Riemann curvature of Regge metrics

From angle defects to finite element approximation

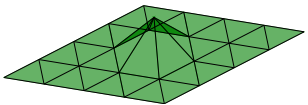
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LOGOS Seminar, University of Vienna
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FWF Austrian Science Fund
Project J 4824-N



NGSolve

Outline

1 Regge metrics and finite elements

2 Generalized curvature and finite element lifting

3 Generalized Riemann curvature tensor and convergence

4 Three-dimensional singular geometries

Part 1

Regge metrics and finite elements

Approximating the metric

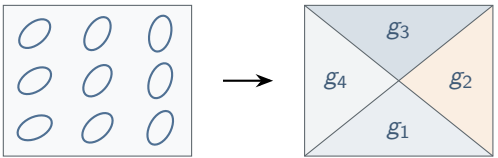
Given: Riemannian target metric $\bar{g}$

Discrete unknown: metric g_h on a simplicial mesh

Lengths, angles, and geodesics computed from g_h

The approximation question

If g_h approximates $\bar{g}$,
what happens to its curvature?

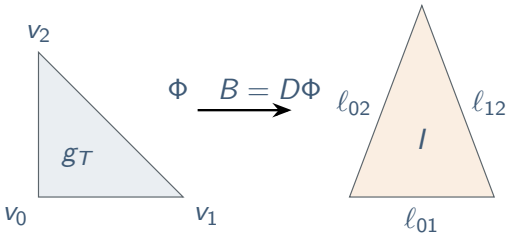


smooth target $\bar{g}$ elementwise metric g_h

Uniform relative metric accuracy controls lengths and distances

Here $g_h > 0$, extension to Lorentzian signature possible

Edge lengths determine a constant metric



fixed coordinates — Euclidean realization

On affine simplex

$$\ell_{ij}^2 = (v_j - v_i)^\top g_T (v_j - v_i).$$

$n(n+1)/2$ edge lengths determine the symmetric metric matrix

Lengths must describe nondegenerate Euclidean simplex: $g_T > 0$

Reference triangle: $v_0 = 0,\ v_1 = e_1,\ v_2 = e_2$

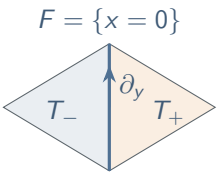
$$g_T = \begin{pmatrix} \ell_{01}^2 & (\ell_{01}^2 + \ell_{02}^2 - \ell_{12}^2)/2 \\ (\ell_{01}^2 + \ell_{02}^2 - \ell_{12}^2)/2 & \ell_{02}^2 \end{pmatrix} \qquad g_T = B^\top B$$

Regge, General relativity without coordinates, Il Nuovo Cimento 19 (1961); Sorkin, Time-evolution problem in Regge calculus, Phys. Rev. D 12 (1975).

Isometric gluing and tangential continuity

The induced metrics agree on each common face

$$g_+(X, Y) = g_-(X, Y) \qquad X, Y \in TF, \quad F = T_+ \cap T_-$$



$$g_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g_+ = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

Both induce $\mathrm{d}y^2$ on F

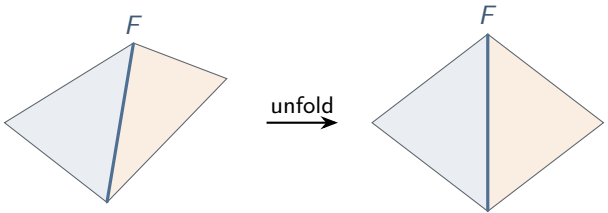
Normal and mixed components may jump

Degree zero: shared edge lengths determine same metric on common face F

Regge metrics: retain this face condition for elementwise smooth metrics

Geodesics across a shared edge

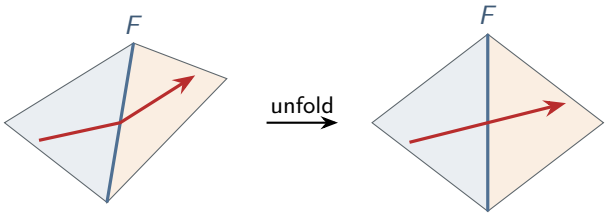
For a **piecewise constant metric**, every triangle is intrinsically flat



1. Develop two adjacent triangles into same Euclidean plane

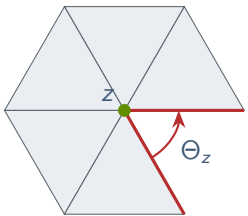
Geodesics across a shared edge

For a **piecewise constant metric**, every triangle is intrinsically flat



1. Develop two adjacent triangles into same Euclidean plane
2. Continue the geodesic as a straight line across the edge
3. Fold back

Angle defect around a vertex



identify two red edges

Develop triangles around vertex z

$$\Theta_z = 2\pi - \sum_{T \ni z} \alpha_{T,z}$$

E.g., five equilateral triangles give angle defect

$$\Theta_z = 2\pi - 5\frac{\pi}{3} = \frac{\pi}{3}$$

Nonzero defect prevents vertex patch from developing into a flat disk

Classical Regge curvature: concentrated on codimension-two hinges

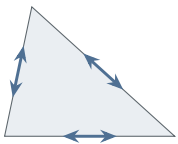
Vertices in 2D, edges in 3D, triangles in 4D, etc.

Regge, General relativity without coordinates, Il Nuovo Cimento 19 (1961)
Christiansen, *On the definition of curvature in Regge calculus*, IMA J. Numer. Anal. 44 (2024)

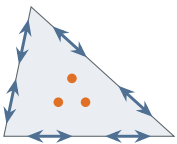
Regge finite elements and moment interpolation

Affine simplices $\quad \mathbb{S}$: symmetric matrices $\quad$ Riemannian metric: $g_h > 0$

$$\mathrm{Reg}_h^k = \{\sigma_h : \ \sigma_h|_T \in \mathcal{P}_k(T;\mathbb{S}), \quad \sigma_h|_{TF\times TF} \text{ is single-valued}\}$$



$k = 0$: 3 edge moments



$k = 1$: 6 edge moments
and 3 cell moments

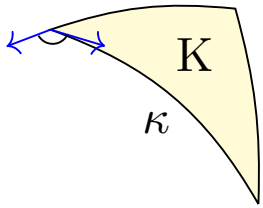
Canonical interpolation in 2D: match tangential metric moments

$$\int_e (g_h - \bar{g})(t,t)\, q \,\mathrm{d}s = 0 \qquad q \in \mathcal{P}_k(e)$$

Euclidean unit tangent t and arc length $\mathrm{d}s$ $\quad$ Cell moments for $k \geq 1$

S. H. Christiansen, *On the linearization of Regge calculus*, Numer. Math. 119 (2011).
L. Li, *Regge Finite Elements with Applications in Solid Mechanics and Relativity*, PhD thesis, University of Minnesota (2018).

Higher-order metrics can curve inside an element



Intrinsic curvature within a single metric triangle

Consider the affine metric

$$g = \mathrm{d}x^2 + (1+x)\,\mathrm{d}y^2, \qquad x > -1$$

With $f = \sqrt{1+x}$,

$$K_g = -\frac{f''}{f} = \frac{1}{4(1+x)^2}$$

Geodesic curvature: $\kappa_g = g(\nabla_\tau \tau, \nu)$

τ: g-unit tangent; ν: inward g-unit normal

$$y = \mathrm{const}: \quad \kappa_g = 0, \qquad x = c: \quad |\kappa_g| = \frac{1}{2(1+c)}$$

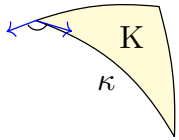
Curvature contributions: bulk curvature, edge jumps, and vertex defects

Berchenko-Kogan, Gawlik, *Finite Element Approximation of the Levi-Civita Connection and Its Curvature in Two Dimensions*, Found. Comput. Math. 24 (2024)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023)

Part 2

Generalized curvature and finite element lifting

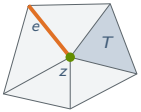
Gauss–Bonnet on glued triangles



$$\int_T K_g \,\mathrm{d}A_g + \int_{\partial T} \kappa_g \,\mathrm{d}s_g + \sum_{z \in \partial T} (\pi - \alpha_{T,z}) = 2\pi$$

Berchenko-Kogan, Gawlik, *Finite Element Approximation of the Levi-Civita Connection and Its Curvature in Two Dimensions*, Found. Comput. Math. 24 (2024)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023)

Gauss–Bonnet on glued triangles



$$\int_T K_g \,\mathrm{d}A_g + \int_{\partial T} \kappa_g \,\mathrm{d}s_g + \sum_{z \in \partial T} (\pi - \alpha_{T,z}) = 2\pi$$

$$\langle \widetilde{K_g \,\mathrm{d}A}, v \rangle = \sum_T \int_T K_g v \,\mathrm{d}A_g + \sum_{e \text{ int.}} \int_e [\![\kappa_g]\!] v \,\mathrm{d}s_g + \sum_{z \text{ int.}} \Theta_z v(z)$$

$$\Theta_z = 2\pi - \sum_{T \ni z} \alpha_{T,z}$$

Element interior	**Edge**	**Vertex**
Gauss curvature	Jump of geodesic curvature	Angle defect

Choose continuous functions v that vanish near the outer boundary

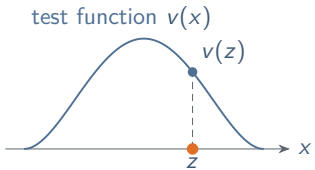
On a closed surface: $\langle \widetilde{K_g \,\mathrm{d}A}, 1 \rangle = 2\pi\chi(M)$

Berchenko-Kogan, Gawlik, *Finite Element Approximation of the Levi-Civita Connection and Its Curvature in Two Dimensions*, Found. Comput. Math. 24 (2024)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023)

Generalized Gauss curvature as a measure/distribution

Write $\mu_h = \widetilde{K_{g_h}\,\mathrm{d}A_{g_h}}$. A continuous test function specifies what we measure:

$$\mu_h(v) = \sum_T \int_T K_{g_h} v \,\mathrm{d}A_{g_h} + \sum_{e \text{ int.}} \int_e [\![\kappa_{g_h}]\!] v \,\mathrm{d}s_{g_h} + \sum_{z \text{ int.}} \Theta_z v(z)$$



$$\mu_{\{z\}} = \Theta_z \delta_z$$
$$\mu_{\{z\}}(v) = \Theta_z\, v(z)$$

Direct assembly of a finite element load vector

The pairing treats all three supports:

A smooth density is integrated over area

A line measure is integrated along an edge

A point mass is evaluated at its vertex

Berchenko-Kogan, Gawlik, *Finite Element Approximation of the Levi-Civita Connection and Its Curvature in Two Dimensions*, Found. Comput. Math. 24 (2024)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023)

Finite element assembly of curvature

Choose a continuous scalar test space $V_h = \operatorname{span}\{\varphi_i\}$ (e.g. Nodal/Lagrange finite elements)

$$g \longrightarrow g_h \in \mathrm{Reg}_h^k \longrightarrow \mu_h \longrightarrow b_i = \mu_h(\varphi_i)$$

1. Interpolate the metric using its (shared tangential) moments
2. Assemble element, edge, and vertex contributions into vector b
3. Use b in a PDE, or reconstruct a field (e.g. for visualization)

Mass-matrix reconstruction (lifting)

$$\int_\Omega K_h v_h \,\mathrm{d}A_{g_h} = \mu_h(v_h) \quad (v_h \in V_h), \qquad M\boldsymbol{K} = \boldsymbol{b}$$

$K_h \in V_h$ is a regular function (discrete L^2 Riesz representative of a functional, thus "lifting")

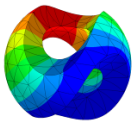
Gawlik, *High-Order Approximation of Gaussian Curvature with Regge Finite Elements*, SIAM J. Numer. Anal. 58 (2020)
Berchenko-Kogan, Gawlik, *Finite Element Approximation of the Levi-Civita Connection and Its Curvature in Two Dimensions*, Found. Comput. Math. 24 (2024)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023)

NGSDiffGeo with NGSolve

Regge metrics, geometric operators, and finite element assembly in Python

```
from ngsolve import *
import ngsdiffgeo as dg

gh = GridFunction(HCurlCurl(mesh, order=2))
gh.Set(g, dual=True)
geometry = dg.RiemannianManifold(gh, normal_sign=-1)
```



NGSolve (www.ngsolve.org)
Meshes, finite element spaces, quadrature, sparse matrices, solvers

NGSDiffGeo
Metric-dependent geometry: `Gauss`, `GeodesicCurvature`, `AngleDefect`, tensor fields, covariant derivatives, etc.

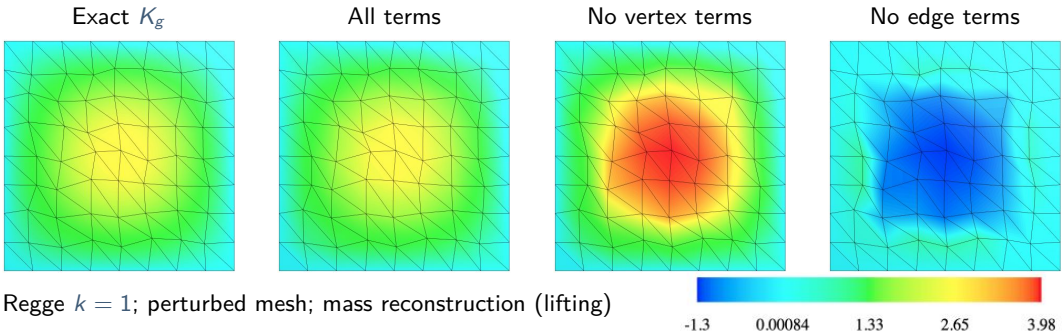
michaelneunteufel.github.io/NGSDiffGeo

GitHub: MichaelNeunteufel/NGSDiffGeo

Experiment 1: reconstructing smooth Gauss curvature

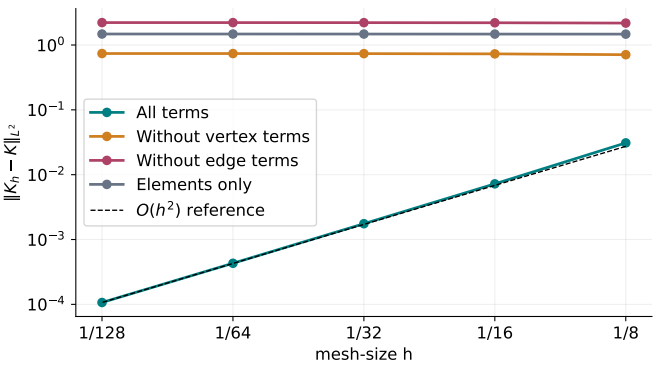
On $\Omega = (0,1)^2$

$$g = e^{2\psi} I, \qquad \psi = 0.2 \sin(\pi x) \sin(\pi y), \qquad K_g = -e^{-2\psi} \Delta\psi = 2\pi^2 \psi e^{-2\psi}$$



Regge $k = 1$; perturbed mesh; mass reconstruction (lifting)

All three contributions matter under refinement



Full assembly

Perturbed meshes; $h = 1/n$

L^2 errors:

n	error
8	$3.09 \cdot 10^{-2}$
16	$7.22 \cdot 10^{-3}$
32	$1.75 \cdot 10^{-3}$
64	$4.30 \cdot 10^{-4}$
128	$1.07 \cdot 10^{-4}$

Omitting either skeleton term gives stagnating errors

A smooth target still needs singular terms of its Regge approximation

Experiment 2: a line of curvature

$$g = dx^2 + f(x)^2 dy^2, \quad f(x) = 1 + a|x|, \quad \Omega = (-1,1)^2, \quad a = 0.5$$

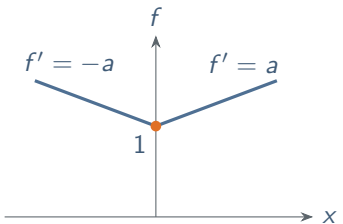
Flat on both sides of $x = 0$

$$K_g = -\frac{f''}{f}, \qquad \mathrm{d}A_g = f \,\mathrm{d}x\,\mathrm{d}y$$

The derivative jumps at $x = 0$ by $2a$:

$$\boxed{\mu = \widetilde{K_g \,\mathrm{d}A_g} = -2a\,\delta_0(dx)\,dy}$$

Continuous metric has line measure of curvature



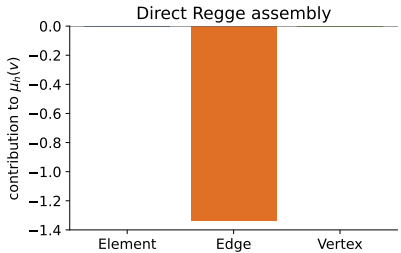
The jump in f'
creates the line measure

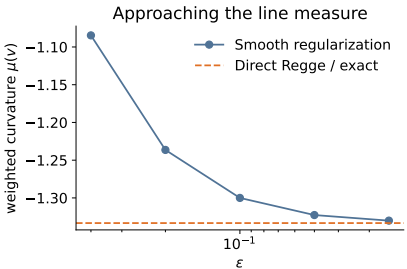
Exact representation: Regge $k = 2$ metric on a mesh fitted to the seam

Direct assembly and smooth regularization

Compare one weighted measurement: $\mu(v) = -2a \int_{-1}^{1} v(0,y)\,\mathrm{d}y = -\frac{4}{3}$

$v = (1-x^2)(1-y^2)$ vanishes on $\partial\Omega$





Direct: assemble the curvature measure
Regge $k = 2$, 128 triangles fitted to $x = 0$
Full assembly: error $1.4 \cdot 10^{-11}$

Smooth: evaluate the integral analytically
$f_\varepsilon = 1 + a(\sqrt{x^2 + \varepsilon^2} - \varepsilon)$
Gap to the dashed line: smoothing error

Generalized scalar curvature in arbitrary dimension

Scalar curvature is the trace of Ricci tensor (in 2D $S = 2K_g$)

For smooth compactly supported scalar tests w:

$$\langle \widetilde{S\,\mathrm{d}V}, w\rangle = \sum_T \int_T S w \,\mathrm{d}V_g + 2\sum_{F \text{ int.}} \int_F [\![H]\!]\, w \,\mathrm{d}A_g + 2\sum_{E \text{ int.}} \int_E \Theta_E w \,\mathrm{d}E_g$$

$\mathrm{d}E_g$: induced $(n-2)$-dimensional measure on each hinge E

H: trace of the second fundamental form, with inward normals (in 2D $H = \kappa_g$)

For the seam metric of Experiment 2, with $\Sigma = \{x = 0\}$:

$$\mu_g := \widetilde{S\,\mathrm{d}A_g} = -4a\,\delta_0(dx)\,dy, \qquad \langle \mu_g, w\rangle = -4a \int_\Sigma w \,\mathrm{d}y$$

Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp. 94 (2025)

Experiment 3: a curvature-coupled scalar field

A scalar field on the seam geometry

$$(-\Delta_g + \xi S_g)u = \lambda u, \qquad S_g = 2K_g, \quad \xi \in \mathbb{R}$$

Weak form: Find $(u, \lambda) \in H_0^1(\Omega) \setminus \{0\} \times \mathbb{R}$ s.t. for all $v \in H_0^1(\Omega)$

$$\int_\Omega g(\nabla_g u, \nabla_g v)\,\mathrm{d}A_g + \xi \widetilde{S\,\mathrm{d}A_g}(u\,v) = \lambda \int_\Omega uv\,\mathrm{d}A_g$$

Here $g = \mathrm{diag}(1, f^2)$, $f = 1 + a|x|$, and $\Sigma = \{x = 0\}$

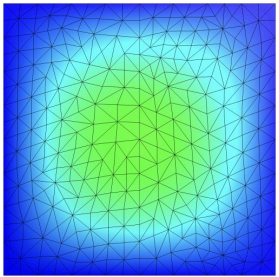
$$\underbrace{\int_\Omega \left(f u_x v_x + f^{-1} u_y v_y\right)\mathrm{d}x\,\mathrm{d}y}_{\text{kinetic term}} + \underbrace{\xi\left(-4a\int_\Sigma (\gamma_\Sigma u)(\gamma_\Sigma v)\,\mathrm{d}y\right)}_{=:c_g(u,v)} = \lambda \underbrace{\int_\Omega fuv\,\mathrm{d}x\,\mathrm{d}y}_{\text{mass}}$$

$\gamma_\Sigma : H^1(\Omega) \to L^2(\Sigma)$: trace; c_g is continuous on $H_0^1 \times H_0^1$

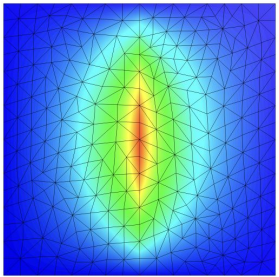
FEM: $(A + \xi C)\boldsymbol{u} = \lambda M \boldsymbol{u}$, $\quad C_{ij} = c_g(\varphi_i, \varphi_j)$

Ground modes on the finite element mesh

No curvature coupling: $\xi = 0$



Attractive curvature coupling: $\xi = 3$

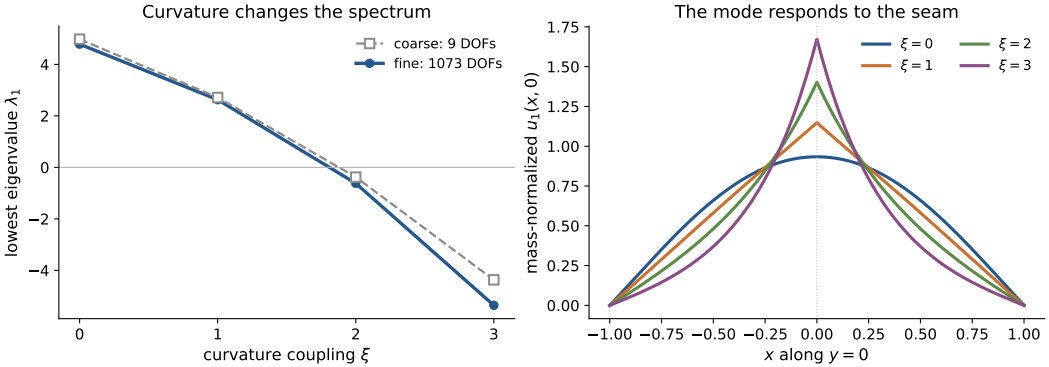




Same mesh and mass normalization: $\int_\Omega u_1^2 \, \mathrm{d}A_g = 1$

Mode becomes concentrated near the seam $x = 0$

The field responds to the curvature seam



$\xi = 0, 1, 2, 3$: the lowest mode concentrates near the seam as coupling increases

Part 3

Generalized Riemann curvature tensor and convergence

From scalar curvature to the Riemann tensor

The full tensor records curvature in every two-plane

Riemannian curvature tensor: $g > 0$. Fix the convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$$
$$\mathrm{Ric}(X,Y) = \sum_i \mathcal{R}(e_i,X,Y,e_i), \qquad S = \sum_i \mathrm{Ric}(e_i,e_i)$$

Orthonormal X, Y: $K(X,Y) = \mathcal{R}(X,Y,Y,X)$. Unit sphere: $K = +1$

In two dimensions
Gauss curvature determines Riemann tensor

In higher dimensions
Different two-planes carry different curvature

The full tensor: bulk, face, and hinge curvature

Admissible tensor test A (analogous to scalar tests v)

$$\widetilde{\mathcal{R}\,\mathrm{d}V}_g(A) = \sum_{T\in\mathcal{T}} \int_T \langle \mathcal{R}_g, A\rangle_g \,\mathrm{d}V_g$$

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted
Gawlik, McKee, *On the Curvature of Regge Metrics*, arXiv:2510.25027v2 (2026)

The full tensor: bulk, face, and hinge curvature

Admissible tensor test A (analogous to scalar tests v)

$$\widetilde{\mathcal{R}\,\mathrm{d}V}_g(A) = \sum_{T\in\mathcal{T}} \int_T \langle \mathcal{R}_g, A\rangle_g \,\mathrm{d}V_g + 4 \sum_{F \text{ int.}} \int_F \langle [\![\mathrm{II}]\!]_F, A_{F\nu\nu F}\rangle_{g_F} \,\mathrm{d}A_g$$

F: facet ν: inward unit normal
$A_{F\nu\nu F}(X,Y) = A(X,\nu,\nu,Y)$

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted
Gawlik, McKee, *On the Curvature of Regge Metrics*, arXiv:2510.25027v2 (2026)

The full tensor: bulk, face, and hinge curvature

Admissible tensor test A (analogous to scalar tests v)

$$\widetilde{\mathcal{R}\,\mathrm{d}V}_g(A) = \sum_{T\in\mathcal{T}} \int_T \langle \mathcal{R}_g, A\rangle_g \,\mathrm{d}V_g + 4 \sum_{F \text{ int.}} \int_F \langle [\![\mathrm{II}]\!]_F, A_{F\nu\nu F}\rangle_{g_F} \,\mathrm{d}A_g + 4 \sum_{E \text{ int.}} \int_E \Theta_E\, A(\eta,\nu,\nu,\eta)\,\mathrm{d}E_g$$

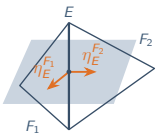
F: facet $\quad$ ν: inward unit normal

$A_{F\nu\nu F}(X,Y) = A(X,\nu,\nu,Y)$

E: codimension-two hinge $\quad$ $\mathrm{d}E_g$: metric measure

η: unit conormal of E within F

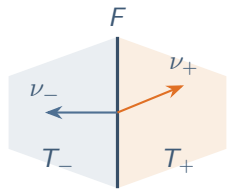
$\Theta_E = 2\pi - \sum_{T\supset E} \theta_E^T$



Cross-section around a 3D hinge

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted
Gawlik, McKee, *On the Curvature of Regge Metrics*, arXiv:2510.25027v2 (2026)

The face term and its test trace



Inward normals need not be opposite

$$\mathrm{II}^{\nu}(X, Y) = -g(\nabla_X \nu, Y)$$

$$[\![\mathrm{II}]\!]_F = \mathrm{II}^{\nu_+} + \mathrm{II}^{\nu_-}$$

Shared first fundamental form g_F
Second fundamental forms may fail to cancel

Geometric jump paired with one shared test trace

$$4 \int_F \langle [\![\mathrm{II}]\!], A_{F\nu\nu F} \rangle_{g_F} \, \mathrm{d}A_g \qquad A_{F\nu\nu F}(X, Y) = A(X, \nu, \nu, Y)$$

Comparing curvature on a fixed test space

Metric-dependent normals, contractions and traces

Hodge identification

$$A = \mathbb{A}_g U, \qquad \mathcal{C}(g)(U) := \widetilde{\mathcal{R}\,\mathrm{d}V_g}(\mathbb{A}_g U)$$

U: symmetric $(n-2, n-2)$ double form with single-valued tangential trace

In 2D
U: continuous scalar function

In 3D
U: symmetric covariant 2-tensor with tangential–tangential continuity

Uhlenbeck trick: Fix U and compare $\mathcal{C}(g_h)(U)$ and $\mathcal{C}(\bar{g})(U)$

Generalized Einstein tensor

$g > 0 \qquad n \geq 3 \qquad G = \mathrm{Ric} - \frac{1}{2} S g$

Test σ: piecewise smooth and symmetric with tangential–tangential continuity

$$\begin{aligned}\langle \widetilde{G \,\mathrm{d}V_g}, \sigma \rangle = &\sum_T \int_T \langle G, \sigma \rangle_g \,\mathrm{d}V_g \\ &+ \sum_{F \text{ int.}} \int_F \langle [\![\mathrm{II} - H g_F]\!], \sigma_F \rangle_{g_F} \,\mathrm{d}A_g \\ &- \sum_{E \text{ int.}} \int_E \Theta_E \langle g_E, \sigma_E \rangle_{g_E} \,\mathrm{d}E_g\end{aligned}$$

$H = \mathrm{tr}_{g_F} \mathrm{II} \qquad \sigma_D, g_D$: tangential restrictions on $D = F, E$

Variational meaning: negative first variation of total generalized scalar curvature

In 3D: $\langle \widetilde{G \,\mathrm{d}V_g}, U \rangle = -\frac{1}{4} \mathcal{C}(g)(U)$

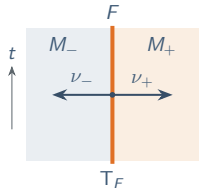
Full Riemann information in a symmetric 2-tensor by Hodge duality

Gawlik, Neunteufel, *Finite element approximation of the Einstein tensor*, IMA J. Numer. Anal. 46 (2026)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Facet curvature and the Israel junction conditions

Continuous piecewise smooth spacetime metric of signature $(-+++)$

Timelike shell F**:** spacelike unit normals $\quad G_N = c = 1$



Schematic spacetime section

Same induced metric: $g_F^+ = g_F^- = g_F$

$$B_F := [\![\mathrm{II} - H g_F]\!], \qquad (G\,\mathrm{d}V_g)_{\mathrm{sing}} = B_F\,\mathrm{d}A_g|_F$$

$\mathrm{II}^\nu = -g(\nabla\nu, \cdot)$ with inward normals

Einstein's equation $G = 8\pi T$

$$\boxed{B_F = 8\pi\,\mathsf{T}_F}$$

T_F: surface stress–energy tensor (energy density and tangential stresses)

Nonzero facet term as a possible physical thin shell

Israel, *Singular hypersurfaces and thin shells in general relativity* (1966)
Khakshournia, Mansouri, *The Art of Gluing Space-Time Manifolds: Methods and Applications* (2023)

Convergence to a smooth target metric

Assumption: $\bar{g}$ sufficiently smooth and uniformly positive definite
g_h: degree-k Regge interpolant on a shape-regular affine mesh family

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)

$$\boxed{\|\mathcal{C}(g_h) - \mathcal{C}(\bar{g})\|_{H^{-2}} \leq Ch^{k+1}}$$

$$k \geq 0 \quad (n = 2), \qquad k \geq 1 \quad (n \geq 3)$$

Small error for every smooth weighted measurement with bounded H^2-size

$$\|F\|_{H^{-2}} = \sup_{0 \neq U \in H_0^2(\Omega;\mathbb{S}^m)} \frac{|\langle F, U \rangle|}{\|U\|_{H^2}}, \qquad m = \frac{n(n-1)}{2}$$

Metric approximation order retained in the weak curvature norm

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Why a weak norm can retain the metric order

One-dimensional model: $e \in L^2$ and $v \in H^2_0$

$$\langle e'', v\rangle = \int e\, v''\,\mathrm{d}x, \qquad |\langle e'', v\rangle| \le \|e\|_{L^2}\|v\|_{H^2}$$

Two derivatives transferred from approximation error to test

Nonlinear curvature: $e_h = g_h - \bar{g}$ $\qquad g(t) = \bar{g} + te_h$

$$\big(\mathcal{C}(g_h) - \mathcal{C}(\bar{g})\big)(U) = \int_0^1 D\mathcal{C}(g(t))[e_h](U)\,\mathrm{d}t$$

Two integrations by parts of the linearization including face and hinge terms

Skeleton terms as the interface contributions required by integration by parts

Gawlik, *High-Order Approximation of Gaussian Curvature with Regge Finite Elements*, SIAM J. Numer. Anal. 58 (2020)
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Experiment 4: smooth Riemann curvature in 3D

$\Omega = (-1,1)^3 \quad \bar{g} = I + \nabla f \otimes \nabla f$

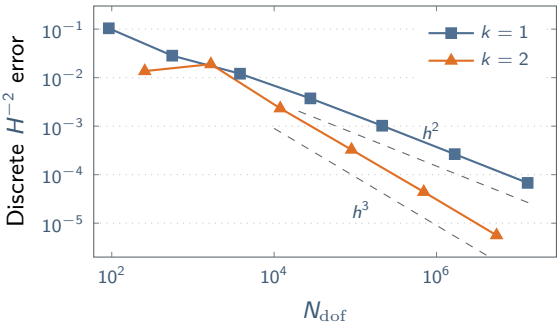
$$f(x,y,z) = \sum_{\xi \in \{x,y,z\}} \left(\frac{\xi^2}{2} - \frac{\xi^4}{12} \right)$$

Perturbed tetrahedral meshes

Curvature functional used:
$\frac{1}{4}\mathcal{C}$

Discrete approximation of an equivalent H^{-2} norm via componentwise biharmonic problems

Reference rates: h^2 for $k = 1$ and h^3 for $k = 2$



Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Part 4

Three-dimensional singular geometries

Experiment 5: the spatial geometry of a cosmic string

Ideal straight string with conical spatial metric

$$g = dr^2 + \alpha^2 r^2 d\theta^2 + dz^2, \qquad \theta \in [0, 2\pi), \quad 0 < \alpha < 1$$

Angle deficit enclosed by a transverse circuit

$$\Delta = 2\pi(1 - \alpha)$$

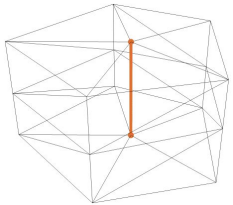
Flat away from the axis with curvature as a line measure

$$\boxed{\langle \widetilde{S\,\mathrm{d}V}, v \rangle = 2\Delta \int_{\text{axis}} v(0, 0, z)\,\mathrm{d}z}$$

Regge interpretation: axis as a chain of mesh edges carrying hinge curvature

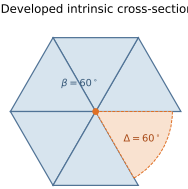
Clarke, Vickers, Wilson, *Generalised Functions and Distributional Curvature of Cosmic Strings*, Class. Quantum Grav. 13 (1996)

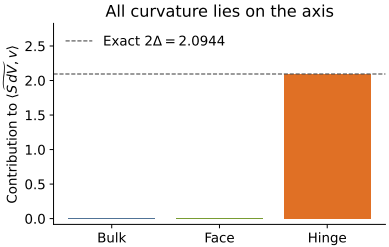
A conical metric on a small tetrahedral mesh



Perturbed chart with 30 tetrahedra

Orange string axis





Five intrinsic 60° sectors: $\alpha = 5/6$ and $\Delta = \pi/3$

Shared tangential lengths with angle deficit on the axis

$v = w(x, y)\, 6z(1 - z)$ with $w = 1$ on the axis and zero on the outer polygon

Weighted curvature: exact $2\pi/3$ absolute error $1.2 \cdot 10^{-14}$

Experiment 6: a thin spherical matter shell

Time symmetry: extrinsic curvature $K = 0$ $g > 0$ $G_N = c = 1$ $\Lambda = 0$

Hamiltonian constraint: $\widetilde{S\,\mathrm{d}V}_g = 16\pi\,\rho\,\mathrm{d}V_g$

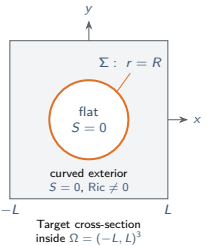
$$g = \psi(r)^4 I \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$\psi(r) = \begin{cases} 1 + \dfrac{m}{2R}, & r \le R \\ 1 + \dfrac{m}{2r}, & r \ge R \end{cases}$$

$R = 1$ (isotropic radius) $m = 0.2$ $L = 2$

Continuous metric with a jump in normal derivative

Flat interior and spatial Schwarzschild exterior



Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity* (2007)
Anglada, Dain, Ortiz, *Inequality between size and charge in spherical symmetry*, Phys. Rev. D 93 (2016)

The entire scalar-curvature measure lies on the shell

Conformal curvature formula

$$S(g) = -8\psi^{-5}\Delta\psi, \qquad \mathrm{d}V_g = \psi^6\,\mathrm{d}x$$

Harmonic on both sides with a jump in normal derivative

$$[\partial_r\psi]_\Sigma = -\frac{m}{2R^2}, \qquad \Delta\psi = -\frac{m}{2R^2}\,\delta_\Sigma^{\mathrm{eucl}}$$

Scalar curvature measure

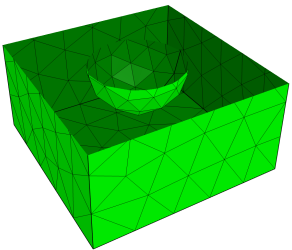
$$\boxed{\langle \widetilde{S\,\mathrm{d}V}, v\rangle = \frac{4m\psi(R)}{R^2}\int_\Sigma v\,\mathrm{d}A_{\mathrm{eucl}}}$$

Hamiltonian constraint: proper matter mass (test $v = 1$)

$$M_{\mathrm{proper}} = \frac{1}{16\pi}\langle \widetilde{S\,\mathrm{d}V}_g, 1\rangle = m\psi(R) = 0.22$$

Elementwise scalar curvature alone misses the matter layer

A continuous polynomial Regge approximation



Affine tetrahedra
Planar approximation of Σ

Continuous nodal $\mathcal{P}_1$ interpolant ψ_h of ψ

$$g_h = \psi_h^4 I \in \mathrm{Reg}_h^4$$

Within elements

$\Delta\psi_h = 0$ and $S(g_h) = 0$

Hinges

No angle defects by full metric continuity

Facets

Jumps in normal derivatives of ψ_h

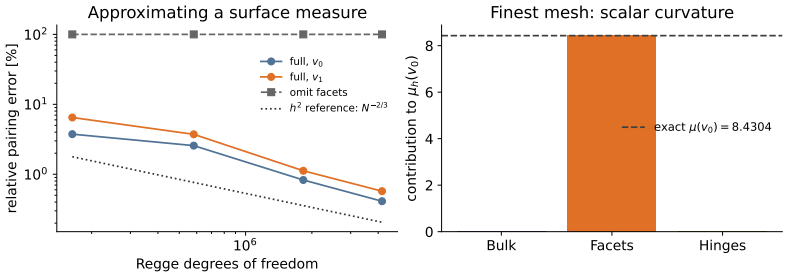
Combined facet measure approximating the shell

Contributions away from Σ possible at finite h

Facet measures approach the spherical matter layer

Fixed tests with zero outer trace and $L = 2$

$v_0 = \prod_{q\in\{x,y,z\}}(1 - q^2/L^2)$ $\quad$ $v_1 = (1 + x^2/R^2)v_0$



Meshes: 1 248 / 4 414 / 14 157 / 32 249 tetrahedra with metric degree four

Full curvature (v_0): relative errors 3.75% → 2.57% → 0.83% → 0.41%

Omitting facets: zero up to numerical cancellation with the shell contribution lost

Summary and outlook

Regge metrics + generalized curvature

- Intrinsic metric approximation with finite elements
- Curvature from bulk + facets + hinges
- Weak convergence for smooth targets
  Singular measures in numerical examples

Towards numerical relativity

- $3+1$ splitting with positive definite spatial Regge metrics
- Constraint equations for unknown initial data
- Coupled fields and geometry
  Convergence for singular targets

NGSDiffGeo + NGSolve: from a metric to curvature assembly and field problems

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NGSDiffGeo + NGSolve: from a metric to curvature assembly and field problems

Thank You for Your Attention!

References: Regge foundations and relativity

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É. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, arXiv:gr-qc/0703035 (2007)

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E. S. Gawlik, J. McKee, *On the Curvature of Regge Metrics*, arXiv:2510.25027v2 (2026)

The common test space

U: symmetric double $(n-2)$-form

$$\mathcal{U}(\mathcal{T}) = \left\{ U \in \mathrm{Sym}^2(\Lambda^{n-2}(\mathcal{T})) : U|_{TF} \text{ is single-valued on each interior } F \right\}$$

Boundary condition: $U|_{TF} = 0$ on boundary facets

Metric-dependent identification, fixed input U

$$\mathbb{A}_g = -\star_g^{\odot 2}, \qquad \mathcal{C}(g)(U) = \widetilde{\mathcal{R}\,\mathrm{d}V}_g(\mathbb{A}_g U)$$

$$(\mathbb{A}_g U)(X,Y,Z,W) = \left\langle U, \star_g(X^\flat \wedge Y^\flat) \odot \star_g(W^\flat \wedge Z^\flat) \right\rangle_g$$

$n = 2$**:** U: continuous scalar

$n = 3$**:** U: symmetric covariant 2-tensor with matching tangential–tangential traces

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Convergence: hypotheses and local rate

$\bar{g}$ smooth and uniformly positive definite $\quad$ $\mathcal{T}_h$ shape regular

Regge interpolants $g_h = \mathcal{I}_h^k \bar{g}$ with

$$\|g_h - \bar{g}\|_{L^\infty} \longrightarrow 0 \qquad \sup_h \max_{T \in \mathcal{T}_h} \|g_h\|_{W^{2,\infty}(T)} < \infty$$

Assumed local interpolation bounds for some $p \in [2, \infty]$

$$|\mathcal{I}_h^k \bar{g} - \bar{g}|_{W^{t,p}(T)} \lesssim h_T^{k+1-t} |\bar{g}|_{W^{k+1,p}(T)} \quad (0 \le t \le k+1)$$

and corresponding $W^{s,\infty}$ bounds for $0 < s \le \min\{k+1, 2\}$

Corollary 5.3

$$\|\mathcal{C}(g_h) - \mathcal{C}(\bar{g})\|_{H^{-2}} \le C \left(\sum_{T \in \mathcal{T}_h} h_T^{p(k+1)} |\bar{g}|^p_{W^{k+1,p}(T)} \right)^{1/p} \qquad \begin{cases} k \ge 0, & n = 2 \\ k \ge 1, & n \ge 3 \end{cases}$$

For $p = \infty$: sum replaced by $\max_T h_T^{k+1} |\bar{g}|_{W^{k+1,\infty}(T)}$

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, Comm. Amer. Math. Soc., accepted

Metric approximation also controls distances

Assumed relative metric bounds on a common domain

$$(1-\varepsilon_h)\bar{g} \le g_h \le (1+\varepsilon_h)\bar{g} \qquad 0 \le \varepsilon_h < 1$$

Every admissible curve c

$$\sqrt{1-\varepsilon_h}\, L_{\bar{g}}(c) \le L_{g_h}(c) \le \sqrt{1+\varepsilon_h}\, L_{\bar{g}}(c)$$

Infimum over curves joining p and q

$$\boxed{\sqrt{1-\varepsilon_h}\, d_{\bar{g}}(p,q) \le d_{g_h}(p,q) \le \sqrt{1+\varepsilon_h}\, d_{\bar{g}}(p,q)}$$

Control of Gromov–Hausdorff distance on a common compact domain

Additional approximation information required for curvature bounds

Length preservation and moment interpolation

Parameterize edge by $x(s) = v_i + sa$, $a = v_j - v_i$, $0 \le s \le 1$

Length-preserving choice g_h^{len}

$$\ell_e = \int_0^1 \sqrt{\bar{g}_{x(s)}(a,a)}\,\mathrm{d}s$$

Realize the length of this edge curve exactly

Canonical Regge interpolant g_h^{mom}

$$\int_0^1 (g_h^{\mathrm{mom}} - \bar{g})_{x(s)}(a,a)\, q(s)\,\mathrm{d}s = 0$$

$$q \in \mathcal{P}_k([0,1])$$

Match tangential metric moments

For $k = 0$

$$g_h^{\mathrm{len}}|_e(a,a) = \left(\int_0^1 \sqrt{\bar{g}_{x(s)}(a,a)}\,\mathrm{d}s\right)^2$$

Left: average the speed, then square

$$g_h^{\mathrm{mom}}|_e(a,a) = \int_0^1 \bar{g}_{x(s)}(a,a)\,\mathrm{d}s$$

Right: average the squared speed

A scalar measurement of the singular curvature

One fixed smooth test under mesh refinement

$$v_0(x,y,z) = \left(1 - \frac{x^2}{L^2}\right)\left(1 - \frac{y^2}{L^2}\right)\left(1 - \frac{z^2}{L^2}\right), \qquad L = 2$$

Zero on the cube boundary and positive on the shell

Exact reference from spherical moments

$$\langle \widetilde{S\,\mathrm{d}V}, v_0 \rangle = 16\pi m\psi(R)\left(1 - \frac{R^2}{L^2} + \frac{R^4}{5L^4} - \frac{R^6}{105L^6}\right) = 8.430389\ldots$$

Second test: $v_1 = (1 + x^2/R^2)v_0$ with exact pairing $4/3$ of this value

Direct integration of facet terms without a smoothing width

Comparison with distributional geometry

A common class of metrics

Continuous piecewise smooth metrics with local regularity

$$g, g^{-1} \in L^\infty, \qquad \partial g \in L^2$$

$\Gamma \in L^2$ and distributional interpretation of

$$\mathrm{Riem} = \partial\Gamma + \Gamma\Gamma$$

What a Lorentzian extension adds

- Preservation of signature and nondegeneracy
- Spacelike, timelike and null facets
  No unit normal of this type on null facets
- Boost angles and causal structure

Continuous seam examples in this class

General Regge metrics: possible jumps in a fixed chart and geometric gluing data

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