Generalized shape operator for nonlinear shells

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Project J 4824-N

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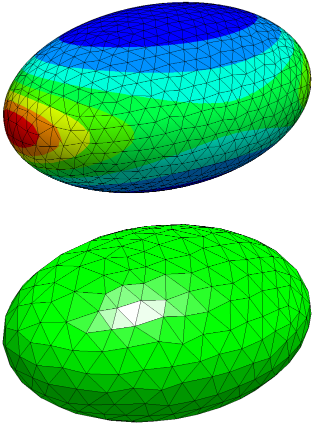
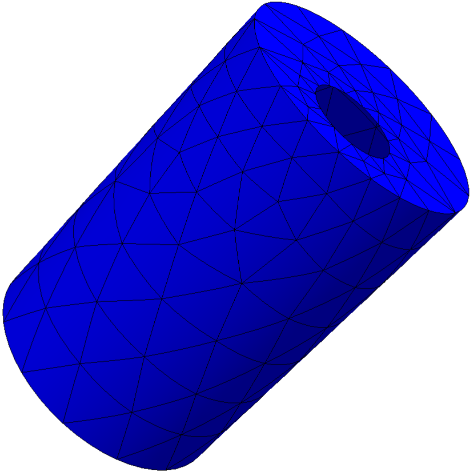
Motivation

Approximate extrinsic curvature of non-smooth surfaces



Application to shells



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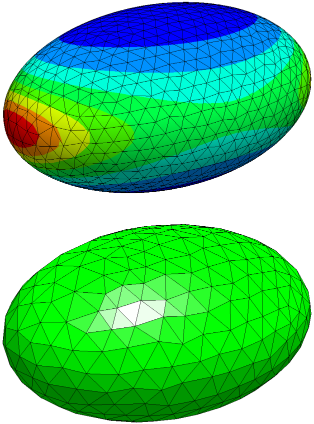

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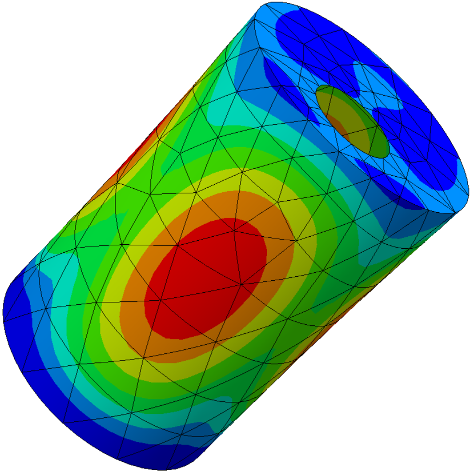
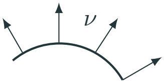


Extrinsic curvature

Weingarten tensor (shape operator)



- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces

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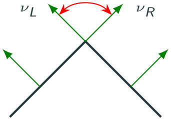

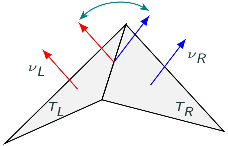

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- Normal vector ν is piecewise constant and jumps

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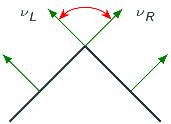

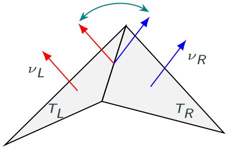

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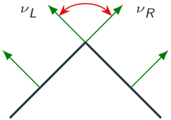

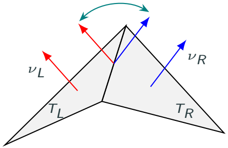

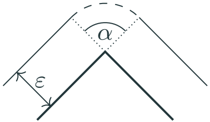

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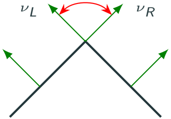
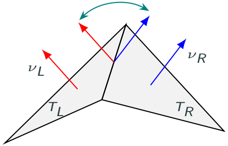
- Dihedral angle formula (from Steiner's offset formula): $\sum_{E\in\mathring{\mathscr{E}}} \alpha_E \, |E|$

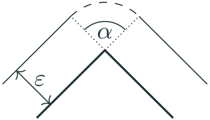
Steiner: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

Grinspun, Gingold, Reisman, Zorin Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

Weingarten tensor (shape operator)







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How to define a generalized Weingarten tensor object? Combine FEM & DDG!

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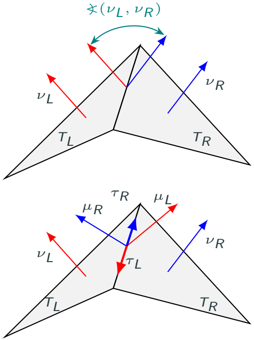
DDG & FEM: Generalized Weingarten tensor

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\widetilde{\nabla\nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}}\int_T \nabla\nu : \boldsymbol{\Psi}\,dx + \sum_{E\in\mathring{\mathscr{E}}}\int_E \sphericalangle(\nu_L,\nu_R)\,\boldsymbol{\Psi}_{\mu\mu}\,ds$$



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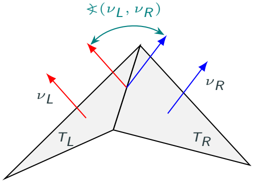
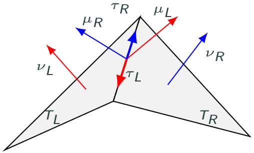
- Test function space

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T},\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \,:\, (\boldsymbol{\sigma}\nu)|_T = 0,\, (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

- Motivation: TDNNS method: $\nabla H(\mathrm{curl}) \subset H(\mathrm{div}\,\mathrm{div})^*$
  $\Sigma \ldots$ Hellan–Herrmann–Johnson space





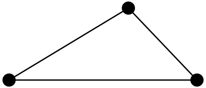
Pechstein, Schöberl: Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity, *M3AS* (2011)

Finite Element Spaces

Lagrange elements:

$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$

$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$



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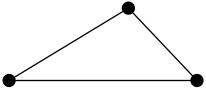

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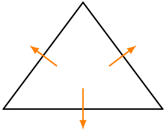

Raviart–Thomas/Brezzi–Douglas–Marini elements:

$$H(\mathrm{div}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}\sigma \in L^2(\Omega)\}$$

$$\mathrm{BDM}_h^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div}, \Omega)$$

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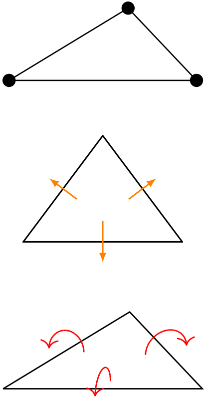

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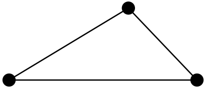

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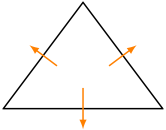

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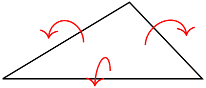

Regge elements:

$$H(\mathrm{curl}\,\mathrm{curl}, \Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\}$$

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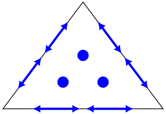
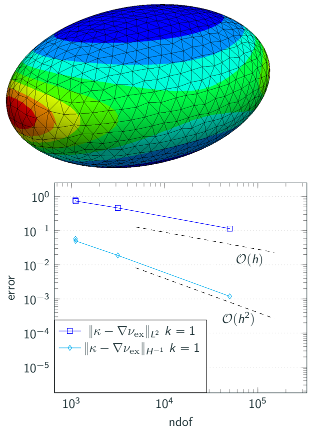


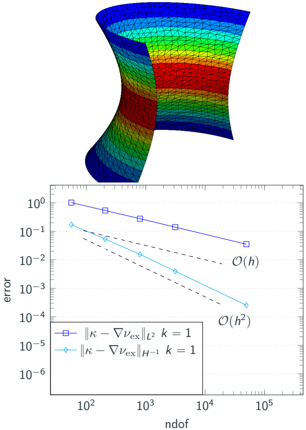
Example curvature approximation

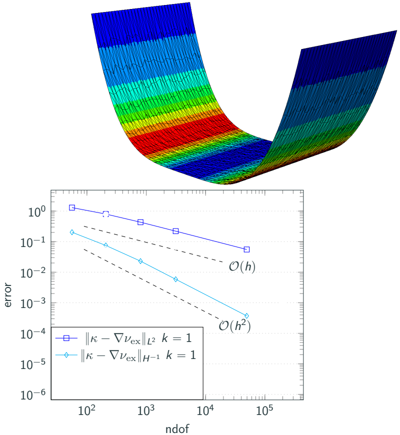
Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \widetilde{\nabla \nu}(\boldsymbol{\sigma}).$$







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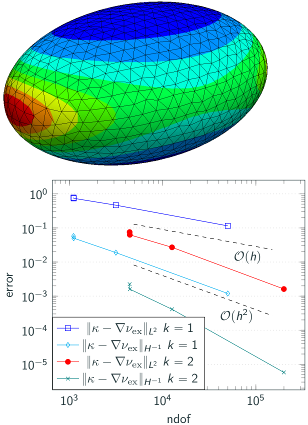
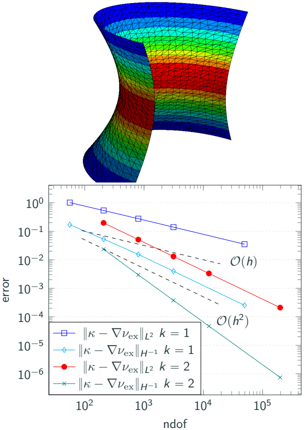
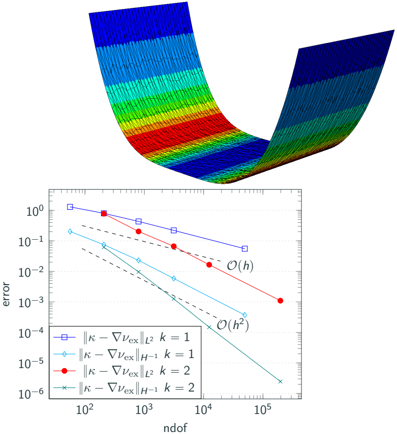


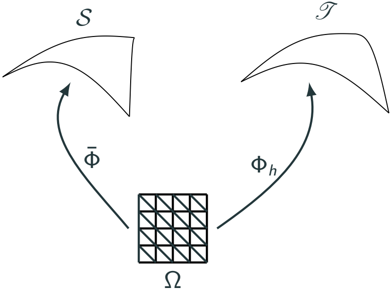




Error analysis (integral representation)

- If $\mathscr{T} \to \mathcal{S}$, does $\boldsymbol{\kappa} \to \nabla \bar{\nu}$?
- Dihedral angle $\sphericalangle(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi} + t(\Phi_h - \bar{\Phi})$ and use integral representation of the error

$$\widetilde{\nabla}\nu(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla \nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt} \widetilde{\nabla}\nu(\boldsymbol{\sigma})\, dt$$



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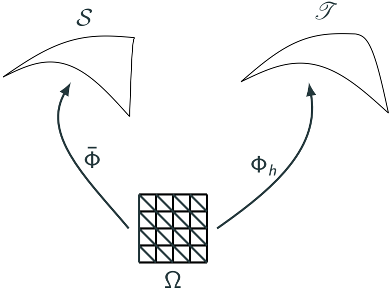

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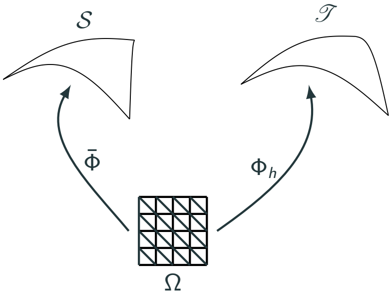

- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ
- Solution: Use fixed reference domain (Uhlenbeck trick)
- Then estimate integrand

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Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi} + t(\Phi_h - \bar{\Phi})$.
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b.
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\|\Phi_h - \bar{\Phi}\|_{H^1} \leqslant C\, h^k.$$

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

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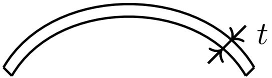
Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\Phi_h = \mathcal{I}_h^{\mathrm{Lag}^k}\bar{\Phi}$ for $k \geqslant 1$. Let $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ be the lifted Weingarten tensor. Then $\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$.

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

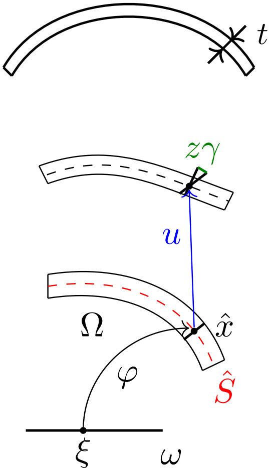
Shells

Thin-walled structures



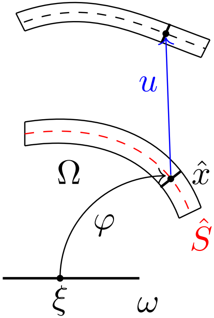
- Reduce 3D elasticity to 2D shell model

Thin-walled structures



- Reduce 3D elasticity to 2D shell model
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z\, \underbrace{(\nu + \gamma) \circ \phi}_{=\tilde{\nu}\circ\phi}(\hat{x})$
- Reissner-Mindlin/Naghdi shell

Thin-walled structures



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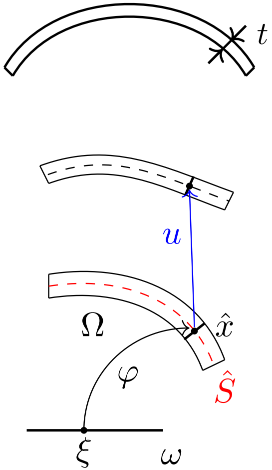
Thin-walled structures



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- Kirchhoff–Love/Koiter shell
- Insert Φ in 3D elasticity and integrate over thickness, neglect higher order terms $\mathcal{O}(t^4)$ (asymptotical analysis)

Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

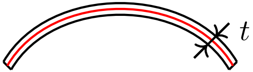
$u\dots$ displacement of mid-surface

$t\dots$ thickness

$\mathcal{M}\dots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



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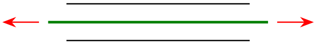

membrane energy

u . . . displacement of mid-surface

t . . . thickness

$\mathcal{M}$ . . . material tensor

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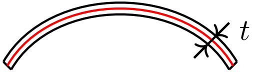

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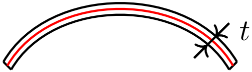
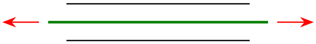
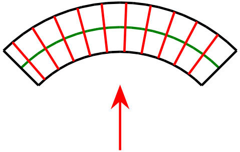




membrane energy



bending energy

FEM + DDG for Koiter/Kirchhoff–Love shell

Lifted shape operator: $\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\Psi}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\Psi}) := \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle(\nu_L, \nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

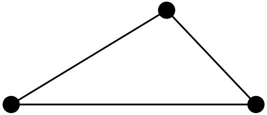
$$\begin{aligned}\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) = {}& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx \\ &+ \sum_{E\in\mathscr{E}} \int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

- Lagrange parameter $\boldsymbol{\sigma} \in \Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}}$ → two-field formulation in $(u, \boldsymbol{\sigma})$

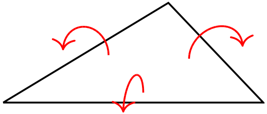
N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

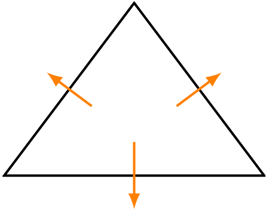
Shell element



Displacement u

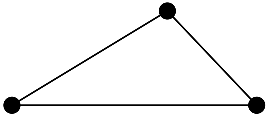


Moment $\boldsymbol{\sigma}$



Hybridization

Shell element



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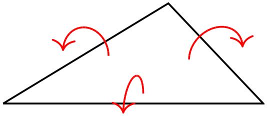

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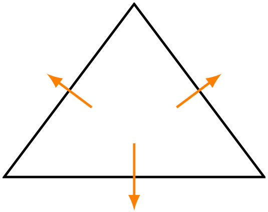

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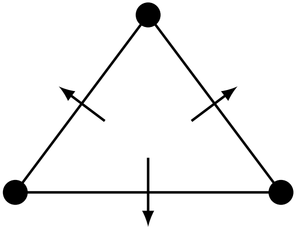
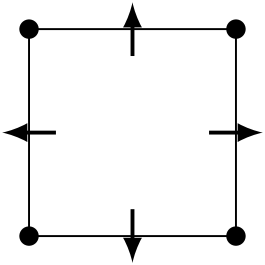


Morley



Quadrilaterial (hybridized)

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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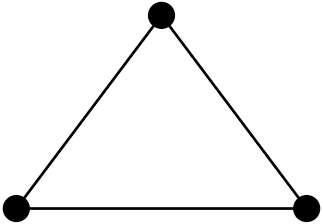

$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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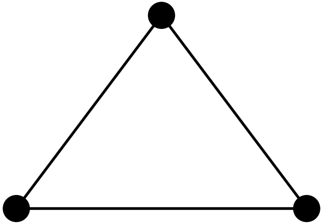

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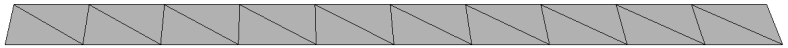

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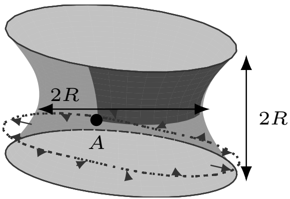

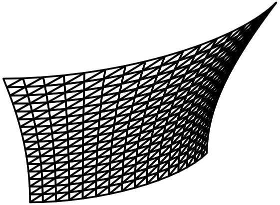

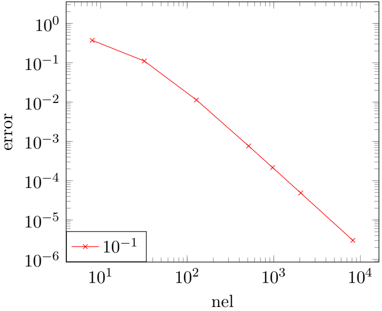

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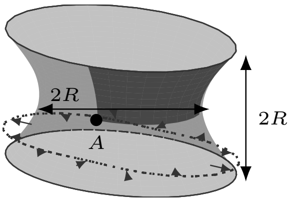
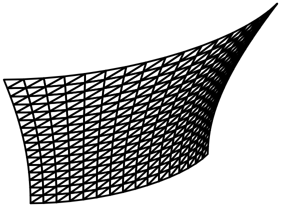
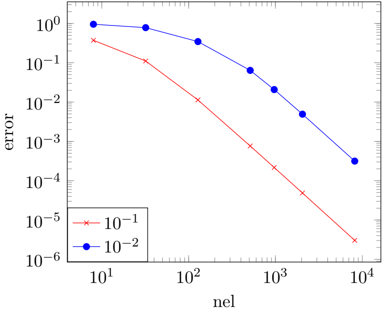
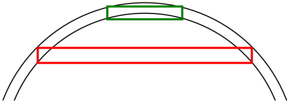






- Pre-asymptotic regime



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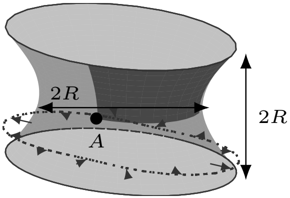

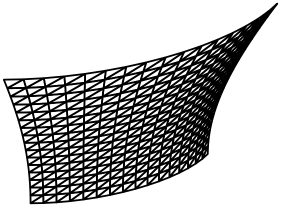

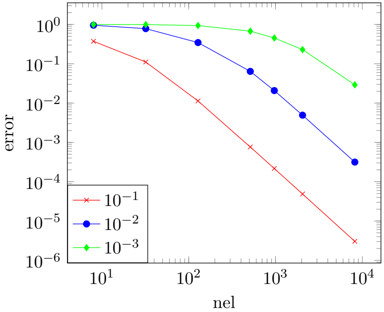

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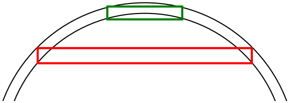

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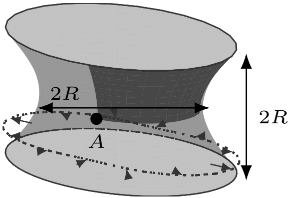

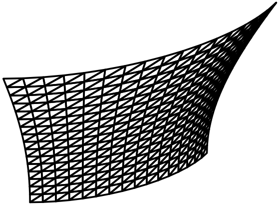

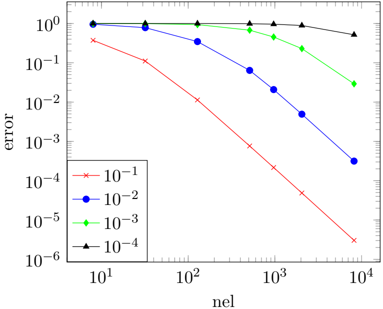

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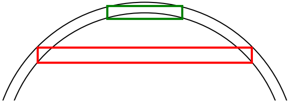
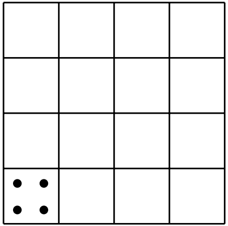


Membrane locking

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

Membrane locking

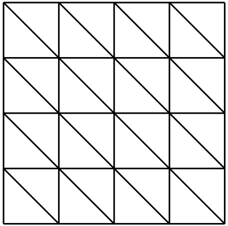
$$\frac{1}{t^2}\|\Pi_{L^2}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

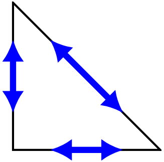


- Reduced integration for quadrilateral meshes

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$





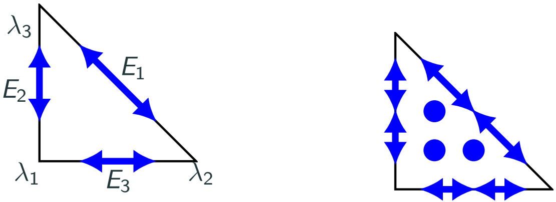
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Regge elements & interpolation operator

$$H(\mathrm{curl\,curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl\,curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$



$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

- Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*
- Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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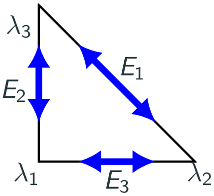

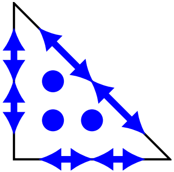

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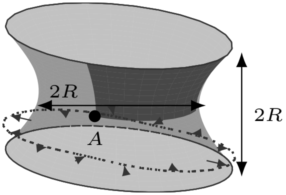
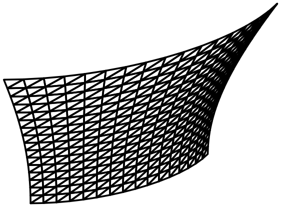
$$\mathcal{I}_{\mathcal{R}}^k : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

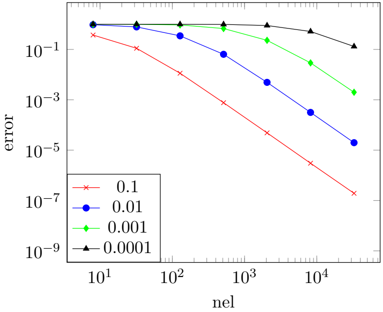
$$\int_E (g - \mathcal{I}_{\mathcal{R}}^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

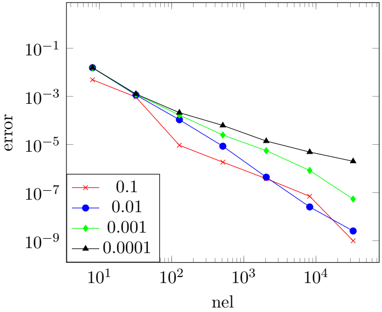
$$\int_T (g - \mathcal{I}_{\mathcal{R}}^k g) : Q\, dx = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking





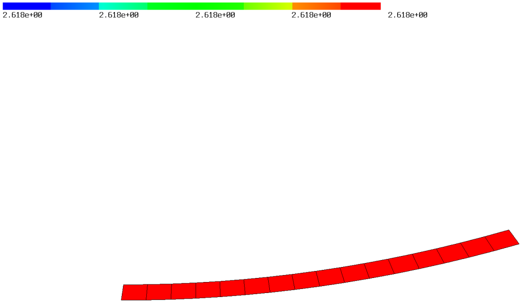


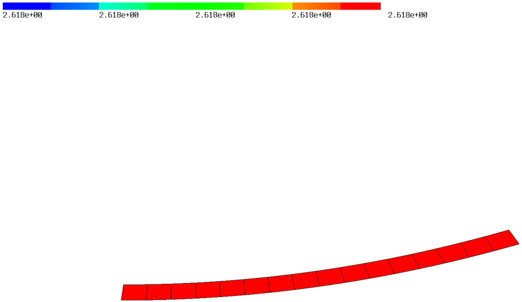


Numerics & Applications

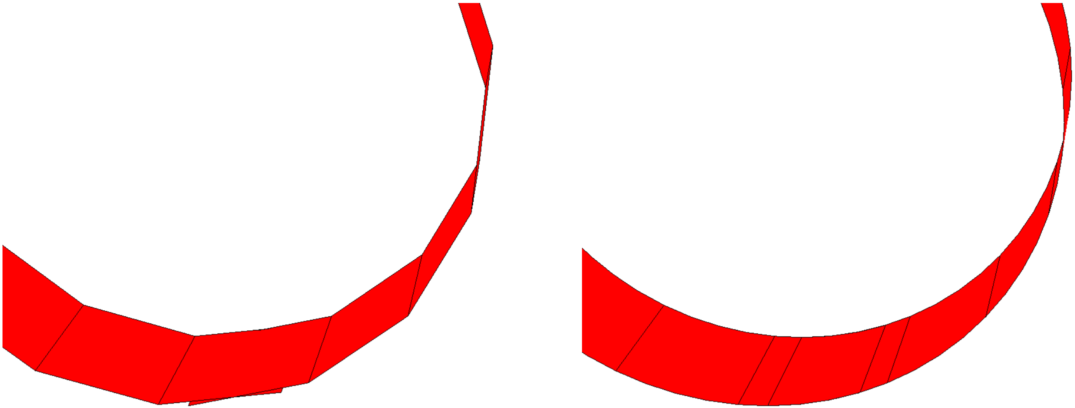
Example (cantilever bending)

NGSolve





Example (cantilever bending)

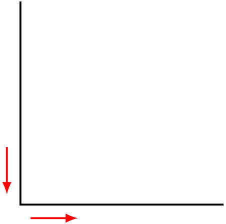


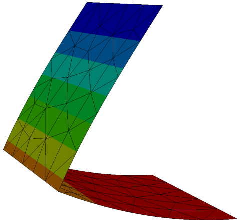
Example (cantilever bending)

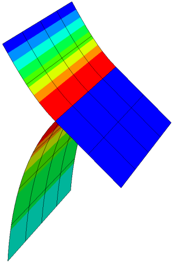
Structures with kinks and branched shells

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

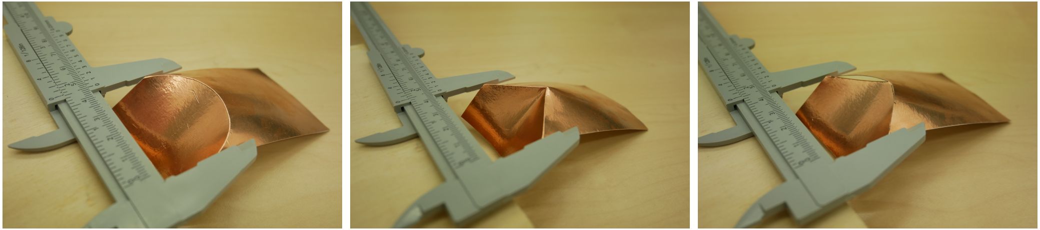
$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \overset{!}{=} 0 \quad \Rightarrow \quad \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$

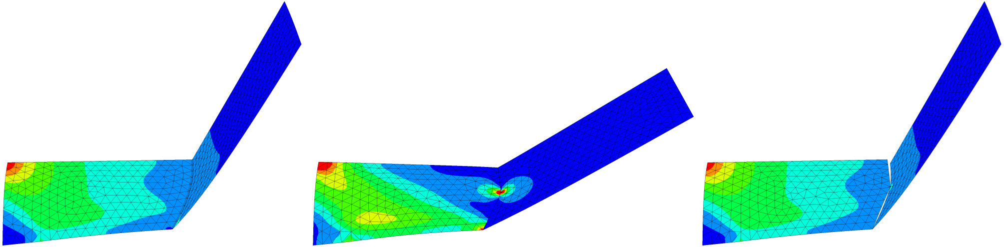






Application (bending-folding model)

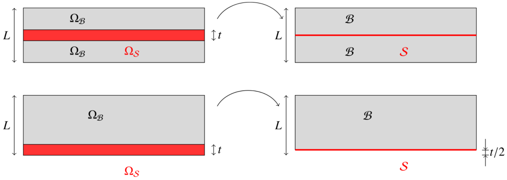


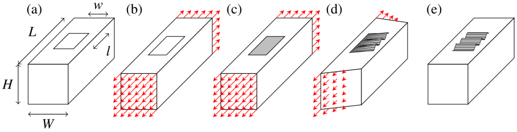


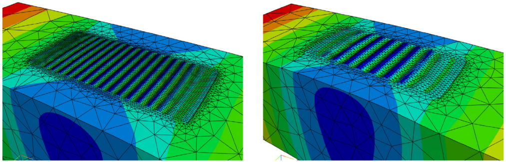
Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces and Free Boundaries (accepted)* (2026).

Coupling shells – 3D elasticity, wrinkling experiment

- Composite materials, blood vessels, etc.
- Lagrange elements for elasticity and shell displacement → easy to couple







Pechstein, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)

Application (cell membrane)

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

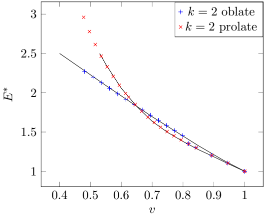
κ_b bending elastic constant

$H = 0.5\,\mathrm{tr}(\nabla\nu)$ mean curvature

$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

Functional: $\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$



N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)

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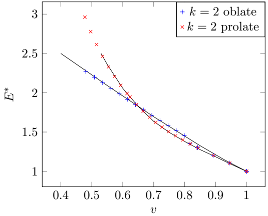

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Summary

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- Bending energy for shell model
- Regge elements for membrane locking
- Applications (coupling, origami, cell membranes)

- Reissner–Mindlin/Naghdi shell
- Linear and nonlinear shell models

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Thank You for Your attention!

Literature

- Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces and Free Boundaries (accepted)* (2026).
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