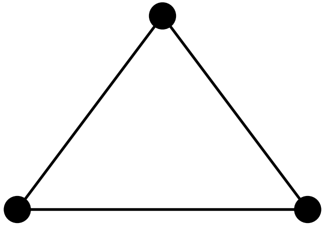


$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

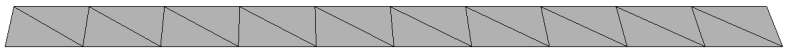
$$\mathcal{W}(u) = t^{-2}E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f}\cdot u, \qquad f = t^3\tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

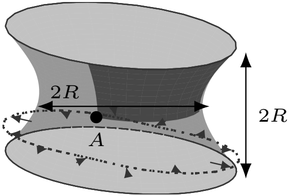
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$

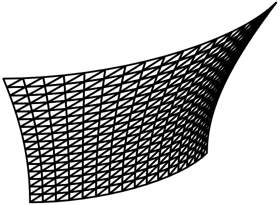


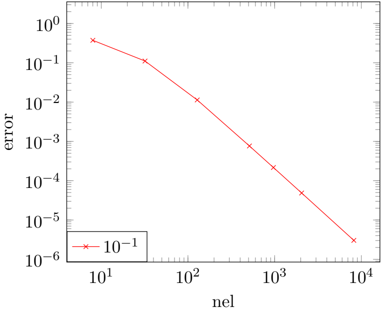
$$\text{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$



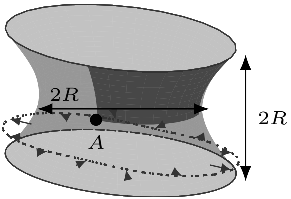
Membrane locking

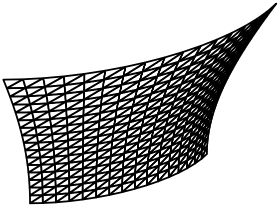


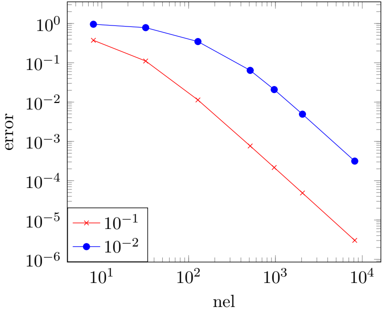




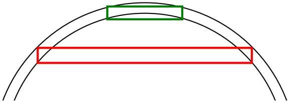
Membrane locking



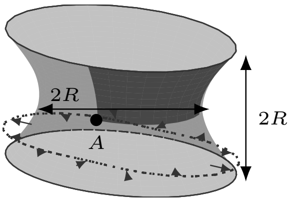


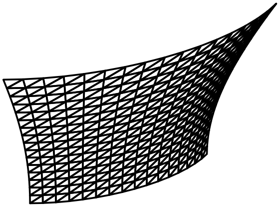


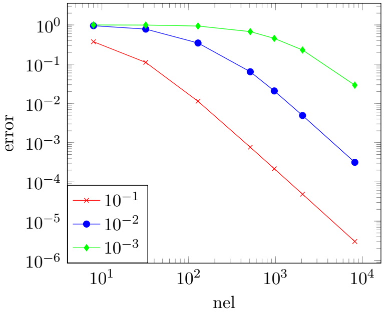
- Pre-asymptotic regime



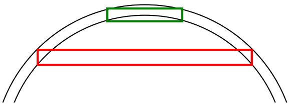
Membrane locking

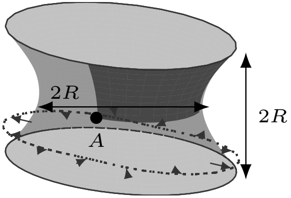


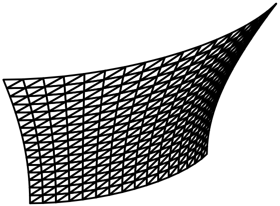


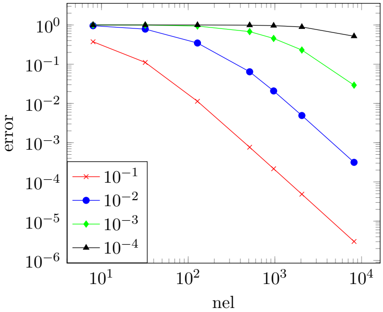


- Pre-asymptotic regime









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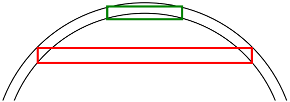
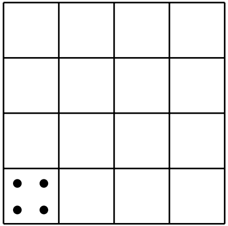


Membrane locking

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

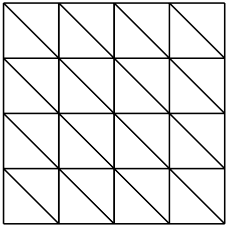
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

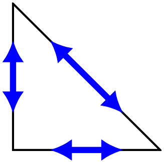


- Reduced integration for quadrilateral meshes

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$



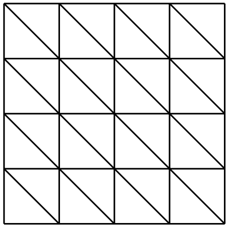


- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Membrane locking

$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u)\|_{\mathcal{M}}^{2} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|_{\mathcal{M}}^{2} - \langle f, u\rangle \\
+ \sum_{T\in\mathscr{T}}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx \\
- \sum_{E\in\mathscr{E}}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds
\end{aligned}
$$



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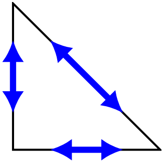
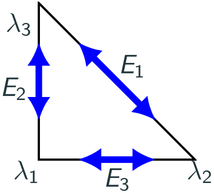


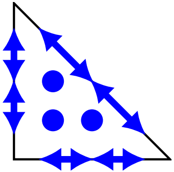
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Regge elements & interpolation operator

$$H(\mathrm{curl}\,\mathrm{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

- CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*
- LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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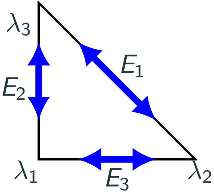

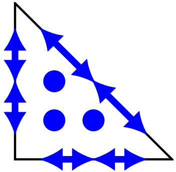

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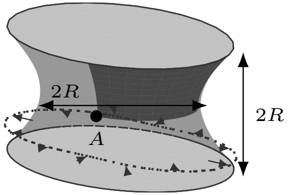
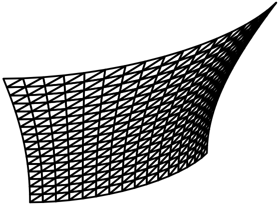
$$\mathcal{I}^k_{\mathcal{R}} : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

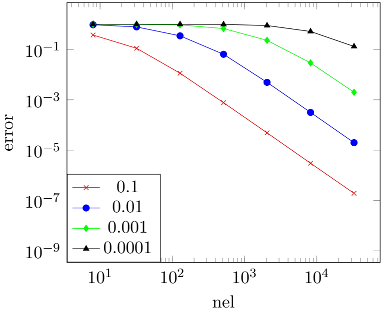
$$\int_E (g - \mathcal{I}^k_{\mathcal{R}} g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

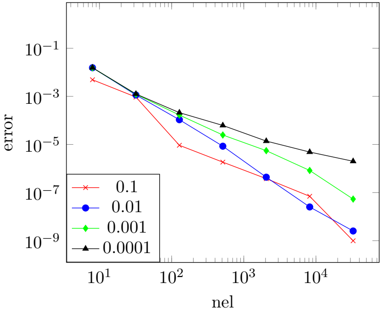
$$\int_T (g - \mathcal{I}^k_{\mathcal{R}} g) : Q\, dx = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking

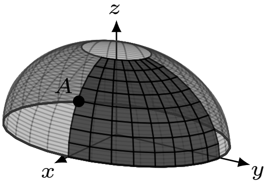


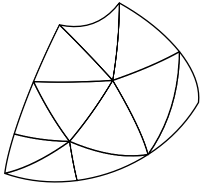


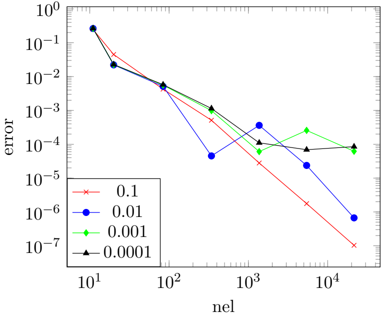


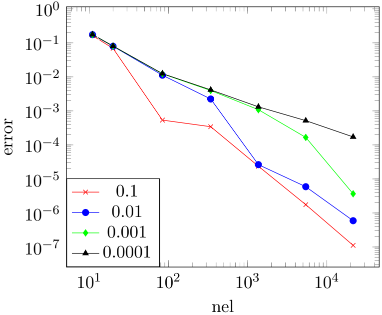


Open hemisphere with clamped ends





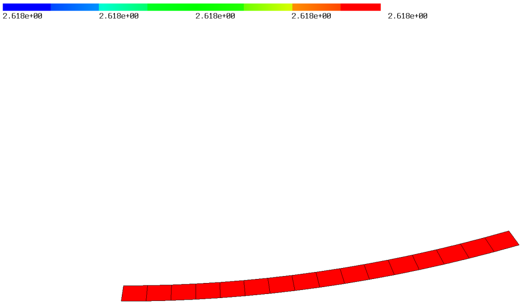


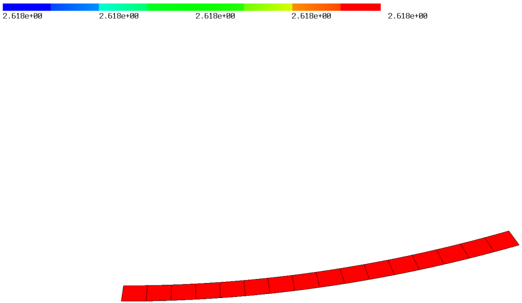


Numerics & Applications

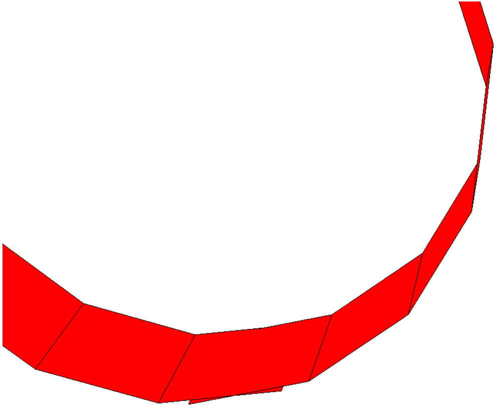
Example (cantilever bending)

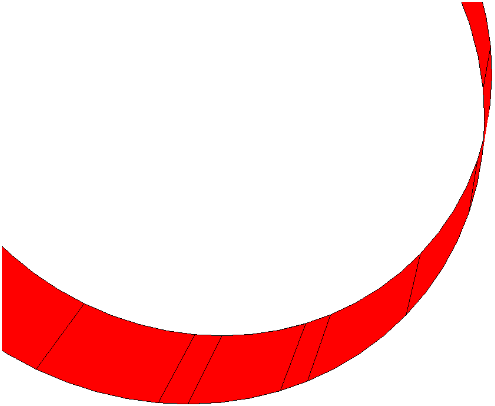
NGSolve



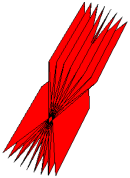


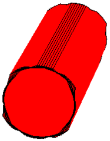
Example (cantilever bending)



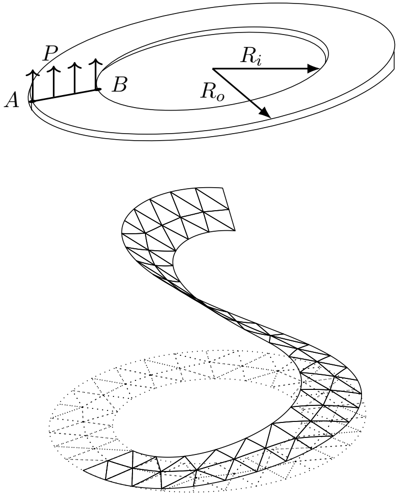


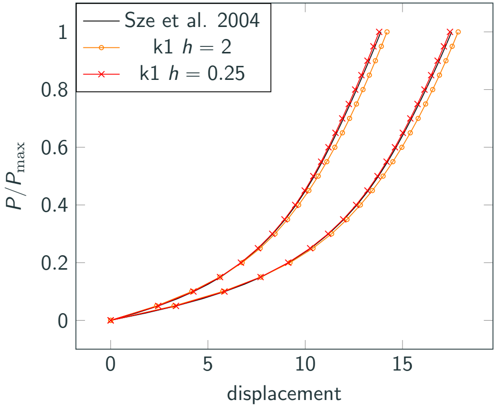
Example (cantilever bending)



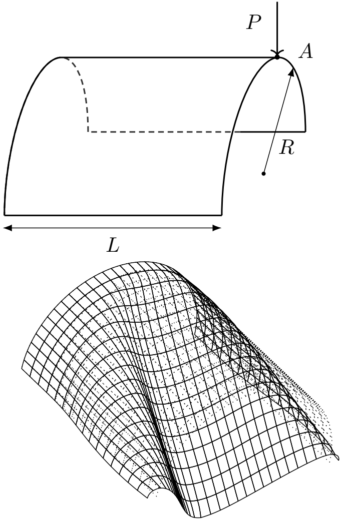


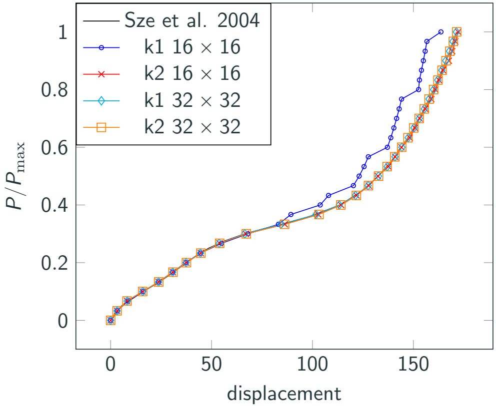
Slit annular plate





Pinched cylinder



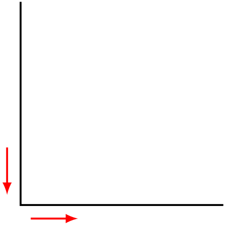


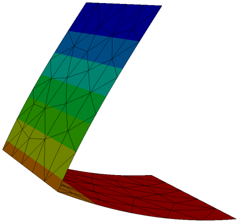
Structures with kinks and branched shells

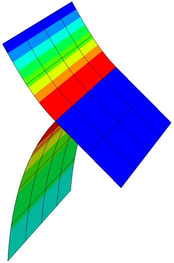
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

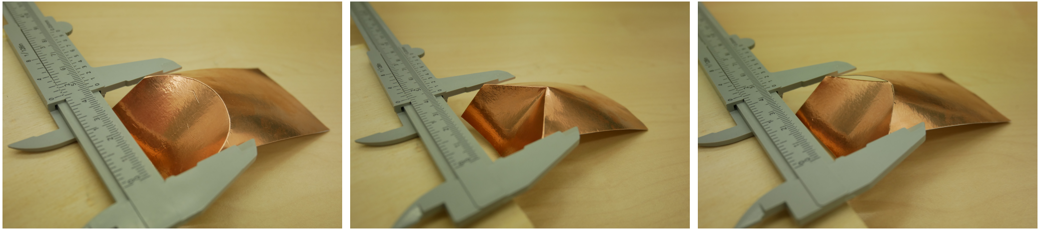
$$\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0$$

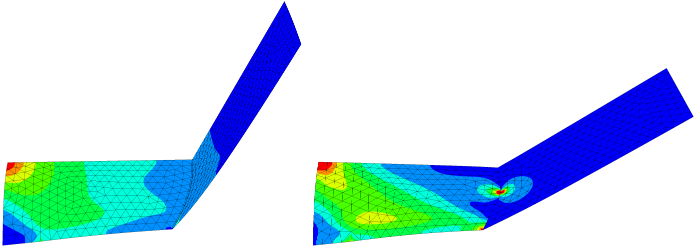
$$\Rightarrow \quad \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R) = 0$$







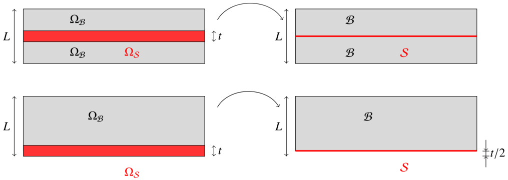


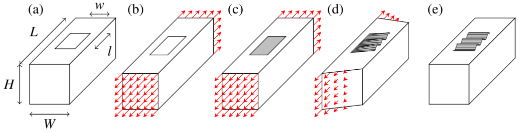


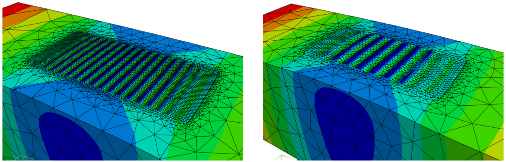
Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces Free Bound.* (2026).

Coupling shells – 3D elasticity, wrinkling experiment

- Composite materials, blood vessels, etc.
- Lagrange elements for elasticity and shell displacement → easy to couple







PECHSTEIN, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)

Summary

- DDG + FEM: Generalized shape operator (Weingarten tensor)
- Bending energy for shell model
- Regge elements for membrane locking
- Applications (coupling, origami)

- Reissner–Mindlin/Naghdi shell
- Linear and nonlinear shell models

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Thank You for Your attention!

Literature

- Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces Free Bound.* (2026).
- Pechstein, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)
- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg (2021).*
- N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis (2021).*
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*