Locking-free mixed finite element methods for thin-walled and nearly incompressible nonlinear continuum mechanics

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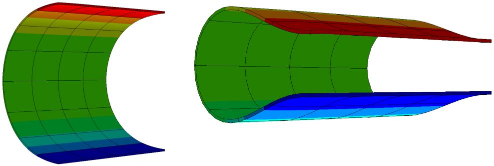
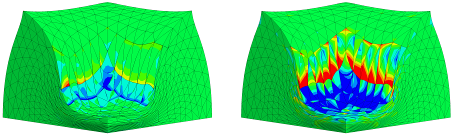




MS052 - Treatment of Locking Across Discretization Techniques

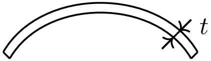
Dedicated to Prof. Ekkehard Ramm on the occasion of his 85th birthday

NGSolve

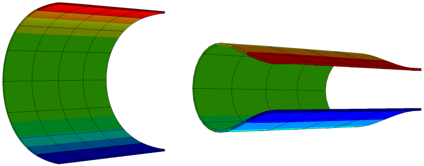
Motivation: Two limits, one numerical symptom

Geometric limit: thin structures

$$\frac{t}{L} \longrightarrow 0$$



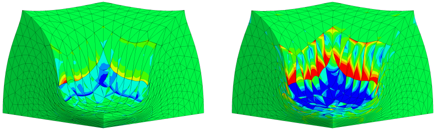
Bending should remain possible, yet the discrete model becomes too stiff.



Material limit: incompressibility

$$\frac{\kappa}{\mu} \longrightarrow \infty, \qquad J = \det \boldsymbol{F} \longrightarrow 1$$

Volume should be preserved, yet the discrete model becomes too stiff or unstable.



Same symptom: artificial stiffness from discrete over-constraint

The constraint is correct, its discrete enforcement is not

Bending-dominated thin-shell limit

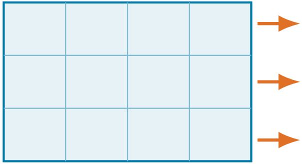
$$\Pi_t(\boldsymbol{u}) = \frac{1}{t^2}\mathcal{U}_{\mathrm{mem}}(\boldsymbol{u}) + \mathcal{U}_{\mathrm{bend}}(\boldsymbol{u}) - \ell(\boldsymbol{u}), \quad t \to 0$$

$$\mathcal{U}_{\mathrm{mem}}(\boldsymbol{u}) = 0 \quad \not\Rightarrow \quad \mathcal{U}_{\mathrm{mem}}(\boldsymbol{u}_h) = 0.$$

The limiting motion exists, but the discrete kinematics may not contain it.

Incompressible limit

$$J(\boldsymbol{F}) = 1, \qquad \frac{\kappa}{\mu} \to \infty$$



too many volume constraints for the available modes

Locking is a mismatch between a physical limiting constraint and its discrete representation

A Stuttgart line of thought (selected)

1993
EAS ↔ HR
solids, plates, shells

1997
large-strain
shell elements

1999
Hu–Washizu
EAS ↔ hybrid stress

2000
DSG
shear-locking-free shells

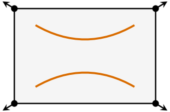
2018
MD: locking-free
by formulation

Kinematic route

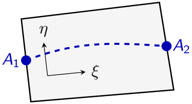
Enrich, sample, project, or reconstruct the strain field: EAS, ANS, MITC, DSG.

Dual route

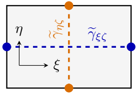
Introduce independent stress fields and weak compatibility: Hellinger–Reissner, hybrid stress, Hu–Washizu.



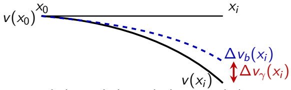
$\varepsilon_h = B u_h + M\alpha$



$\widetilde{\gamma}_{\xi\zeta} = \mathcal{I}_{\mathrm{ANS}}\{\gamma_{\xi\zeta}(A_i)\}$



$\widetilde{\gamma}_{\xi\zeta} = \mathcal{I}_{\xi}\{\gamma_{\xi\zeta}(A_i)\}$
$\widetilde{\gamma}_{\eta\zeta} = \mathcal{I}_{\eta}\{\gamma_{\eta\zeta}(B_i)\}$



$\Delta v_\gamma(x_i) = v(x_i) - v(x_0) - \Delta v_b(x_i)$
$\Delta v_{\gamma,h} = \mathcal{I}_h\{\Delta v_\gamma(x_i)\}, \quad \widetilde{\gamma}_h = \Delta v'_{\gamma,h}$

Andelfinger–Ramm 1993; Bischoff–Ramm 1997; Bischoff–Ramm–Braess 1999; Bletzinger–Bischoff–Ramm 2000; Bieber–Oesterle–Ramm–Bischoff 2018

Same variational idea: Hu–Washizu three-field formulation

$$\boxed{\begin{array}{c}\text{global motion}\\ \boldsymbol{\Phi} = \boldsymbol{X} + \boldsymbol{u}\end{array}} \xleftrightarrow[]{\text{weak compatibility}} \boxed{\begin{array}{c}\text{local kinematics}\\ \boldsymbol{F}\end{array}} \xleftrightarrow[]{\text{material law}} \boxed{\begin{array}{c}\text{dual stress}\\ \boldsymbol{P}\end{array}}$$

Hu–Washizu functional

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P}) = \int_\Omega \Psi(\boldsymbol{F})\,\mathrm{d}V - \langle \boldsymbol{F} - \underbrace{(\boldsymbol{I} + \nabla_X \boldsymbol{u})}_{\nabla_X \boldsymbol{\Phi}}, \boldsymbol{P} \rangle - W_{\mathrm{ext}}(\boldsymbol{u})$$

$\boldsymbol{u}$: global kinematics and boundary conditions

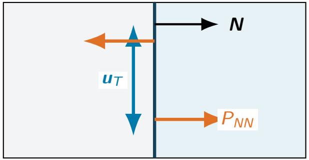
$\boldsymbol{F}$: regular tensor passed to the material law

$\boldsymbol{P}$: compatibility multiplier and interface force

TDNNS and MCS: two complementary mixed methods

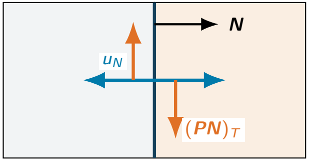
Linear starting points: choose continuity from interface work

TDNNS



tangential displacement $\boldsymbol{u}_T$
nominal normal stress P_{NN}

MCS → NDTNS



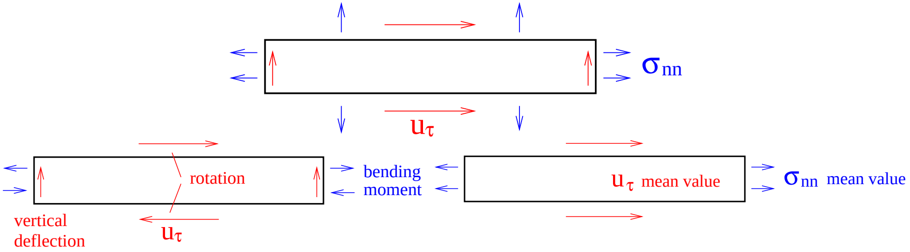
normal displacement u_N
nominal tangential traction $(\boldsymbol{PN})_T$

$$\boldsymbol{\Pi}_T = \boldsymbol{I} - \boldsymbol{N}\otimes\boldsymbol{N}, \qquad u_N = \boldsymbol{u}\cdot\boldsymbol{N}, \qquad \boldsymbol{u}_T = \boldsymbol{\Pi}_T\boldsymbol{u},$$
$$P_{NN} = \boldsymbol{N}\cdot\boldsymbol{PN}, \quad (\boldsymbol{PN})_T = \boldsymbol{\Pi}_T\boldsymbol{PN}, \qquad (\boldsymbol{PN})\cdot\delta\boldsymbol{u} = P_{NN}\delta u_N + (\boldsymbol{PN})_T\cdot\delta\boldsymbol{u}_T.$$

Complementary choices: TDNNS keeps $(\boldsymbol{u}_T, P_{NN})$, NDTNS keeps $(u_N, (\boldsymbol{PN})_T)$

Pechstein & Schöberl, *M3AS* 21 (2011); Gopalakrishnan, Lederer & Schöberl, *IMA JNA* 40 (2020)

Linear starting points: choose continuity from interface work



$$\mathbf{\Pi}_T = \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}, \qquad u_N = \boldsymbol{u} \cdot \boldsymbol{N}, \qquad \boldsymbol{u}_T = \mathbf{\Pi}_T \boldsymbol{u},$$

$$P_{NN} = \boldsymbol{N} \cdot \boldsymbol{P}\boldsymbol{N}, \quad (\boldsymbol{P}\boldsymbol{N})_T = \mathbf{\Pi}_T \boldsymbol{P}\boldsymbol{N}, \qquad (\boldsymbol{P}\boldsymbol{N}) \cdot \delta \boldsymbol{u} = P_{NN} \delta u_N + (\boldsymbol{P}\boldsymbol{N})_T \cdot \delta \boldsymbol{u}_T.$$

Complementary choices: TDNNS keeps $(\boldsymbol{u}_T, P_{NN})$, NDTNS keeps $(u_N, (\boldsymbol{P}\boldsymbol{N})_T)$

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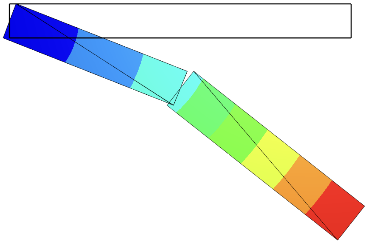
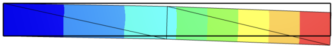
TDNNS and MCS: two complementary mixed methods

Linear starting points: choose continuity from interface work





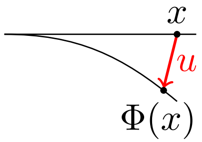
$$\boldsymbol{\Pi}_T = \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}, \qquad u_N = \boldsymbol{u} \cdot \boldsymbol{N}, \qquad \boldsymbol{u}_T = \boldsymbol{\Pi}_T \boldsymbol{u},$$

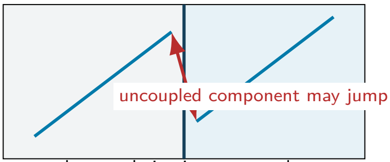
$$P_{NN} = \boldsymbol{N} \cdot \boldsymbol{P}\boldsymbol{N}, \quad (\boldsymbol{P}\boldsymbol{N})_T = \boldsymbol{\Pi}_T \boldsymbol{P}\boldsymbol{N}, \qquad (\boldsymbol{P}\boldsymbol{N}) \cdot \delta\boldsymbol{u} = P_{NN}\delta u_N + (\boldsymbol{P}\boldsymbol{N})_T \cdot \delta\boldsymbol{u}_T.$$

Complementary choices: TDNNS keeps $(\boldsymbol{u}_T, P_{NN})$, NDTNS keeps $(u_N, (\boldsymbol{P}\boldsymbol{N})_T)$

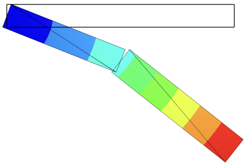
Pechstein & Schöberl, *M3AS* 21 (2011); Gopalakrishnan, Lederer & Schöberl, *IMA JNA* 40 (2020)

Finite strain creates a regularity gap





element derivatives are regular; the global derivative also contains interface contributions



$$\boldsymbol{F} = \boldsymbol{I} + \nabla_X \boldsymbol{u}, \quad \boldsymbol{P} = \partial_{\boldsymbol{F}} \Psi(\boldsymbol{F})$$

If u_N jumps, the global gradient is a distribution. Nonlinear products of that distribution are undefined. The elementwise gradient alone omits the jump kinematics.

Keep the locking-free continuity $\Rightarrow$ lift the kinematics

TDNNS at finite strain: lift the kinematics

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{F},\boldsymbol{P}) = \int_\Omega \Psi(\boldsymbol{F})\,\mathrm{d}V - \langle \boldsymbol{F} - \boldsymbol{I} - \nabla_X \boldsymbol{u}, \boldsymbol{P}\rangle - W_{\mathrm{ext}}(\boldsymbol{u})$$

$$\begin{cases} \nabla_X \boldsymbol{u} = (\nabla_X \boldsymbol{u})_K + [\![u_N]\!]\,\delta_F \boldsymbol{N} \otimes \boldsymbol{N} \\ [\![P_{NN}]\!] = 0 \end{cases}$$

Material law

$$\boldsymbol{P} = \partial_{\boldsymbol{F}} \Psi(\boldsymbol{F})$$

The constitutive routine sees a regular local $\boldsymbol{F}$.

Compatibility

$$\boldsymbol{F} \simeq \boldsymbol{I} + \nabla_X \boldsymbol{u}$$

The stress multiplier enforces the relation weakly.

Equilibrium

$$-\operatorname{Div} \boldsymbol{P} = \boldsymbol{B}$$

Interface work supplies the physically correct coupling.

Three variants of the lifting idea:

F**-based:** lift $\boldsymbol{F} = \nabla_X \boldsymbol{\Phi}$; use $\Psi(\boldsymbol{F})$

C**-based:** lift $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$; use $\widehat{\Psi}(\boldsymbol{C})$

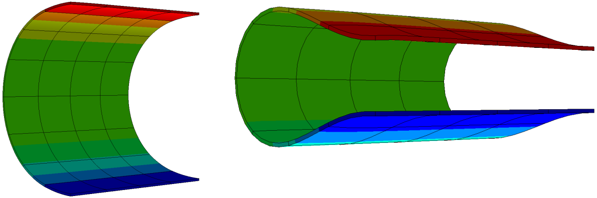
FC**-based:** lift $\boldsymbol{F}$, then project $\boldsymbol{F}^\top \boldsymbol{F}$ to independent $\boldsymbol{C}$

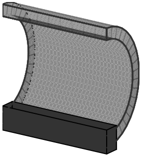
local: $\boldsymbol{F}$, hybridized $\boldsymbol{P}$

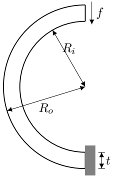
$\downarrow$ static condensation

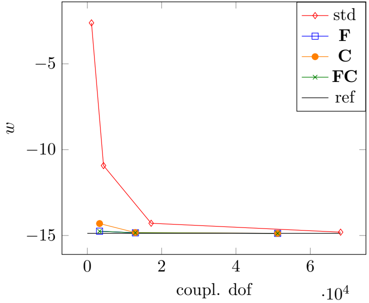
global: $\boldsymbol{u}_T$, facet $\widehat{u}_N$

3D thin-walled cylinder







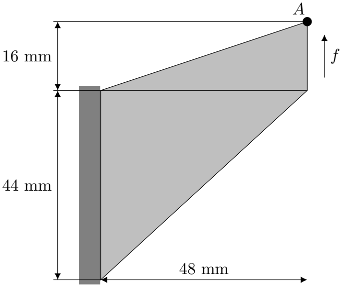


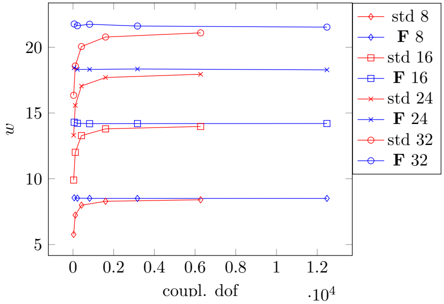
Vertical displacement w_A, $t = 0.2\,\mathrm{mm}$, $p = 2$

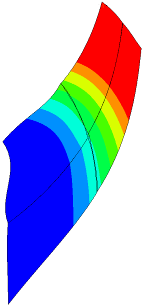
method	coupled dofs	w_A
standard	1,125	−2.613
F-based	3,300	−14.749
standard	68,157	−14.806

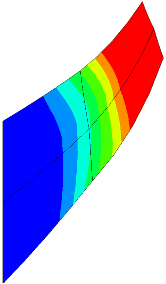
Reference: $w_A \approx -14.9$ (standard $p = 4$, finest grid)

Cooks Membrane





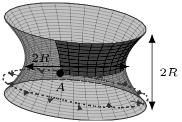


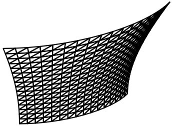


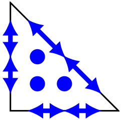
$$\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h) \quad \Rightarrow \quad \mathcal{I}_{\mathrm{Regge}}\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h)$$

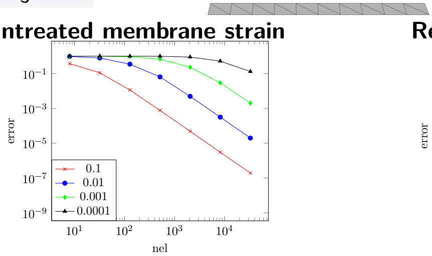
Constraint count ($p = 2$)

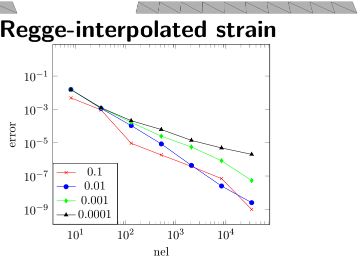
$$\frac{\#dofs}{\#constraints} = \frac{3\#V+3\#E}{2\#E+3\#T} = 1 = \frac{3}{3}$$











Relative radial-deflection error for $t = 10^{-1}, \ldots, 10^{-4}$ and $p = 2$

Regge interpolation has a MITC tying-point relation

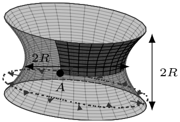
Neunteufel & Schöberl, *CMAME* 373 (2021)

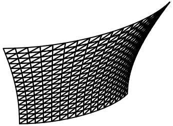
Membrane-locking: Regge interpolation of the membrane strain

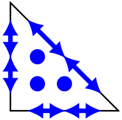
$$\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h) \quad \Rightarrow \quad \mathcal{I}_{\mathrm{Regge}}\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h)$$

Constraint count ($p = 2$)

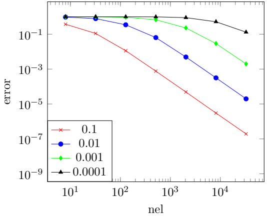
$$\frac{\#dofs}{\#constraints} = \frac{3\#V+3\#E}{2\#E+3\#T} = 1 = \frac{3}{3}$$



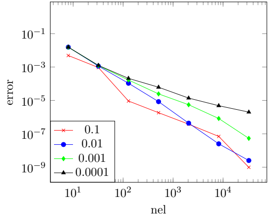




untreated membrane strain



Regge-interpolated strain



Relative radial-deflection error for $t = 10^{-1}, \ldots, 10^{-4}$ and $p = 2$

Regge interpolation has a MITC tying-point relation

Neunteufel & Schöberl, *CMAME* 373 (2021)

Reissner–Mindlin plate: similar projection idea

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Thin limit: incompatible discrete kernel

$$\gamma(w,\boldsymbol{\theta}) = \nabla w - \boldsymbol{\theta},$$
$$\mathcal{E}_t(w,\boldsymbol{\theta}) = \tfrac{1}{2}a(\boldsymbol{\theta},\boldsymbol{\theta}) + \tfrac{\lambda}{2t^2}\|\gamma(w,\boldsymbol{\theta})\|_0^2 - \ell(w),$$
$$t \to 0: \ \gamma = 0, \quad \nabla W_h \not\subset \Theta_h \ \Rightarrow \ \text{locking}$$

MITC: compatible $H(\mathrm{curl})$ shear

$$\gamma_h^{\mathrm{raw}} = \nabla w_h - \boldsymbol{\theta}_h,$$
$$\widetilde{\gamma}_h = R_h \gamma_h^{\mathrm{raw}} \in \Gamma_h,$$
$$\mathcal{E}_{\mathrm{shear},h} \sim t^{-2}\|\widetilde{\gamma}_h\|_0^2, \quad \Gamma_h : \text{tangential continuous}$$

Three compatible choices make the reduction locking-free

1. Stable rotation pair (Θ_h, Q_h)

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{\eta}_h \in \Theta_h} \frac{(q_h, \mathrm{rot}\,\boldsymbol{\eta}_h)}{\|q_h\|_0 \|\boldsymbol{\eta}_h\|_1} \geq \beta > 0$$

β independent of h

2. Commuting reduction R_h

$$R_h : [H^1]^2 \to \Gamma_h,$$
$$\pi_h(\mathrm{rot}\,\boldsymbol{\eta}) = \mathrm{rot}(R_h \boldsymbol{\eta})$$

Nédélec / rotated Raviart–Thomas

3. Curl-free displacement kernel

$$\nabla W_h = \{\boldsymbol{\gamma}_h \in \Gamma_h : \mathrm{rot}\,\boldsymbol{\gamma}_h = 0\}$$

Gradients are represented exactly in Γ_h

Plate: $\gamma_h^{\mathrm{raw}} \xrightarrow{R_h} \Gamma_h$ **(tangential trace)** $\iff$ **shell:** $\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h) \xrightarrow{\mathcal{I}_{\mathrm{Regge}}}$ **Regge_h (tangential–tangential trace)**

Bathe, Bucalem & Brezzi, *Eng. Comput.* 7 (1990); Brezzi, Fortin & Stenberg, *M^3AS* 1 (1991)

From MCS to nearly incompressible finite elasticity

$$\mathcal{L}_\kappa(\boldsymbol{u}, \boldsymbol{F}, \boldsymbol{P}, p) = \int_\Omega \left[\Psi(\boldsymbol{F}) - p\big(J(\boldsymbol{F}) - 1\big) - \frac{p^2}{2\kappa} \right] \mathrm{d}V - \big\langle \boldsymbol{F} - \boldsymbol{I} - \nabla_X \boldsymbol{u}, \boldsymbol{P} \big\rangle - W_{\mathrm{ext}}(\boldsymbol{u})$$

$$\begin{cases} \nabla_X \boldsymbol{u} = (\nabla_X \boldsymbol{u})_K + [\![\boldsymbol{u}_T]\!]\, \delta_F \boldsymbol{N} \otimes \boldsymbol{\Pi}_T \\ [\![(\boldsymbol{P}\boldsymbol{N})_T]\!] = 0 \end{cases}$$

$J - 1 + p/\kappa = 0$, the limit $\kappa \to \infty$ gives exact incompressibility

$\boldsymbol{u}$		$\boldsymbol{F}$		$\boldsymbol{P}$		p
normal-continuous motion	⟷	local kinematics	⟷	$(\boldsymbol{P}\boldsymbol{N})_T$ continuous	⟷	volume reaction

What comes from MCS?

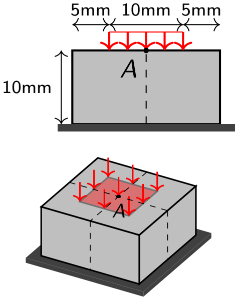
Normal-displacement / tangential-traction continuity and a robust incompressible small-strain limit.

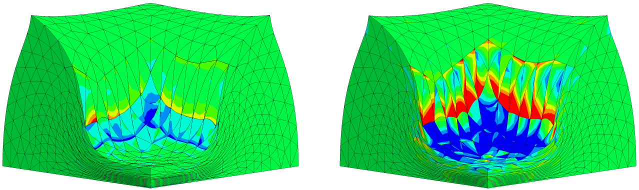
What does Hu–Washizu add?

A regular finite-strain $\boldsymbol{F}$ for the material law and the nonlinear volume constraint.

Fu, Neunteufel, Schöberl & Zdunek, *CMAME* 444 (2025). MCS denotes the Stokes origin, NDTNS the finite elasticity extension

Jacobian determinant J on the deformed configuration at final load $f = 1000$





Pointwise J: NDTNS 0.503 – 1.406, STP -1.189 – 4.140.

Nonlinear robustness

NDTNS and STP reach $f = 1000$.
Taylor–Hood-type method fails earlier when negative Jacobians appear.

Projected volume ratio J_0, 4189 tetrahedra

method	min J_0	max J_0
NDTNS	0.9733	1.025
STP	0.7241	1.294

Fu, Neunteufel, Schöberl & Zdunek, *CMAME* 444 (2025); Simo, Taylor & Pister *CMAME* 51 (1985)

Summary

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1 **Different approaches to tackle locking**
Hu–Washizu formulation as powerful tool to design and relate methods

2 **Extend TDNNS and MCS to finite elasticity**
Beneficial properties analyzed in linear setting carry over to nonlinear setting

3 **Regge interpolation to avoid membrane locking**
Relations to MITC, similarities to shear locking in Reissner–Mindlin plates

4 Physics of fields important to design robust methods
Investigate similarities between different methods and locking phenomena

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Thank You for Your attention!

Literature

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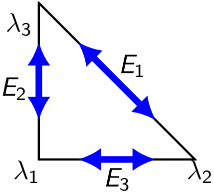
N., Pechstein, Schöberl: Three-field mixed finite element methods for nonlinear elasticity, *Comput. Methods Appl. Mech. Engrg.* 382 (2021), 113857.

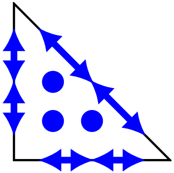
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg.* 373 (2021), 113524.

Fu, N., Schöberl, Zdunek: A four-field mixed formulation for incompressible finite elasticity, *Comput. Methods Appl. Mech. Engrg.* 444 (2025), 118082.

Construction tangential-tangential continuous Regge elements

$$\mathrm{Reg}_h^p := \{\varepsilon \in [\mathcal{P}^p(\mathcal{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![\boldsymbol{\tau}^\top \varepsilon\, \boldsymbol{\tau}]\!] = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad \boldsymbol{\tau}_j^\top \varphi_{E_i} \boldsymbol{\tau}_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

Canonical Regge interpolant

$$\mathcal{I}^p_{\mathrm{Regge}} : C^0(\Omega) \to \mathrm{Reg}_h^p$$

$$\int_E \boldsymbol{\tau}^T (\boldsymbol{G} - \mathcal{I}^p_{\mathrm{Regge}} \boldsymbol{G}) \boldsymbol{\tau}\, q \,\mathrm{d}A = 0 \text{ for all } q \in \mathcal{P}^p(E)$$

$$\int_K (\boldsymbol{G} - \mathcal{I}^p_{\mathrm{Regge}} \boldsymbol{G}) : \boldsymbol{Q} \,\mathrm{d}V = 0 \text{ for all } \boldsymbol{Q} \in \mathcal{P}^{p-1}(K, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Christiansen, *Numer. Math.* 119 (2011); Li, PhD thesis, University of Minnesota (2018); Neunteufel, PhD thesis, TU Wien (2021)

Small-strain finite element settings

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TDNNS

$$\boldsymbol{u} \in \mathrm{Ned}_h^p, \qquad [\![\boldsymbol{u}_T]\!] = 0$$
$$\boldsymbol{\sigma} \in \mathrm{HHJ}_h^p, \qquad [\![\sigma_{NN}]\!] = 0$$

The duality pairing $\langle \sigma, \nabla \boldsymbol{u} \rangle$ may be written elementwise as

$$\sum_K \int_K \boldsymbol{\sigma} : \nabla \boldsymbol{u} \, \mathrm{d}V - \sum_F \int_F \sigma_{NN} [\![u_N]\!] \, \mathrm{d}A.$$

MCS / NDTNS

$$\boldsymbol{u} \in \mathrm{BDM}_h^p, \qquad [\![u_N]\!] = 0$$
$$\boldsymbol{\sigma} \in \mathrm{MCS}_h^p, \qquad [\![(\boldsymbol{\sigma} \boldsymbol{N})_T]\!] = 0$$

The normal displacement couples directly. The tangential displacement is recovered through hybrid/interface variables.

$$\mathrm{Ned}_h^p := \{\boldsymbol{u} \in [\mathcal{P}^p(\mathcal{T}_h)]^d \mid [\![\boldsymbol{\tau}^\top \boldsymbol{u}]\!] = 0 \text{ for all edges } E, \ [\![\boldsymbol{u}_T]\!] = 0 \text{ for all facets } F\}$$
$$\mathrm{HHJ}_h^p := \{\boldsymbol{\sigma} \in [\mathcal{P}^p(\mathcal{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![\sigma_{NN}]\!] = 0 \text{ for all facets } F\}$$

$$\mathrm{BDM}_h^p := \{\boldsymbol{u} \in [\mathcal{P}^p(\mathcal{T}_h)]^d \mid [\![u_N]\!] = 0 \text{ for all facets } F\}$$
$$\mathrm{MCS}_h^p := \{\boldsymbol{\sigma} \in [\mathcal{P}^p(\mathcal{T}_h)]^{d\times d} \mid [\![(\boldsymbol{\sigma} \boldsymbol{N})_T]\!] = 0 \text{ for all facets } F\}$$

Pechstein & Schöberl, *M3AS* 21 (2011); Gopalakrishnan, Lederer & Schöberl, *IMA JNA* 40 (2020)

Targeted strain reconstruction

$$\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h) \quad \mapsto \quad \mathcal{I}^{p-1}_{\mathrm{Regge}}\boldsymbol{E}_{\mathrm{mem}}(\boldsymbol{u}_h).$$

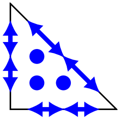
The canonical interpolant preserves tangential–tangential strain moments on edges:

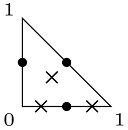
$$\int_E \boldsymbol{\tau}^\top(\boldsymbol{E}-\mathcal{I}^k_{\mathrm{Regge}}\boldsymbol{E})\boldsymbol{\tau}\, q \,\mathrm{d}A = 0, \qquad q \in \mathcal{P}^k(E),$$

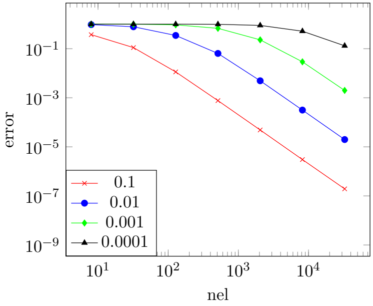
$$\int_K (\boldsymbol{E}-\mathcal{I}^k_{\mathrm{Regge}}\boldsymbol{E}) : \boldsymbol{Q}\,\mathrm{d}V = 0, \qquad \boldsymbol{Q} \in \mathcal{P}^{k-1}(K,\mathbb{R}^{2\times 2}_{\mathrm{sym}}).$$

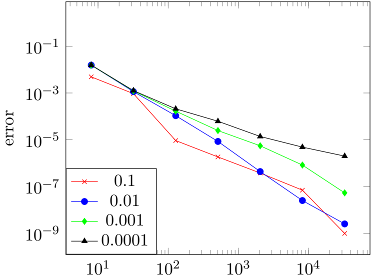
Closely connected to MITC-type strain interpolation.

Extendable to quadrilateral and mixed meshes.



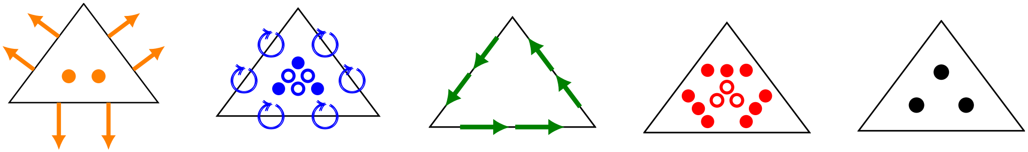






Neunteufel & Schöberl, *CMAME* 373 (2021); Kim & Bathe, *Comput. Struct.* 87 (2009)

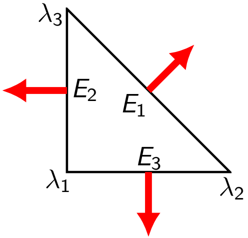
Lowest-order hybridized NDTNS element

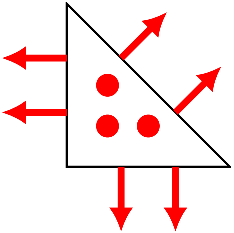


- Hybridization makes all stress degrees of freedom local
- The trace-free split avoids lifting the already regular divergence of the displacement
- The local fields $\boldsymbol{F}$, $\boldsymbol{P}$, and p are condensed, global system contains displacement-type unknowns only

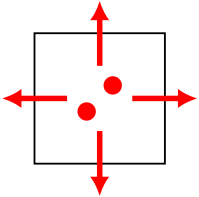
Fu, Neunteufel, Schöberl & Zdunek, *CMAME* 444 (2025)

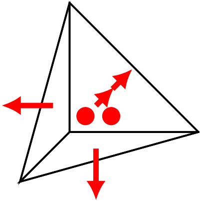
$$\mathrm{HHJ}_h^p := \{\boldsymbol{\sigma} \in [\mathcal{P}^p(\mathcal{T}_h)]_{\mathrm{sym}}^{d\times d} \mid [\![\sigma_{NN}]\!] = 0 \text{ for all facets } F\}$$





$$\varphi_{E_i} = \nabla^\perp \lambda_j \odot \nabla^\perp \lambda_k, \qquad \boldsymbol{N}_j^\top \varphi_{E_i} \boldsymbol{N}_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla^\perp \lambda_j \odot \nabla^\perp \lambda_k$$





Pechstein & Schöberl *M3AS* 21 (2011); Comodi *Math. Comp.* 52 (1989)