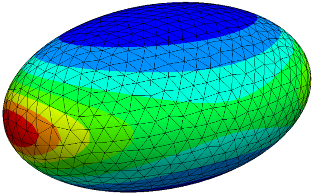
Combining Finite Elements with Discrete Differential Geometry for Curvature Approximation and Nonlinear Shell Analysis

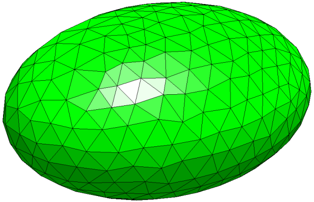
Michael Neunteufel (TU Wien)
Jay Gopalakrishnan (Portland State University)
Joachim Schöberl (TU Wien)

FWF Austrian Science Fund
Project J 4824-N

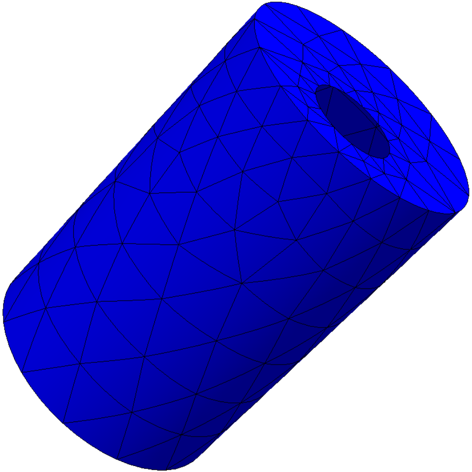
RICAM, Seminar, Linz, June 25th, 2026

Approximate extrinsic curvature of non-smooth surfaces



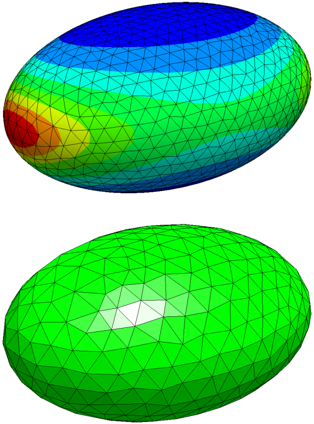


Application to shells

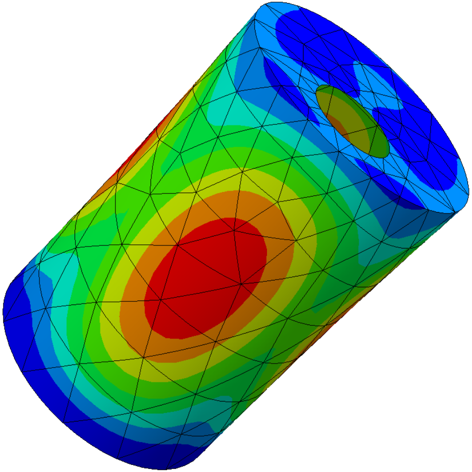


Motivation

Approximate extrinsic curvature of non-smooth surfaces

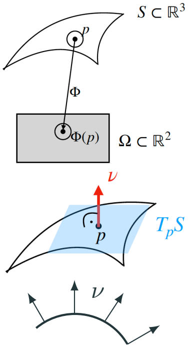


Application to shells



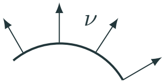
Extrinsic curvature

- Surface $\mathcal{S}$ embedded in $\mathbb{R}^3$
- Normal vector $\nu : \mathcal{S} \to \mathbb{S}^2$
- Shape operator, Weingarten tensor, second fundamental form $\nabla\nu$
- Eigenvalues 0, κ_1, κ_2



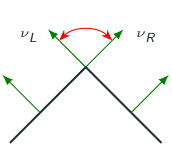
Mean curvature $H = 0.5(\kappa_1 + \kappa_2) = 0.5\,\mathrm{tr}\,(\nabla\nu)$ $\Rightarrow$ extrinsic curvature
Gauss curvature $K = \kappa_1\kappa_2 = \det(\nabla\nu + \nu \otimes \nu)$ $\Rightarrow$ intrinsic curvature

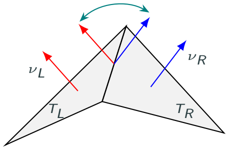
Intrinsic curvature is independent of the embedding (surrounding space)



- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces

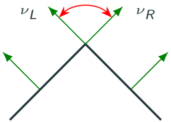
Weingarten tensor (shape operator)

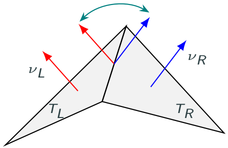




- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps

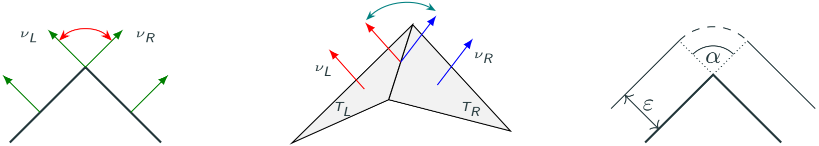
Weingarten tensor (shape operator)





- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps
- How to define and approximate the Weingarten tensor? $\Rightarrow$ Discrete differential geometry

Weingarten tensor (shape operator)

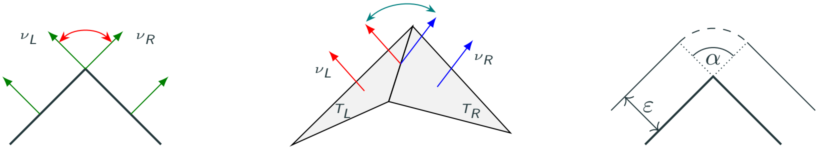


- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps
- How to define and approximate the Weingarten tensor? $\Rightarrow$ Discrete differential geometry
- Dihedral angle formula (from Steiner's offset formula): $\sum_{E\in\mathring{\mathscr{E}}} \alpha_E\, |E|$

Steiner: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

Grinspun, Gingold, Reisman, Zorin Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

Weingarten tensor (shape operator)



- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps
- How to define and approximate the Weingarten tensor? $\Rightarrow$ Discrete differential geometry
- Dihedral angle formula (from Steiner's offset formula): $\sum_{E\in\mathring{\mathcal{E}}} \alpha_E\,|E|$

How to define a generalized Weingarten tensor object? Combine FEM & DDG!

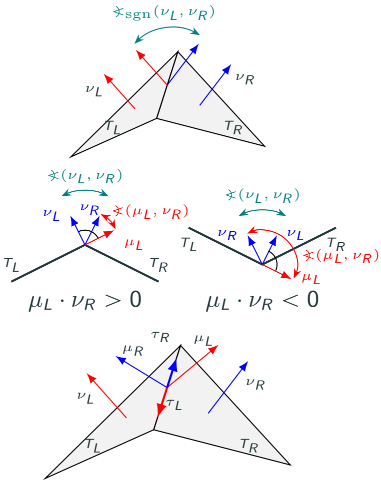
STEINER: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

GRINSPUN, GINGOLD, REISMAN, ZORIN Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\widetilde{\nabla\nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}}\int_T \nabla\nu : \boldsymbol{\Psi}\,dx + \sum_{E\in\mathring{\mathscr{E}}}\int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\,\boldsymbol{\Psi}_{\mu\mu}\,ds$$

- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$



DDG & FEM: Generalized Weingarten tensor

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

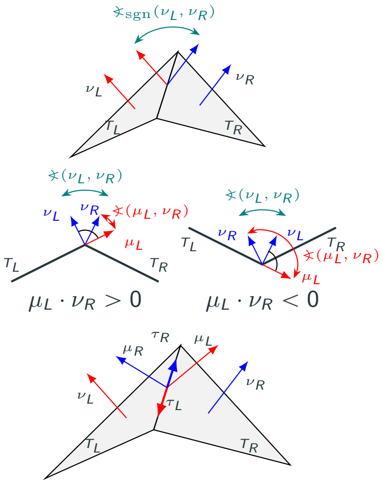
$$\widetilde{\nabla\nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}}\int_T \nabla\nu : \boldsymbol{\Psi}\,dx + \sum_{E\in\mathring{\mathscr{E}}}\int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\,\boldsymbol{\Psi}_{\mu\mu}\,ds$$

- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$
- Test function space

$$\Sigma = \{\boldsymbol{\sigma}\in L^2(\mathscr{T},\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \,:\, (\boldsymbol{\sigma}\nu)|_T = 0,\ (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

- Motivation: TDNNS method: $\nabla H(\mathrm{curl}) \subset H(\mathrm{div}\,\mathrm{div})$*

  $\Sigma \ldots$ Hellan–Herrmann–Johnson space



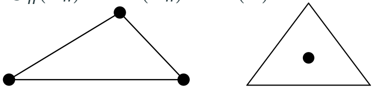
Pechstein, Schöberl: Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity, *M3AS* (2011)

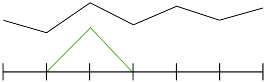
Finite element spaces: scalar, vector, tensor

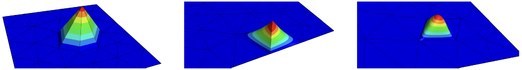
Scalar-valued spaces

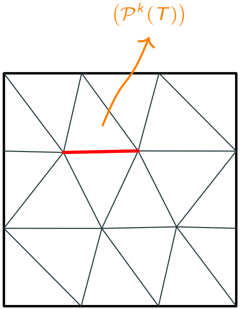
$$
\begin{aligned}
H^1(\Omega) &= \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\},\\
\mathrm{Lag}_h^k(\mathscr{T}_h) &= \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
L^2(\Omega) &= \{u : \Omega \to \mathbb{R} \,|\, u^2 \text{ integrable}\},\\
\mathcal{DG}_h^k(\mathscr{T}_h) &= \mathcal{P}^k(\mathscr{T}_h) \subset L^2(\Omega)
\end{aligned}
$$









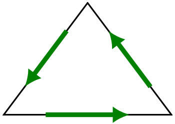
Examples: Density, pressure, temperature
Finite elements: Lagrange (H^1, continuous), discontinuous Galerkin (L^2, discontinuous)

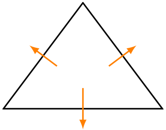
Finite element spaces: scalar, vector, tensor

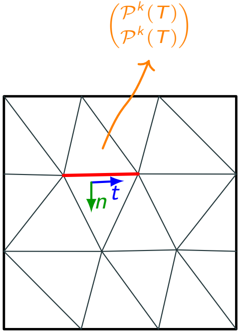
Vector-valued spaces

$$
\begin{aligned}
H(\mathrm{curl},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{curl}\,\sigma \in [L^2(\Omega)]^{2d-3}\},\\
\mathcal{N}^k_{II,h} &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_t]\!]_F = 0\} \subset H(\mathrm{curl},\Omega),
\end{aligned}
$$

$$
\begin{aligned}
H(\mathrm{div},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}\sigma \in L^2(\Omega)\},\\
\mathrm{BDM}^k_h &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div},\Omega)
\end{aligned}
$$







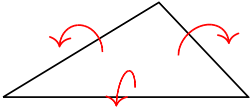
Examples: Deformation, velocity, momentum
Finite elements: Nédélec ($H(\mathrm{curl})$, tangential), Raviart–Thomas/BDM ($H(\mathrm{div})$, normal)

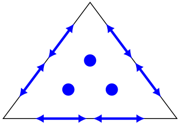
Finite element spaces: scalar, vector, tensor

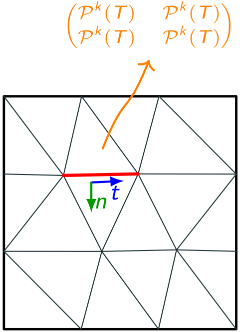
Tensor-valued spaces

$$H(\mathrm{divdiv},\Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{div}\,\mathrm{div}\,\sigma \in H^{-1}(\Omega)\},$$
$$M_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![n^T\sigma n]\!]_F = 0\},$$

$$H(\mathrm{curl}\,\mathrm{curl},\Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\},$$
$$\mathrm{Reg}_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^T\sigma t]\!]_F = 0\}$$







Examples: Strain, stress, metric
Finite elements: Regge ($H(\mathrm{curl}\,\mathrm{curl})$, tt),
Hellan–Herrmann–Johnson ($H(\mathrm{divdiv})$, nn)

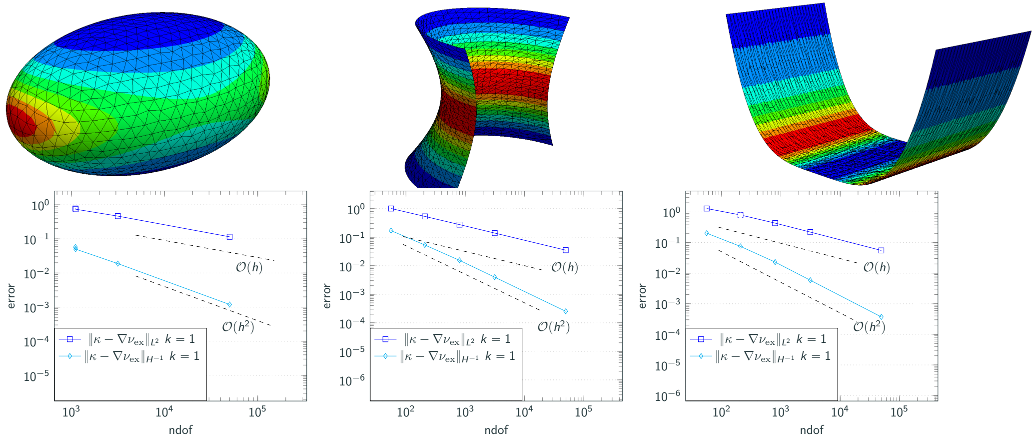
Example curvature approximation

TECHNISCHE UNIVERSITÄT WIEN

Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$



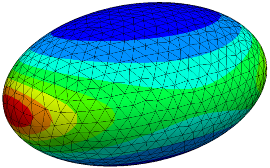
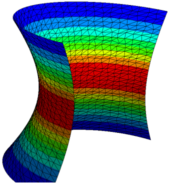
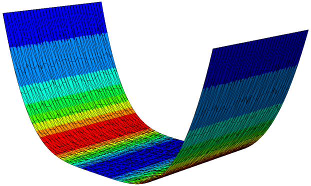
Example curvature approximation

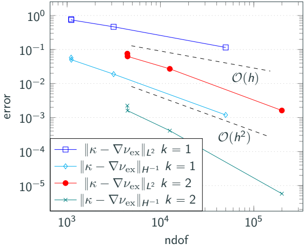
TECHNISCHE UNIVERSITÄT WIEN

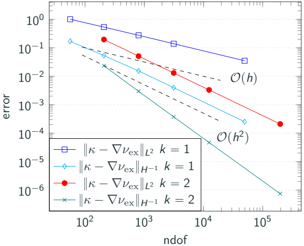
Lifting of distributional Weingarten tensor

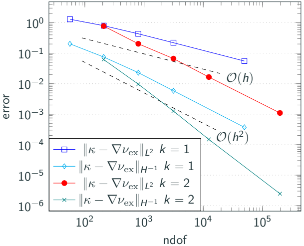
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$







Lifted curvature & integral representation

Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T \in \mathscr{T}} \int_T \nabla \nu : \boldsymbol{\sigma} \, dx + \sum_{E \in \mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) \, \boldsymbol{\sigma}_{\mu\mu} \, ds.$$

Lifted curvature & integral representation

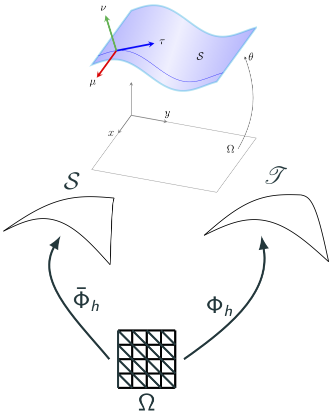
Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$

- If $\mathscr{T} \to \mathcal{S}$, does $\boldsymbol{\kappa} \to \nabla\bar{\nu}$?
- Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$ and use integral representation of the error

$$\widetilde{\nabla}\nu(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma})\, dt$$



Lifting of distributional Weingarten tensor

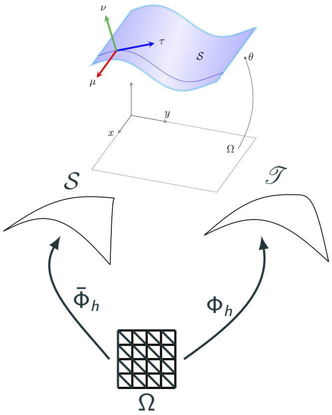
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$

- If $\mathscr{T} \to \mathcal{S}$, does $\boldsymbol{\kappa} \to \nabla\bar{\nu}$?
- Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$ and use integral representation of the error

$$\widetilde{\nabla}\nu(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma})\, dt$$

- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ



Lifted curvature & integral representation

Lifting of distributional Weingarten tensor

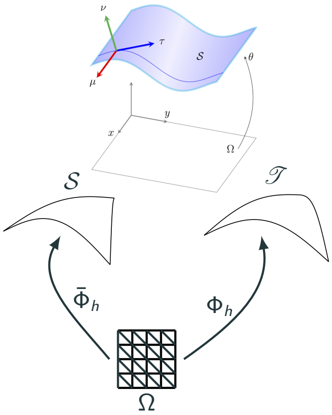
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$

- If $\mathscr{T} \to \mathcal{S}$, does $\boldsymbol{\kappa} \to \nabla\bar{\nu}$?
- Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$ and use integral representation of the error

$$\widetilde{\nabla}\nu(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma})\, dt$$

- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ
- Solution: Use fixed reference domain (Uhlenbeck trick)
- Then estimate integrand



TECHNISCHE UNIVERSITÄT WIEN

Evolution of quantities in direction $X \circ \Phi = \hat{X} := \dot{\Phi}$ (shape optimization techniques)

Lemma (linearization of geometric quantities)

$$\frac{d}{dt} J = (\mathrm{div}_{\mathcal{S}} X) \circ \Phi\, J \qquad \text{surface determinant}$$

$$\frac{d}{dt}(\nu \circ \Phi) = -((\nabla_{\mathcal{S}} X)^T \nu) \circ \Phi \qquad \text{surface normal}$$

$$\frac{d}{dt}(\boldsymbol{\sigma} \circ \Phi) = (-2\mathrm{div}_{\mathcal{S}}(X)\boldsymbol{\sigma} + 2\mathrm{sym}(\nabla_{\mathcal{S}} X \boldsymbol{\sigma})) \circ \Phi \qquad \text{HHJ test function}$$

$$\frac{d}{dt}(\boldsymbol{\sigma}_{\mu\mu} \circ \Phi) = (-2(\nabla_{\mathcal{S}} X)_{\tau\tau} \boldsymbol{\sigma}_{\mu\mu}) \circ \Phi$$

$$\frac{d}{dt}(\nabla_{\mathcal{S}} \nu \circ \Phi) = \left(\sum_{i=1}^{3} \nu_i \nabla_{\mathcal{S}}^2 X_i - (\nabla_{\mathcal{S}} X)^T \nabla_{\mathcal{S}} \nu + \nabla_{\mathcal{S}} \nu ((\nabla_{\mathcal{S}} X^T \nu) \otimes \nu - \nabla_{\mathcal{S}} X) \right) \circ \Phi$$

$$\frac{d}{dt}(\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) \circ \Phi) = [\![(\nabla_{\mathcal{S}} X)_{\nu\mu}]\!] \circ \Phi \qquad \text{dihedral angle}$$

$F = \hat{\nabla} \hat{X}$, $J = \sqrt{\det(F^T F)}$, $\boldsymbol{\sigma} \circ \Phi = J^{-2} F \hat{\boldsymbol{\sigma}} F^T$

TECHNISCHE UNIVERSITÄT WIEN

Theorem (Gopalakrishnan, N.)

There holds for $\boldsymbol{\sigma} \in \Sigma$ and $X = \dot{\Phi}$

$$\frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, X) + b(\Phi; \boldsymbol{\sigma}, X),$$

where with $\mathcal{H}_\nu(X) = \sum_{i=1}^3 \mathrm{hesse}(X_i)\nu_i$

$$a(\Phi; \boldsymbol{\sigma}, X) = \sum_{T\in\mathscr{T}} \int_T -\mathrm{div}(X)\nabla\nu : \boldsymbol{\sigma} - \sum_{E\in\mathring{\mathscr{E}}} \int_E (\nabla X)_{\tau\tau} \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\boldsymbol{\sigma}_{\mu\mu},$$

$$b(\Phi; \boldsymbol{\sigma}, X) = \sum_{T\in\mathscr{T}} \int_T -\mathcal{H}_\nu(X) : \boldsymbol{\sigma} + \sum_{E\in\mathring{\mathscr{E}}} \int_E [\![(\nabla X)_{\nu\mu}]\!]_E \boldsymbol{\sigma}_{\mu\mu}.$$

Bilinear form $b(\Phi; \boldsymbol{\sigma}, X)$ is closely related to the surface Hellan–Herrmann–Johnson method

Walker: The Kirchhoff plate equation on surfaces: the surface Hellan–Herrmann–Johnson method, *IMA J. Numer. Anal.* (2021)

Perform all estimates on the reference domain: Transform bilinear forms back

Lemma (pull-back)

$$a(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat{T}\in\hat{\mathscr{T}}}\int_{\hat{T}} -J^{-1}\mathrm{tr}(\hat{\nabla}\hat{X}F^{\dagger})\hat{S} : \hat{\boldsymbol{\sigma}} - \sum_{\hat{E}\in\mathring{\hat{\mathscr{E}}}}\int_{\hat{E}} J_{\mathrm{bnd}}^{-3}(\hat{\nabla}_{\hat{\tau}}\hat{X})\cdot(F\hat{\tau})\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\circ\Phi\hat{\boldsymbol{\sigma}}_{\hat{\mu}\hat{\mu}},$$

$$b(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat{T}\in\hat{\mathscr{T}}}\int_{\hat{T}} -\sum_{i=1}^{3} J^{-1}(F_1\times F_2)\left(\hat{\nabla}^2\hat{X}_i - \sum_{\alpha=1}^{2}(\hat{\nabla}_\alpha\hat{X}_i)\Gamma^\alpha\right) : \hat{\boldsymbol{\sigma}}$$

$$+\sum_{\hat{E}\in\mathring{\hat{\mathscr{E}}}}\int_{\hat{E}}\left[\!\left[\sum_{i=1}^{3}J^{-1}(F_1\times F_2)\cdot\left(\hat{\nabla}_{\hat{\mu}}\hat{X}_i - J_{\mathrm{bnd}}^{2}(F^TF)_{\hat{\tau}\hat{\mu}}\hat{\nabla}_{\hat{\tau}}\hat{X}_i\right)\right]\!\right]_E \hat{\boldsymbol{\sigma}}_{\hat{\mu}\hat{\mu}}$$

Christoffel symbols of second kind: $\Gamma^\alpha_{\beta\gamma} = \sum_{\ell=1}^3 F^{\dagger}_{\alpha\ell}\hat{\nabla}_\beta F_{\ell\gamma}$

$\hat{S} = \sum_{i=1}^3 \nu_i\circ\Phi\hat{\nabla}^2\Phi_i$ pull-back of Weingarten tensor: $\nabla_{\mathcal{S}}\nu\circ\Phi = -F^{\dagger^T}\hat{S}F^{\dagger}$

$\boldsymbol{\sigma}\circ\Phi = J^{-2}F\hat{\boldsymbol{\sigma}}F^T, \quad \nu\circ\Phi = J^{-1}F_1\times F_2$

Note: $\hat{X} = \dot{\Phi} = \Phi_h - \bar{\Phi}_h \Rightarrow$ gives convergence

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leqslant C\, h^k.$$

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leqslant C\, h^k.$$

Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ always converges in H^{-1}!

GOPALAKRISHNAN, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leqslant C\, h^k.$$

Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ always converges in H^{-1}!

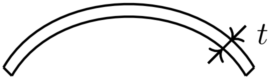
Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\Phi_h = \mathcal{I}_h^{\mathrm{Lag}^k}\bar{\Phi}_h$ for $k \geqslant 1$. Let $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ be the lifted Weingarten tensor. Then $\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$.

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

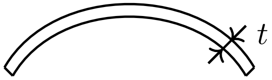
Shells

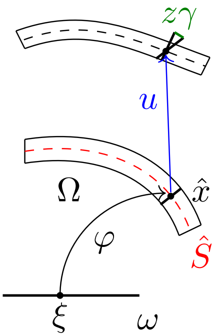
Thin-walled structures



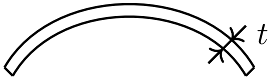
- Reduce 3D elasticity to 2D shell model

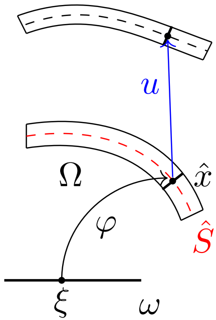
Thin-walled structures



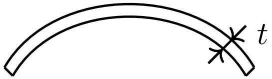


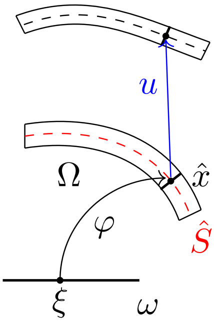
- Reduce 3D elasticity to 2D shell model
- $\Omega = \{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in [-\frac{t}{2}, \frac{t}{2}]\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z \underbrace{(\nu + \gamma) \circ \phi}_{=\tilde{\nu}\circ\phi}(\hat{x})$
- Reissner-Mindlin/Naghdi shell





- Reduce 3D elasticity to 2D shell model
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z\,\nu \circ \phi(\hat{x})$
- Kirchhoff–Love/Koiter shell





- Reduce 3D elasticity to 2D shell model
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z\,\nu \circ \phi(\hat{x})$
- Kirchhoff–Love/Koiter shell
- Insert Φ in 3D elasticity and integrate over thickness, neglect higher order terms $\mathcal{O}(t^4)$ (asymptotical analysis)

Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

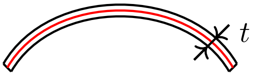
u ... displacement of mid-surface

t ... thickness

$\mathcal{M}$ ... material tensor

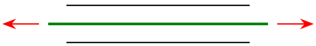
$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$



Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$



membrane energy

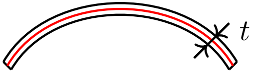
u . . . displacement of mid-surface

t . . . thickness

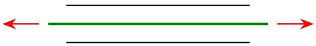
$\mathcal{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$



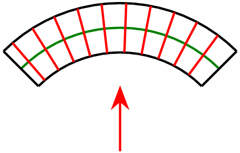
$u\ldots$ displacement of mid-surface

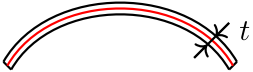
$t\ldots$ thickness

$\mathcal{M}\ldots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$





FEM + DDG for Koiter/Kirchhoff–Love shell

Lifted shape operator: $$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\Psi}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\Psi}) := \sum_{T\in\mathscr{T}}\int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}}\int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\,\boldsymbol{\Psi}_{\mu\mu}\, ds$$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

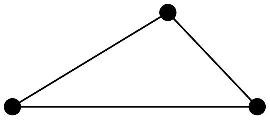
$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) &= \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f,u\rangle\\ &+ \sum_{T\in\mathscr{T}}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx\\ &- \sum_{E\in\mathscr{E}}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

- Lagrange parameter $\boldsymbol{\sigma}\in\Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u,\boldsymbol{\sigma})$

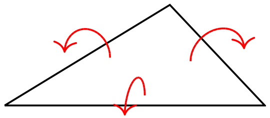
N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

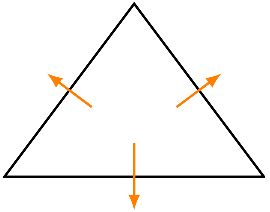
Shell element



Displacement u

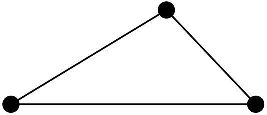


Moment $\boldsymbol{\sigma}$

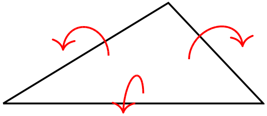


Hybridization

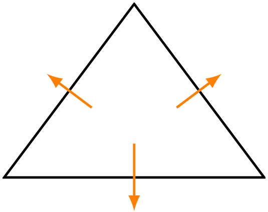
Shell element



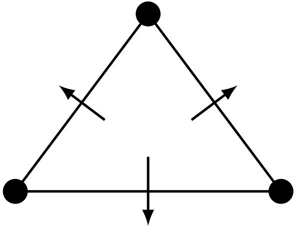
Displacement u



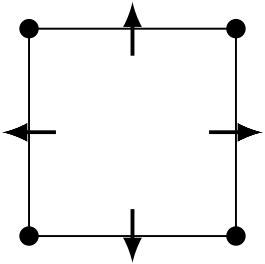
Moment $\boldsymbol{\sigma}$



Hybridization



Morley



Quadrilateral (hybridized)

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

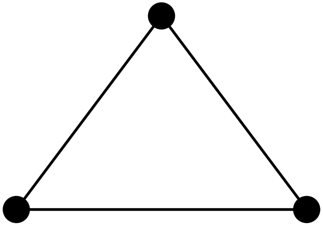
$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



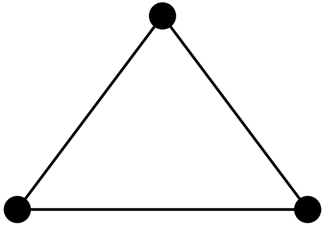
$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

Membrane locking

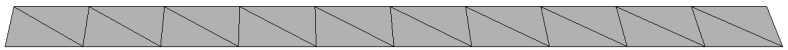
$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

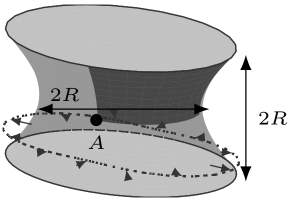
Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

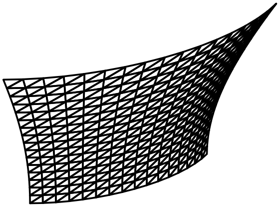
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$

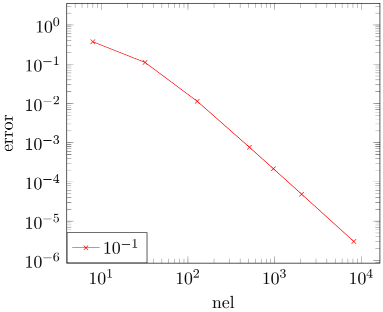


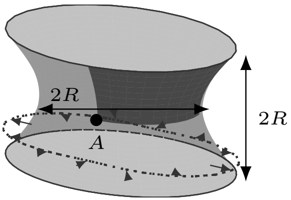
$$\text{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

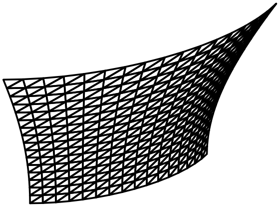


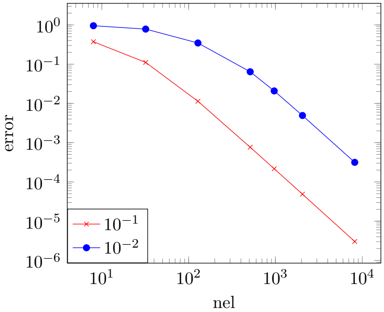

2R
2R
A



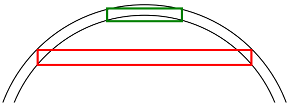

error
nel
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^1
10^2
10^3
10^4
10^{-1}

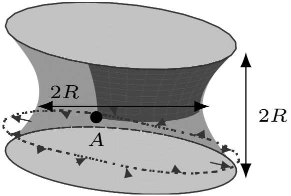


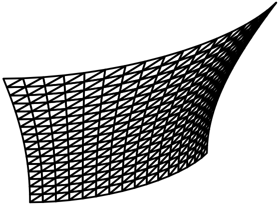


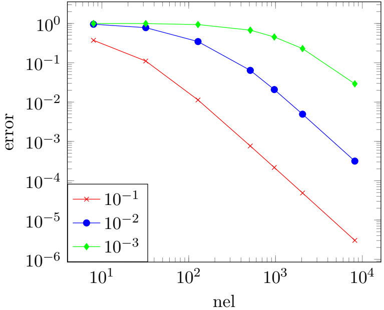


- Pre-asymptotic regime

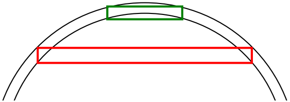


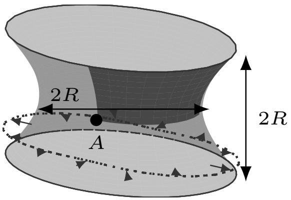


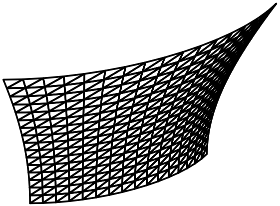


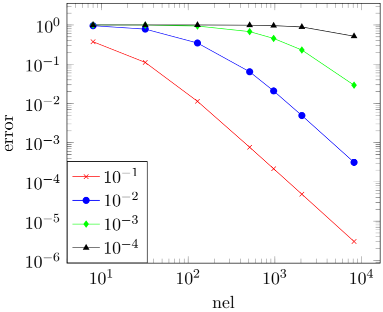


- Pre-asymptotic regime

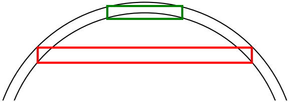








- Pre-asymptotic regime

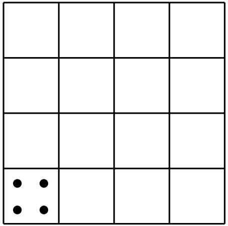


Membrane locking

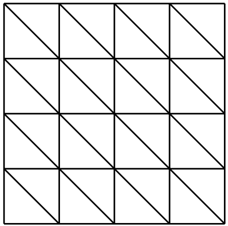
$$\frac{1}{t^2}\|\quad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

Membrane locking

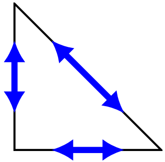
$$\frac{1}{t^2}\|\Pi_{L^2}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$



- Reduced integration for quadrilateral meshes



$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

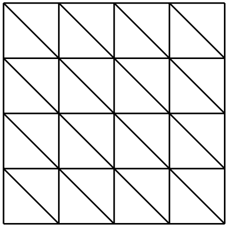


- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

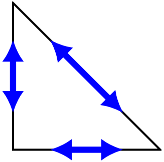
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Membrane locking

$$\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u)\|_{\mathcal{M}}^{2} + \frac{t^{3}}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|_{\mathcal{M}}^{2} - \langle f,u\rangle$$
$$+ \sum_{T\in\mathscr{T}}\int_{T}\left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^{T}\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\,dx$$
$$- \sum_{E\in\mathscr{E}}\int_{E}(\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$



- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

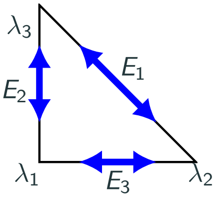


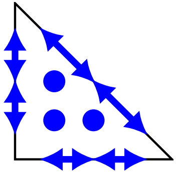
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Regge elements & interpolation operator

$$H(\mathrm{curl}\,\mathrm{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

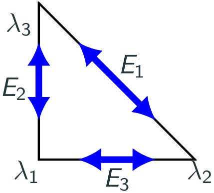
Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

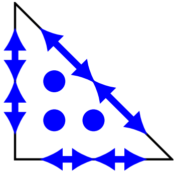
Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

$$H(\operatorname{curl}\operatorname{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \operatorname{curl}\operatorname{curl}\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





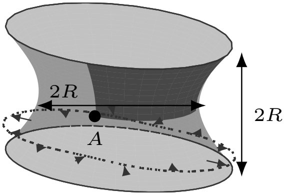
$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

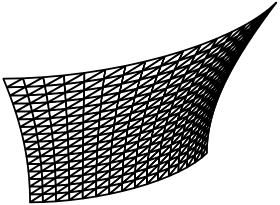
$$\mathcal{I}^k_{\mathcal{R}} : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

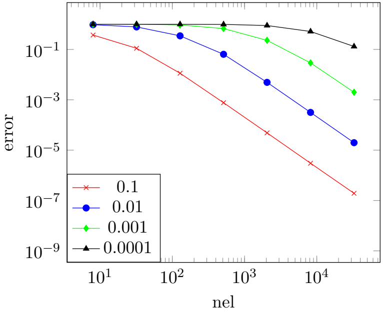
$$\int_E (g - \mathcal{I}^k_{\mathcal{R}} g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

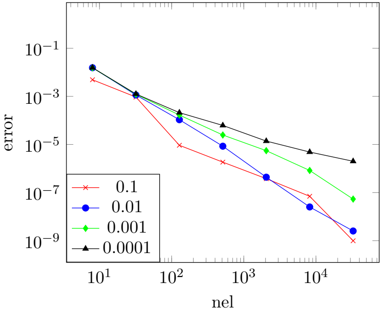
$$\int_T (g - \mathcal{I}^k_{\mathcal{R}} g) : Q\, dx = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking





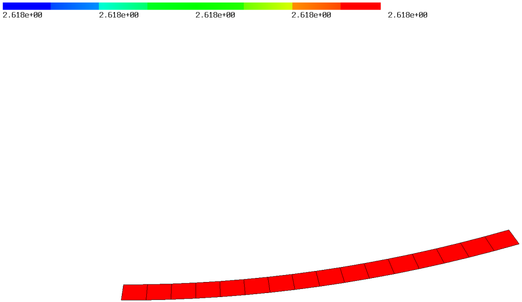


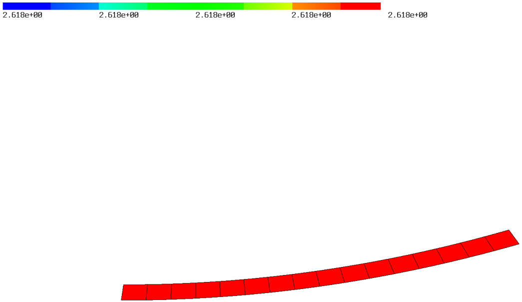


Numerics & Applications

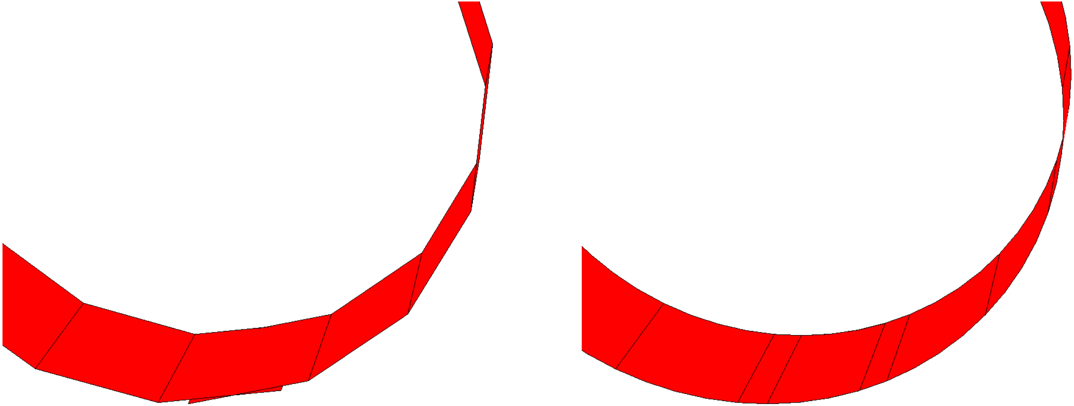
Example (cantilever bending)

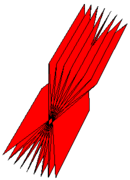
NGSolve

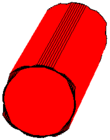




Example (cantilever bending)



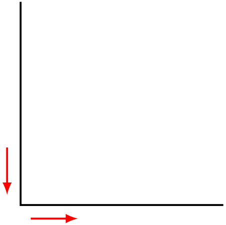


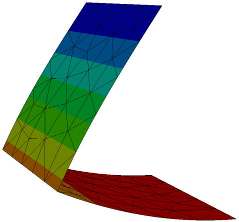


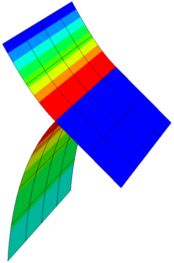
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0$$

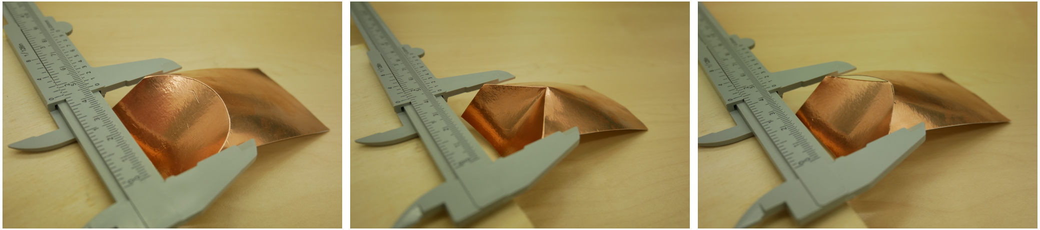
$$\Rightarrow \quad \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R) = 0$$

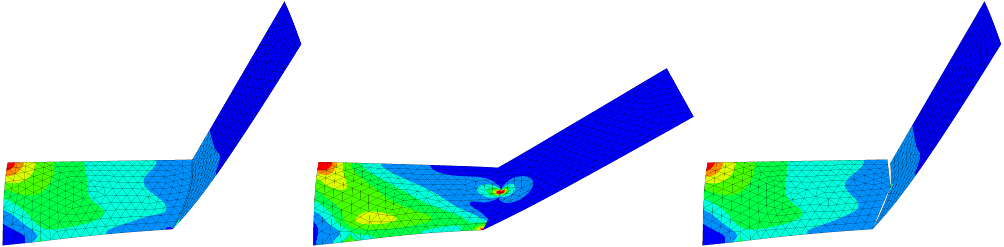






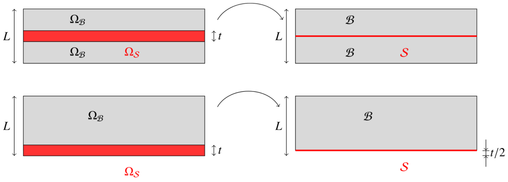
Application (bending-folding model)

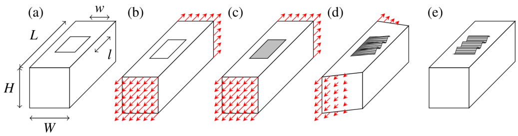


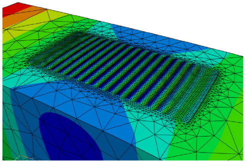


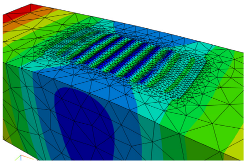
Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces Free Bound.* (2026).

- Composite materials, blood vessels, etc.
- Lagrange elements for elasticity and shell displacement → easy to couple









PECHSTEIN, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)

Application (cell membrane)

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

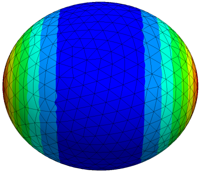
κ_b bending elastic constant

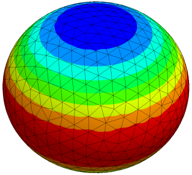
$H = 0.5\,\mathrm{tr}(\nabla\nu)$ mean curvature

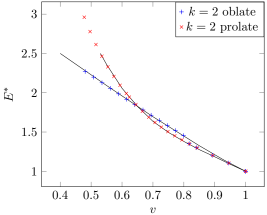
$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

Functional: $\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$







N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Struct. Multidiscip. Optim.* (2021)

Application (cell membrane)

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

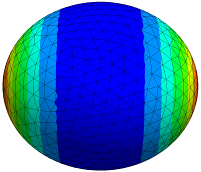
κ_b bending elastic constant

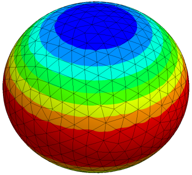
$H = 0.5\,\mathrm{tr}(\nabla \nu)$ mean curvature

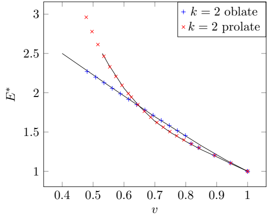
$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

Functional: $\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$







N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Struct. Multidiscip. Optim.* (2021)

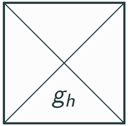
Intrinsic curvature

Riemannian manifold and Regge finite elements

- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}})$, then K is defined pointwise

Riemannian manifold and Regge finite elements

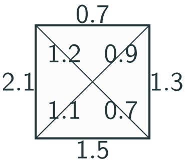
- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}})$, then K is defined pointwise



Regge, *General relativity without coordinates*, Il Nuovo Cimento (1955-1965), (1961).

Sorkin, *Time-evolution problem in Regge calculus*, Phys. Rev. D 12 (1975).

- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, then K is defined pointwise
- Regge: Approximate the metric by squared edge lengths



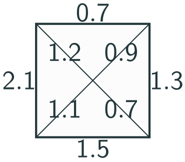
Regge, *General relativity without coordinates*, Il Nuovo Cimento (1955-1965), (1961).
Sorkin, *Time-evolution problem in Regge calculus*, Phys. Rev. D 12 (1975).

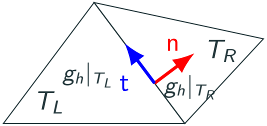
Riemannian manifold and Regge finite elements

TECHNISCHE UNIVERSITÄT WIEN

- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}})$, then K is defined pointwise

- Regge: Approximate the metric by squared edge lengths
- Sorkin: g_h tt continuous $g_h|_{T_L}(t,t) = g_h|_{T_R}(t,t)$ but
  $g_h|_{T_L}(t,n) \neq g_h|_{T_R}(t,n)$ and $g_h|_{T_L}(n,n) \neq g_h|_{T_R}(n,n)$

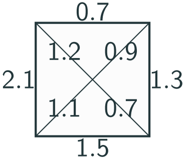




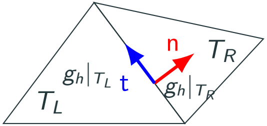
Regge, *General relativity without coordinates*, Il Nuovo Cimento (1955-1965), (1961).

Sorkin, *Time-evolution problem in Regge calculus*, Phys. Rev. D 12 (1975).

TECHNISCHE UNIVERSITÄT WIEN



- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, then K is defined pointwise



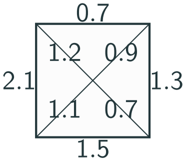
- Regge: Approximate the metric by squared edge lengths
- Sorkin: g_h tt continuous $g_h|_{T_L}(t,t) = g_h|_{T_R}(t,t)$ but
  $g_h|_{T_L}(t,n) \neq g_h|_{T_R}(t,n)$ and $g_h|_{T_L}(n,n) \neq g_h|_{T_R}(n,n)$

Regge finite element space:

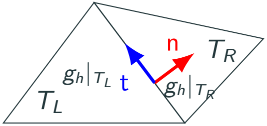
$$\mathrm{Reg}_h^k = \{g_h \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \,:\, g_h \text{ is tt-continuous}\}$$

Christiansen, *On the linearization of Regge calculus, Numerische Mathematik, (2011).*

Li, *Regge Finite Elements with Applications in Solid Mechanics and Relativity*, PhD thesis, University of Minnesota (2018).



- Riemannian manifold (Ω, g)
  $\Omega \subset \mathbb{R}^N$, g metric
- $N = 2$. Gaussian curvature $K = f(g, Dg, D^2 g)$
- $g \in C^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, then K is defined pointwise



- Regge: Approximate the metric by squared edge lengths
- Sorkin: g_h tt continuous $g_h|_{T_L}(t,t) = g_h|_{T_R}(t,t)$ but
  $g_h|_{T_L}(t,n) \neq g_h|_{T_R}(t,n)$ and $g_h|_{T_L}(n,n) \neq g_h|_{T_R}(n,n)$

Regge finite element space:

$$\mathrm{Reg}_h^k = \{g_h \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \ : \ g_h \text{ is tt-continuous}\}$$

Question: Discrete curvature $K_h = f(g_h, Dg_h, D^2 g_h)$?

Christiansen, *On the linearization of Regge calculus, Numerische Mathematik, (2011).*

Li, *Regge Finite Elements with Applications in Solid Mechanics and Relativity*, PhD thesis, University of Minnesota (2018).

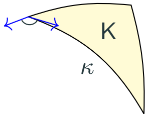
Generalized Gaussian curvature via Gauss–Bonnet

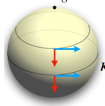
TECHNISCHE UNIVERSITÄT WIEN

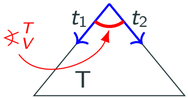
Gauss–Bonnet

$$\int_M K\,\omega_M\,dx + \int_{\partial M} \kappa_g\,\omega_{\partial M}\,ds + \sum_i \alpha_i = 2\pi\chi(M)$$

- Gaussian curvature K
- Geodesic curvature $\kappa_g = g(\nabla_\tau \tau, \mu)$
- Exterior angles α_i at vertices
- $\chi(M) = \#V - \#E + \#F$ Euler characteristic of M
- ω_M, $\omega_{\partial M}$ volume forms on M, ∂M







Sum over triangulation $\mathscr{T}$

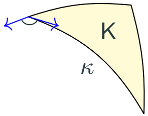
$$\sum_{T\in\mathscr{T}} \int_T K|_T\,\omega_T\,dx + \sum_{E\in\mathscr{E}} \int_E [\![\kappa_g]\!]_E\,\omega_E\,ds + \sum_{V\in\mathscr{V}} \Theta_V = 2\pi, \qquad \Theta_V = 2\pi - \sum_{T\supset V} \measuredangle_V^T$$

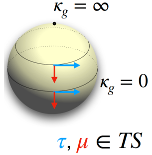
Generalized Gaussian curvature via Gauss–Bonnet

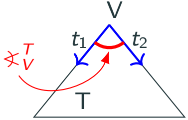
Gauss–Bonnet

$$\int_M K\,\omega_M\,dx + \int_{\partial M} \kappa_g\,\omega_{\partial M}\,ds + \sum_i \alpha_i = 2\pi\chi(M)$$

- Gaussian curvature K
- Geodesic curvature $\kappa_g = g(\nabla_\tau \tau, \mu)$
- Exterior angles α_i at vertices
- $\chi(M) = \#V - \#E + \#F$ Euler characteristic of M
- ω_M, $\omega_{\partial M}$ volume forms on M, ∂M







Generalized Gaussian curvature

$$\widetilde{K\omega}(u) = \sum_{T\in\mathscr{T}} \int_T K|_T\, u\,\omega_T + \sum_{E\in\mathscr{E}} \int_E [\![\kappa_g]\!]_E\, u\,\omega_E + \sum_{V\in\mathscr{V}} \Theta_V\, u(V), \qquad u \in \mathrm{Lag}_h \subset H^1$$

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, SMAI J. Comput. Math. 9 (2023) 151–195.

Generalized Riemann curvature tensor

TECHNISCHE UNIVERSITÄT WIEN

- Intermediate results: scalar curvature ($\mathbb{R}^N$), Einstein tensor ($\mathbb{R}^N$)
- Riemann curvature tensor in arbitrary dimension:
  $\mathcal{R}(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$

$$\widetilde{K\omega} = \sum_T K|_T\,\omega_T + \sum_E [\![\kappa_g]\!]\,\omega_E\,\delta_E + \sum_V \Theta_V\,\delta_V$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\widetilde{\mathcal{R}\omega} = \sum_T \mathcal{R}|_T\,\omega_T + \sum_F [\![I\!I]\!]\,\omega_F\,\delta_F + \sum_E \Theta_E\,\omega_E\,\delta_E$$

Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp. (2024).

Gawlik, Neunteufel, *Finite element approximation of the Einstein tensor*, IMA Numer. Anal. (2025).

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

Generalized Riemann curvature tensor

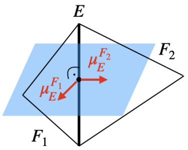
- Intermediate results: scalar curvature ($\mathbb{R}^N$), Einstein tensor ($\mathbb{R}^N$)
- Riemann curvature tensor in arbitrary dimension:
  $\mathcal{R}(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$

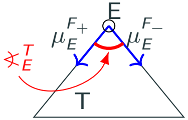
$$\widetilde{K\omega} = \sum_T K|_T\,\omega_T + \sum_E [\![\kappa_g]\!]\,\omega_E\,\delta_E + \sum_V \Theta_V\,\delta_V$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\widetilde{\mathcal{R}\omega} = \sum_T \mathcal{R}|_T\,\omega_T + \sum_F [\![I\!I]\!]\,\omega_F\,\delta_F + \sum_E \Theta_E\,\omega_E\,\delta_E$$

- Angle defect generalizes directly to higher dimensions





Orthogonal surface to E

Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp. (2024).

Gawlik, Neunteufel, *Finite element approximation of the Einstein tensor*, IMA Numer. Anal. (2025).

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

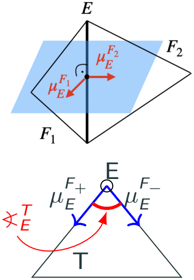
Generalized Riemann curvature tensor

- Intermediate results: scalar curvature ($\mathbb{R}^N$), Einstein tensor ($\mathbb{R}^N$)
- Riemann curvature tensor in arbitrary dimension:
  $\mathcal{R}(X, Y, Z, W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$

$$\widetilde{K\omega} = \sum_T K|_T\,\omega_T + \sum_E [\![\kappa_g]\!]\,\omega_E\,\delta_E + \sum_V \Theta_V\,\delta_V$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\widetilde{\mathcal{R}\omega} = \sum_T \mathcal{R}|_T\,\omega_T + \sum_F [\![I\!I]\!]\,\omega_F\,\delta_F + \sum_E \Theta_E\,\omega_E\,\delta_E$$



Orthogonal surface to E

- Angle defect generalizes directly to higher dimensions
- $I\!I(X, Y) = -g(\nabla_X \nu, Y)$ second fundamental form; 2D $I\!I(t, t) = \kappa_g$

Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp. (2024).

Gawlik, Neunteufel, *Finite element approximation of the Einstein tensor*, IMA Numer. Anal. (2025).

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

Generalized Riemann curvature tensor

Definition (generalized Riemann curvature tensor)

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![II]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, A_{\mu\nu\nu\mu}\, \omega_E .$$

Theorem (convergence)

Let $(g_h)_{h>0} \in \mathrm{Reg}_h^k$ with $g_h \to \bar{g}$. Then, for $k \geqslant 0$ in 2-D or $k \geqslant 1$ in N-D,
$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \lesssim \|\!|g_h - \bar{g}|\!\|_2$.

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

Generalized Riemann curvature tensor

Definition (generalized Riemann curvature tensor)

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \mathcal{R}, A\rangle \omega_T + 4\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, A_{\mu\nu\nu\mu}\,\omega_E.$$

Theorem (convergence)

Let $(g_h)_{h>0}\in \mathrm{Reg}_h^k$ with $g_h\to\bar{g}$. Then, for $k\geqslant 0$ in 2-D or $k\geqslant 1$ in N-D,

$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \lesssim \|\!|g_h-\bar{g}|\!\|_2 \lesssim \mathcal{O}(h^{k+1})$ if $\|g_h-\bar{g}\|_{W^{t,\infty}} \lesssim h^{s-t}\|\bar{g}\|_{W^{s,\infty}}$.

Result: Regge elements for approximating metrics and computing curvatures!

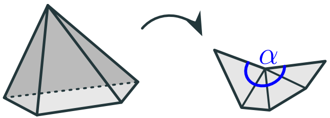
Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

Definition (generalized Riemann curvature tensor)

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \mathcal{R}, A\rangle \omega_T + 4\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, A_{\mu\nu\nu\mu}\,\omega_E.$$

Theorem (convergence)

Let $(g_h)_{h>0}\in \mathrm{Reg}_h^k$ with $g_h\to\bar{g}$. Then, for $k\geqslant 0$ in 2-D or $k\geqslant 1$ in N-D,
$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \lesssim |\!|\!| g_h-\bar{g}|\!|\!|_2 \lesssim \mathcal{O}(h^{k+1})$ if $\|g_h-\bar{g}\|_{W^{t,\infty}}\lesssim h^{s-t}\|\bar{g}\|_{W^{s,\infty}}$.



Result: Angle defects on surfaces always converge in H^{-2} to the Gaussian curvature!

Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603 (2023).

Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp. (2024).

Proof strategy

1. Metric-independent test function U, $A = \mathbb{A}_g(U)$
2. Rewrite the error in integral representation:
$$\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h}(U) - (\mathcal{R}\omega\mathbb{A})_{\bar g}(U) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U)\,dt \qquad \text{with } g(t) = \bar g + t(g_h - \bar g).$$
3. Linearize to perform the numerical analysis:
$$\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U) = a(g;\dot g, U) + b(g;\dot g, U) \qquad \text{sum of bilinear forms } a \text{ and } b.$$
4. Identify the generalized covariant incompatibility operator:
$$b(g;\dot g, U) = -2\,\widetilde{\operatorname{Inc}\dot g}(A), \qquad \text{with } \dot g = g_h - \bar g \text{ and } A = \mathbb{A}_g(U).$$
5. $a(g;\dot g, U)$ has no covariant derivatives of $\dot g$ (easier to estimate).

Example: generalized Gaussian curvature

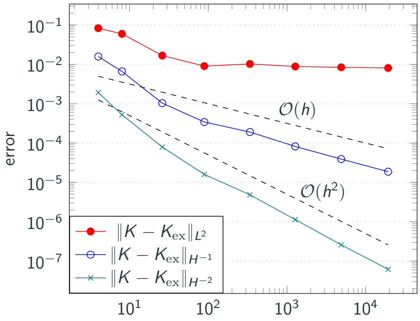
L^2-Riesz representation (lifting) of the generalized Gaussian curvature

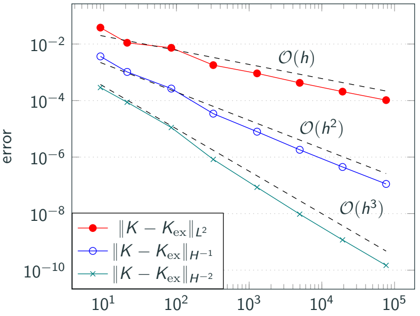
Find $K_h \in \mathrm{Lag}_h^k \subset H^1(\Omega)$ such that for all $u_h \in \mathrm{Lag}_h^k$

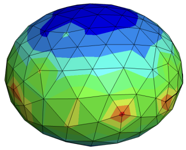
$$\int_\Omega K_h u_h \,\omega = \widetilde{K\omega}(u_h) = \sum_T \int_T K|_T u_h \,\omega_T + \sum_E \int_E [\![\kappa_g]\!]\, u_h \,\omega_E + \sum_V \Theta_V \, u_h(V)$$

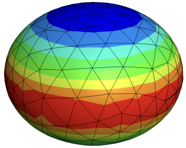
Convergence rate

If $g_h = \mathcal{I}_{\mathcal{R}}^k g$, $K_h \in \mathrm{Lag}_h^{k+1}$, then $\|K_h - K_{\mathrm{ex}}\|_{H^{-2}} \lesssim \mathcal{O}(h^{k+2})$.









Summary

- DDG + FEM: Generalized shape operator (Weingarten tensor)
- Bending energy for shell model
- Regge elements for membrane locking
- Applications (coupling, origami, cell membranes)
- Intrinsic curvature approximation (Gauss, Riemann)

- Reissner–Mindlin/Naghdi shell
- Linear and nonlinear shell models

TECHNISCHE UNIVERSITÄT WIEN

- DDG + FEM: Generalized shape operator (Weingarten tensor)
- Bending energy for shell model
- Regge elements for membrane locking
- Applications (coupling, origami, cell membranes)
- Intrinsic curvature approximation (Gauss, Riemann)

- Reissner–Mindlin/Naghdi shell
- Linear and nonlinear shell models

Thank You for Your attention!

Literature

- Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- Gopalakrishnan, Neunteufel, Schöberl, Wardetzky, *Generalizing Riemann curvature to Regge metrics*, arxiv:2311.01603.
- Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces Free Bound.*
- Pechstein, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.*
- Gawlik, Neunteufel, *Finite element approximation of scalar curvature in arbitrary dimension*, Math. Comp.
- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.*
- Gawlik, Neunteufel, *Finite element approximation of the Einstein tensor*, IMA Numer. Anal.
- N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP*
- Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Struct. Multidiscip. Optim.*
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg.*
- N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis*
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct.*