Analysis of the Generalized Shape Operator

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Project J 4824-N

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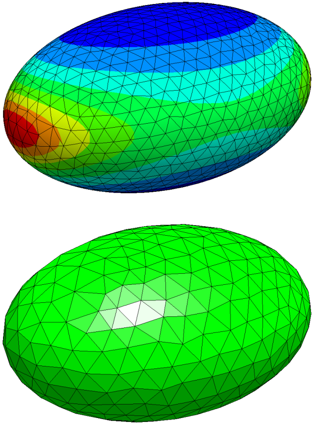
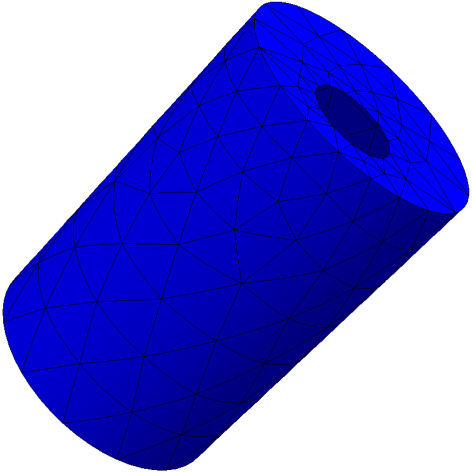
Motivation

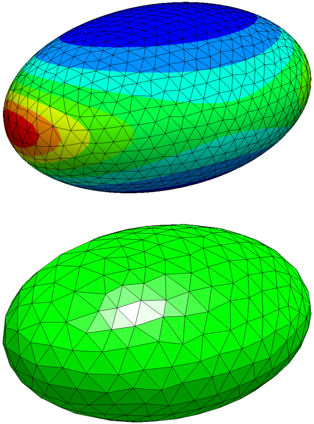
Approximate extrinsic curvature of non-smooth surfaces



Application to shells



Approximate extrinsic curvature of non-smooth surfaces



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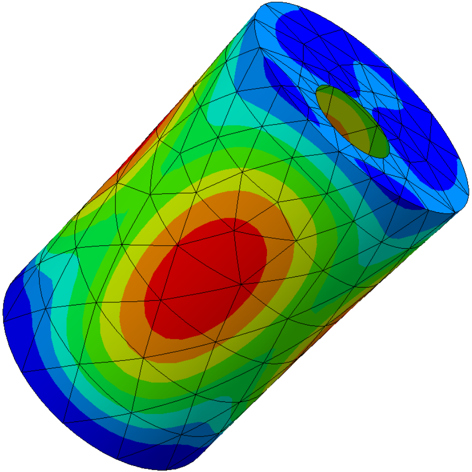
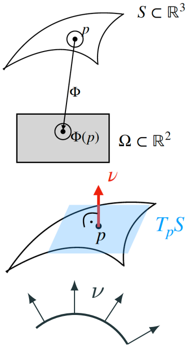


Extrinsic curvature

Curvature types

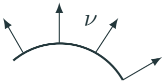
- Surface $\mathcal{S}$ embedded in $\mathbb{R}^3$
- Normal vector $\nu : \mathcal{S} \to \mathbb{S}^2$
- Shape operator, Weingarten tensor, second fundamental form $\nabla\nu$
- Eigenvalues 0, κ_1, κ_2



Mean curvature $H = 0.5(\kappa_1 + \kappa_2) = 0.5\,\mathrm{tr}\,(\nabla\nu)$ $\Rightarrow$ extrinsic curvature
Gauss curvature $K = \kappa_1\kappa_2 = \det(\nabla\nu + \nu \otimes \nu)$ $\Rightarrow$ intrinsic curvature

Intrinsic curvature is independent of the embedding (surrounding space)

Weingarten tensor (shape operator)



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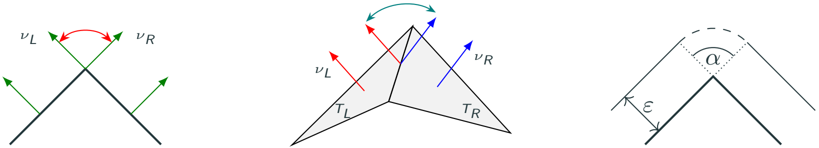

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- Dihedral angle formula (from Steiner's offset formula): $\sum_{E\in\mathring{\mathscr{E}}} \alpha_E\,|E|$

STEINER: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

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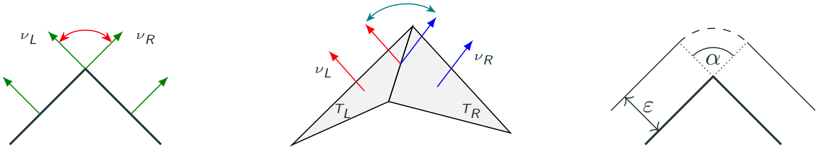

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How to define a generalized Weingarten tensor object? Combine FEM & DDG!

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DDG & FEM: Generalized Weingarten tensor

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\widetilde{\nabla\nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}}\int_T \nabla\nu : \boldsymbol{\Psi}\,dx + \sum_{E\in\mathring{\mathscr{E}}}\int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\,\boldsymbol{\Psi}_{\mu\mu}\,ds$$

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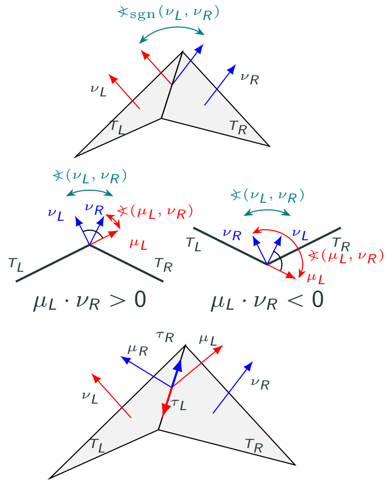

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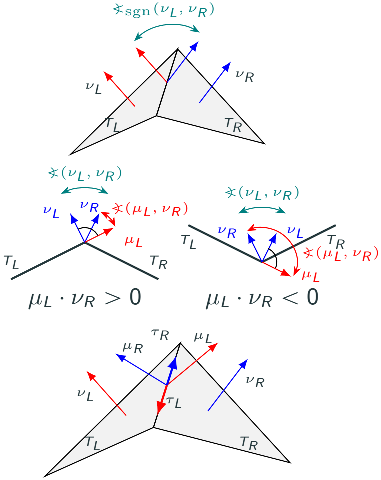
- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$
- Test function space

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T},\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \,:\, (\boldsymbol{\sigma}\nu)|_T = 0,\ (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

- Motivation: TDNNS method: $\nabla H(\mathrm{curl}) \subset H(\mathrm{div}\,\mathrm{div})^*$
  $\Sigma \ldots$ Hellan–Herrmann–Johnson space

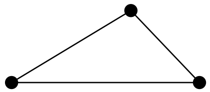


Pechstein, Schöberl: Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity, *M3AS* (2011)

Lagrange elements:

$H^1(\Omega) = \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$

$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$



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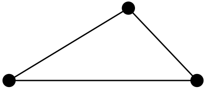
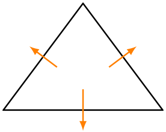


Raviart–Thomas/Brezzi–Douglas–Marini elements:

$$H(\mathrm{div}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}\sigma \in L^2(\Omega)\}$$

$$\mathrm{BDM}_h^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div}, \Omega)$$



Finite Element Spaces

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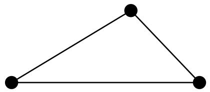

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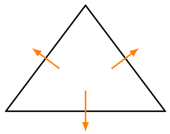

Hellan–Herrmann–Johnson elements:

$H(\mathrm{divdiv},\Omega) = \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \,|\, \mathrm{divdiv}\boldsymbol{\sigma} \in H^{-1}(\Omega)\}$

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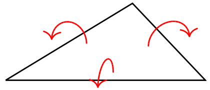

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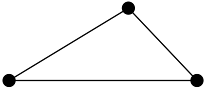

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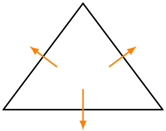

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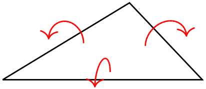
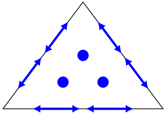


Regge elements:

$$H(\mathrm{curl\,curl}, \Omega) = \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{curl\,curl}\,\boldsymbol{\sigma} \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k(\mathscr{T}_h) = \{\boldsymbol{\sigma} \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^T \boldsymbol{\sigma} t]\!]_F = 0\}$$



Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$

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- Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$ and use integral representation of the error

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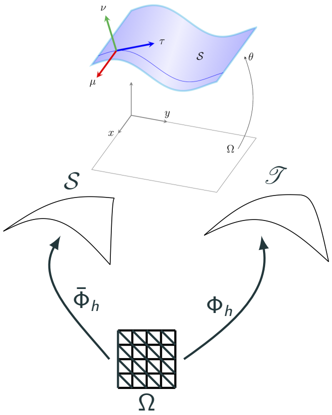

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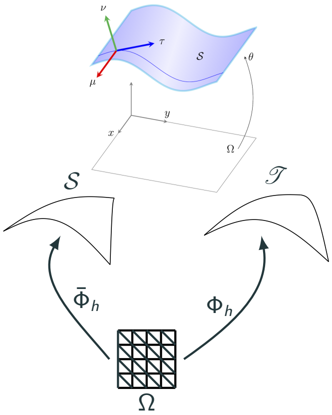

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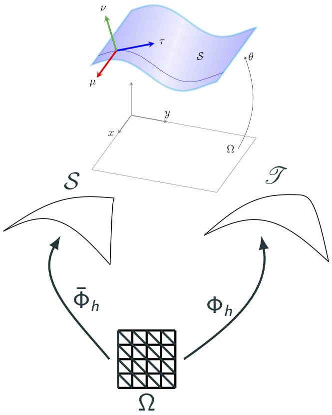
- Solution: Use fixed reference domain (Uhlenbeck trick)
- Then estimate integrand



Linearization

Evolution of quantities in direction $X \circ \Phi = \hat{X} := \dot{\Phi}$ (shape optimization techniques)

Lemma (linearization of geometric quantities)

$$\frac{d}{dt} J = (\mathrm{div}_{\mathcal{S}} X) \circ \Phi \, J$$ surface determinant

$$\frac{d}{dt}(\nu \circ \Phi) = -((\nabla_{\mathcal{S}} X)^T \nu) \circ \Phi$$ surface normal

$$\frac{d}{dt}(\boldsymbol{\sigma} \circ \Phi) = (-2\mathrm{div}_{\mathcal{S}}(X)\boldsymbol{\sigma} + 2\mathrm{sym}(\nabla_{\mathcal{S}} X \boldsymbol{\sigma})) \circ \Phi$$ HHJ test function

$$\frac{d}{dt}(\boldsymbol{\sigma}_{\mu\mu} \circ \Phi) = (-2(\nabla_{\mathcal{S}} X)_{\tau\tau} \boldsymbol{\sigma}_{\mu\mu}) \circ \Phi$$

$$\frac{d}{dt}(\nabla_{\mathcal{S}} \nu \circ \Phi) = \left(\sum_{i=1}^{3} \nu_i \nabla_{\mathcal{S}}^2 X_i - (\nabla_{\mathcal{S}} X)^T \nabla_{\mathcal{S}} \nu + \nabla_{\mathcal{S}} \nu ((\nabla_{\mathcal{S}} X^T \nu) \otimes \nu - \nabla_{\mathcal{S}} X) \right) \circ \Phi$$

$$\frac{d}{dt}(\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) \circ \Phi) = [\![(\nabla_{\mathcal{S}} X)_{\nu\mu}]\!] \circ \Phi$$ dihedral angle

$F = \hat{\nabla} \hat{X}$, $J = \sqrt{\det(F^T F)}$

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Theorem (Gopalakrishnan, N.)

There holds for $\boldsymbol{\sigma} \in \Sigma$ and $X = \dot{\Phi}$

$$\frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma}) = a(\Phi;\boldsymbol{\sigma},X) + b(\Phi;\boldsymbol{\sigma},X),$$

where with $\mathcal{H}_\nu(X) = \sum_{i=1}^3 \mathrm{hesse}(X_i)\nu_i$

$$a(\Phi;\boldsymbol{\sigma},X) = \sum_{T\in\mathscr{T}}\int_T -\mathrm{div}(X)\nabla\nu : \boldsymbol{\sigma} - \sum_{E\in\mathring{\mathscr{E}}}\int_E (\nabla X)_{\tau\tau}\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\boldsymbol{\sigma}_{\mu\mu},$$

$$b(\Phi;\boldsymbol{\sigma},X) = \sum_{T\in\mathscr{T}}\int_T -\mathcal{H}_\nu(X) : \boldsymbol{\sigma} + \sum_{E\in\mathring{\mathscr{E}}}\int_E [\![(\nabla X)_{\nu\mu}]\!]_E\boldsymbol{\sigma}_{\mu\mu}.$$

Bilinear form $b(\Phi;\boldsymbol{\sigma},X)$ is closely related to the surface Hellan–Herrmann–Johnson method

Walker: The Kirchhoff plate equation on surfaces: the surface Hellan–Herrmann–Johnson method, *IMA J. Numer. Anal.* (2021)

Perform all estimates on the reference domain: Transform bilinear forms back

Lemma (pull-back)

$$a(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat{T}\in\hat{\mathscr{T}}}\int_{\hat{T}} -J^{-1}\mathrm{tr}(\hat{\nabla}\hat{X}F^{\dagger})\hat{S} : \hat{\boldsymbol{\sigma}} - \sum_{\hat{E}\in\mathring{\hat{\mathscr{E}}}}\int_{\hat{E}} J_{\mathrm{bnd}}^{-3}(\hat{\nabla}_{\hat{\tau}}\hat{X})\cdot(F\hat{\tau})\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\circ\Phi\hat{\boldsymbol{\sigma}}_{\hat{\mu}\hat{\mu}},$$

$$b(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat{T}\in\hat{\mathscr{T}}}\int_{\hat{T}} -\sum_{i=1}^{3} J^{-1}(F_1\times F_2)\left(\hat{\nabla}^2\hat{X}_i - \sum_{\alpha=1}^{2}(\hat{\nabla}_\alpha\hat{X}_i)\Gamma^\alpha\right) : \hat{\boldsymbol{\sigma}}$$

$$+\sum_{\hat{E}\in\mathring{\hat{\mathscr{E}}}}\int_{\hat{E}} \left[\!\left[\sum_{i=1}^{3} J^{-1}(F_1\times F_2)\cdot\left(\hat{\nabla}_{\hat{\mu}}\hat{X}_i - J_{\mathrm{bnd}}^{2}(F^TF)_{\hat{\tau}\hat{\mu}}\hat{\nabla}_{\hat{\tau}}\hat{X}_i\right)\right]\!\right]_E \hat{\boldsymbol{\sigma}}_{\hat{\mu}\hat{\mu}}$$

Christoffel symbols of second kind: $\Gamma^\alpha_{\beta\gamma} = \sum_{\ell=1}^{3} F^{\dagger}_{\alpha\ell}\hat{\nabla}_\beta F_{\ell\gamma}$

$\hat{S} = \sum_{i=1}^{3}\nu_i\circ\Phi\hat{\nabla}^2\Phi_i$ pull-back of Weingarten tensor: $\nabla_{\mathcal{S}}\nu\circ\Phi = -F^{\dagger^T}\hat{S}F^{\dagger}$

$\boldsymbol{\sigma}\circ\Phi = J^{-2}F\hat{\boldsymbol{\sigma}}F^T, \quad \nu\circ\Phi = J^{-1}F_1\times F_2$

Note: $\hat{X} = \dot{\Phi} = \Phi_h - \bar{\Phi}_h \Rightarrow$ gives convergence

Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\,dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\,dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t)) + b(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t);\boldsymbol{\sigma},\dot{\Phi}(t))$ and $b(\Phi(t);\boldsymbol{\sigma},\dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leqslant C\,h^k.$$

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

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1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
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3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leq h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leq C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leq C\, h^k.$$

Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ always converges in H^{-1}!

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

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Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Shells

Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

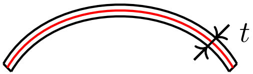
u . . . displacement of mid-surface

t . . . thickness

$\mathcal{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$



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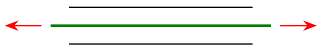

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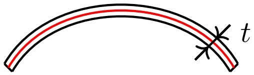

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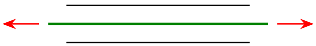

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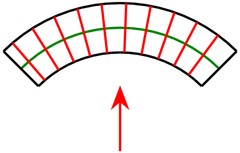

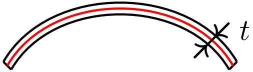

Lifted shape operator: $\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\Psi}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\Psi}) := \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\begin{aligned}\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) &= \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx \\ &+ \sum_{E\in\mathscr{E}} \int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

- Lagrange parameter $\boldsymbol{\sigma} \in \Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u, \boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

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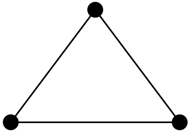

$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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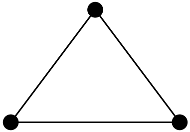

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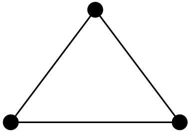

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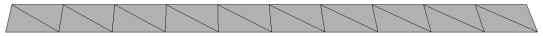
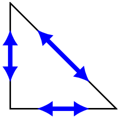


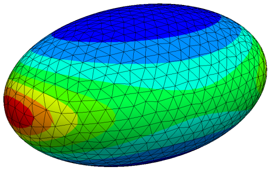


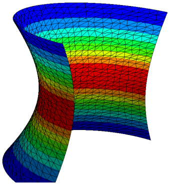
$$\begin{aligned}\mathcal{L}(u, \boldsymbol{\kappa}^{\text{diff}}, \boldsymbol{\sigma}) = {} & \frac{t}{2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}(u)\|_{\mathcal{M}}^2 + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\text{diff}}\|_{\mathcal{M}}^2 - \langle f, u\rangle \\ & + \sum_{T\in\mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\text{diff}} - (\boldsymbol{F}^T \nabla(\nu \circ \phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx \\ & + \sum_{E\in\mathscr{E}} \int_E (\sphericalangle_{\text{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\text{sgn}}(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

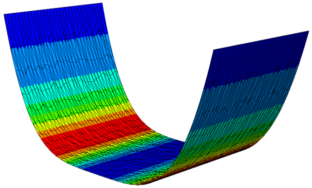
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

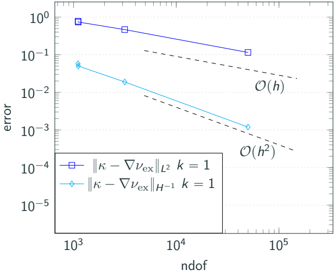
Numerics & Applications

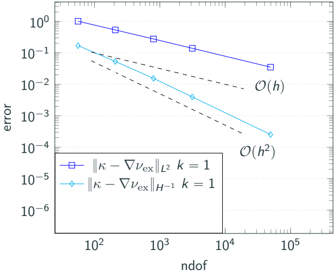
Curvature approximation

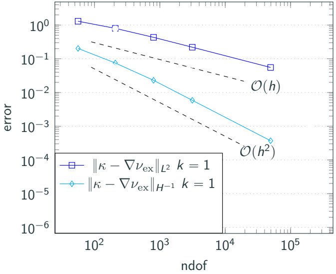




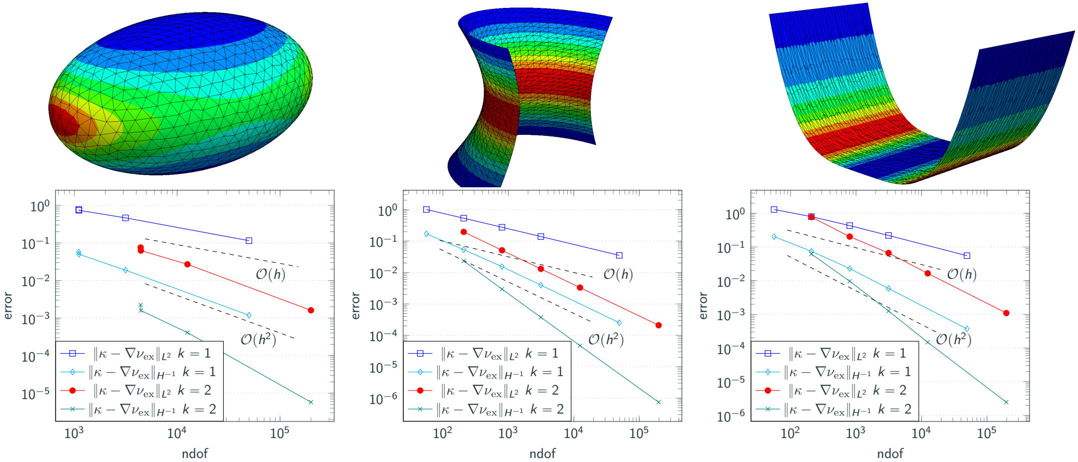






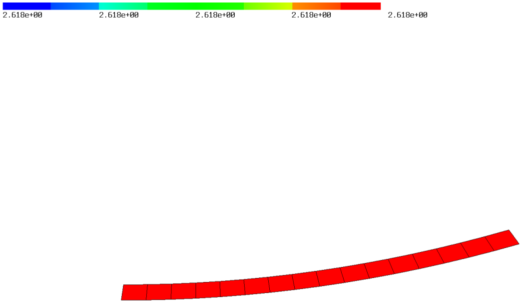


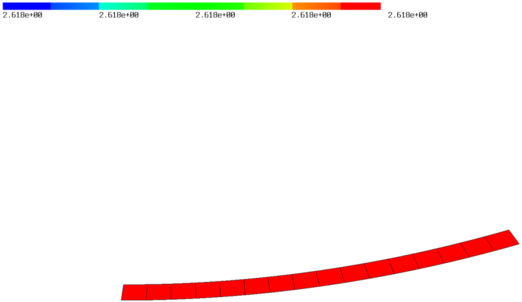
Curvature approximation



Example (cantilever bending)

NGSolve





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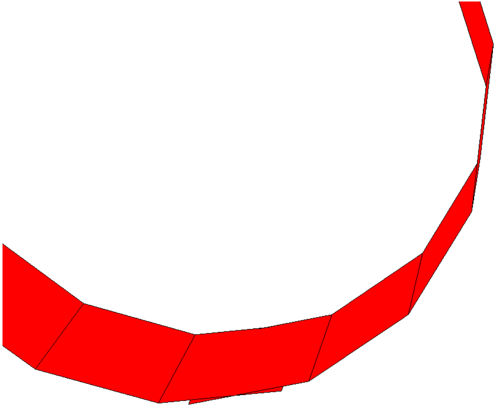

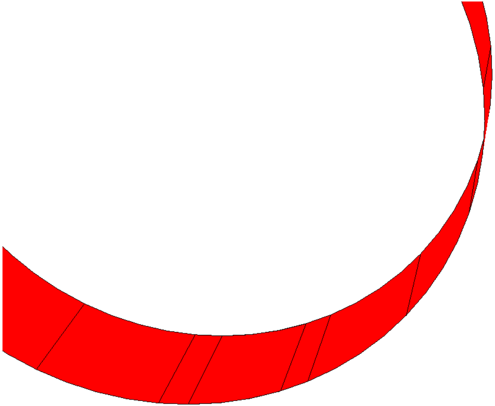

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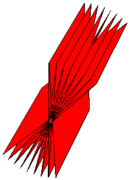
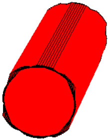
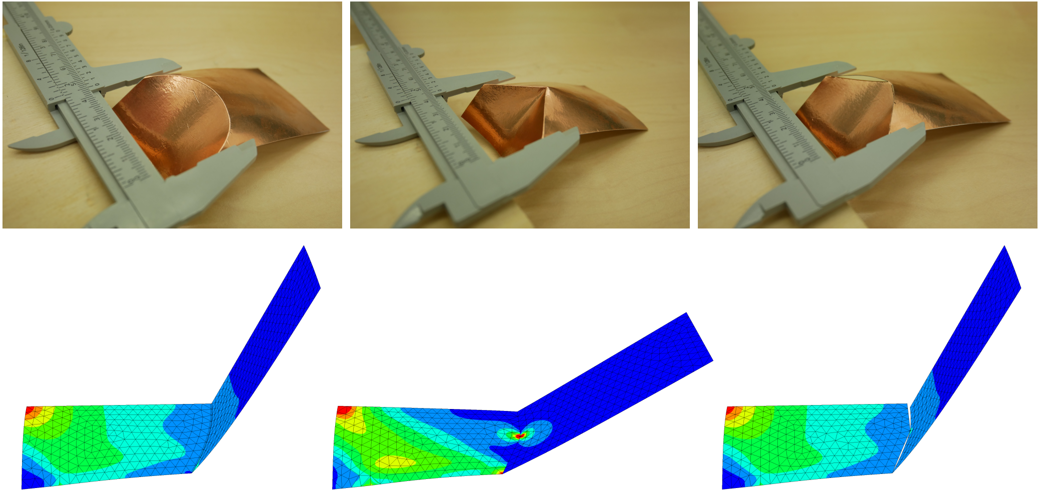




Application (bending-folding model)



Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces and Free Boundaries* (2026).

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

κ_b bending elastic constant

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$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

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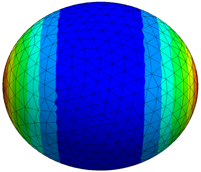

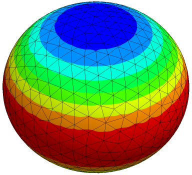

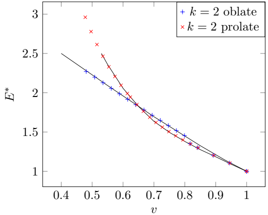

N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)

Application (cell membrane)

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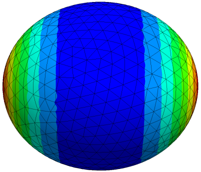

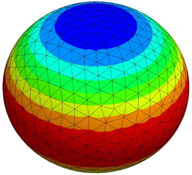

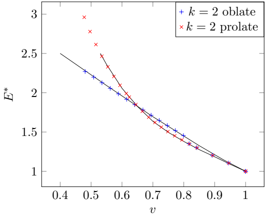

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Summary & Outlook

- DDG + FEM: Generalized shape operator (Weingarten tensor)
- Bending energy for shell model
- Applications (origami, cell membranes)

- Extension to higher dimensional hypersurfaces
- Extension to submanifolds embedded in Riemannian manifolds

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Thank You for Your attention!

Literature

- GOPALAKRISHNAN, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- BARTELS, BONITO, HORNUNG, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces and Free Boundaries* (2026).
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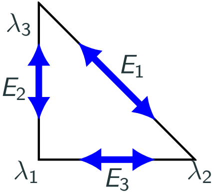
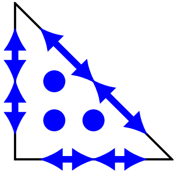
- N., SCHÖBERL, STURM, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)
- GANGL, STURM, N., SCHÖBERL, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)
- N., SCHÖBERL: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg (2021).*
- N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis (2021).*
- N., SCHÖBERL: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

Regge elements & interpolation operator

$$H(\mathrm{curl\,curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl\,curl}\,\boldsymbol{\sigma} \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i\,\nabla\lambda_j \odot \nabla\lambda_k$$

Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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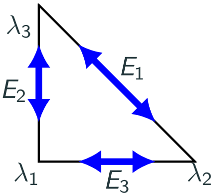
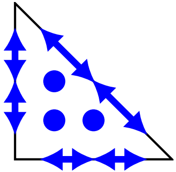
$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

$$\mathcal{I}_{\mathcal{R}}^k : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

$$\int_E (g - \mathcal{I}_{\mathcal{R}}^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{I}_{\mathcal{R}}^k g) : Q\, dx = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$