Finite Element Approximation of the Shape Operator via Discrete Differential Geometry

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Jay Gopalakrishnan (Portland State University)

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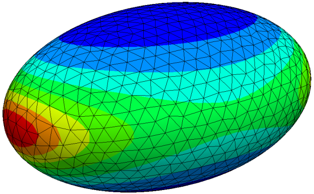
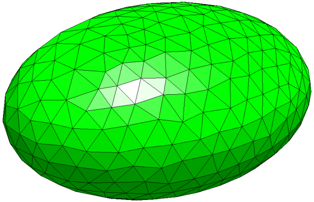
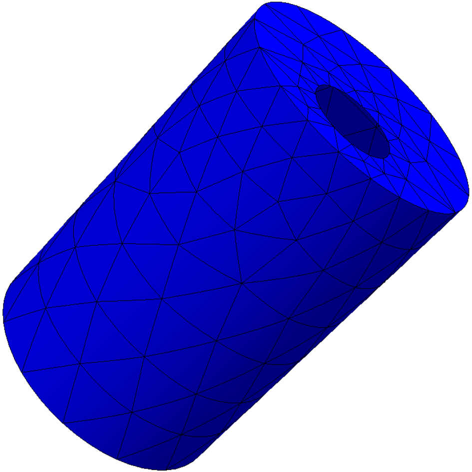
Approximate extrinsic curvature of non-smooth surfaces





Application to shells



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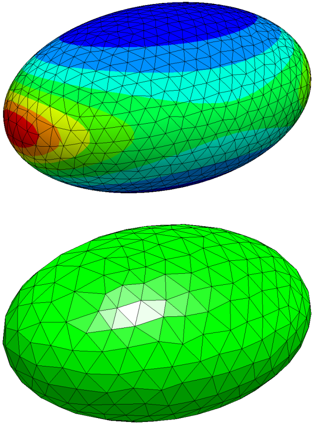

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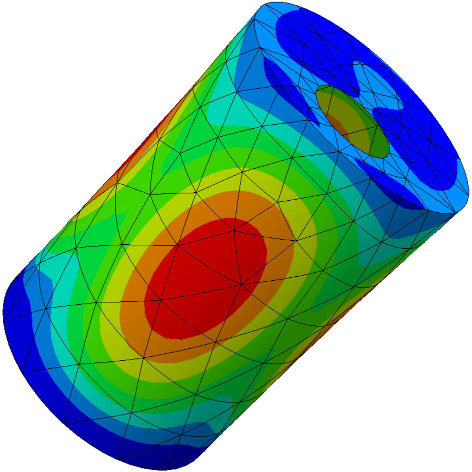
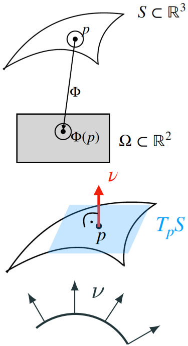


Extrinsic curvature

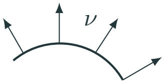
- Surface $\mathcal{S}$ embedded in $\mathbb{R}^3$
- Normal vector $\nu : \mathcal{S} \to \mathbb{S}^2$
- Shape operator, Weingarten tensor, second fundamental form $\nabla\nu$
- Eigenvalues 0, κ_1, κ_2



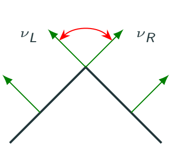
Mean curvature $H = 0.5(\kappa_1 + \kappa_2) = 0.5\,\mathrm{tr}\,(\nabla\nu)$ $\Rightarrow$ extrinsic curvature
Gauss curvature $K = \kappa_1\kappa_2 = \det(\nabla\nu + \nu \otimes \nu)$ $\Rightarrow$ intrinsic curvature

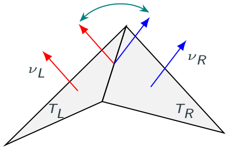
Intrinsic curvature is independent of the embedding (surrounding space)

Weingarten tensor (shape operator)



- Weingarten tensor $\nabla\nu$, ν normal vector, well-defined for C^1 surfaces





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- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps

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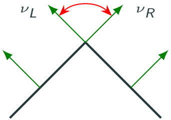
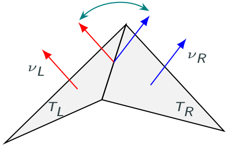
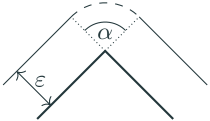
Weingarten tensor (shape operator)







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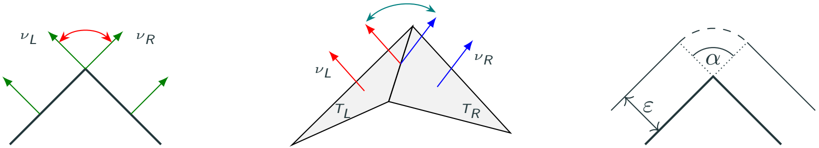
- Dihedral angle formula (from Steiner's offset formula): $\sum_{E\in\mathring{\mathscr{E}}} \alpha_E\,|E|$

STEINER: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

GRINSPUN, GINGOLD, REISMAN, ZORIN Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

Weingarten tensor (shape operator)



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How to define a generalized Weingarten tensor object? Combine FEM & DDG!

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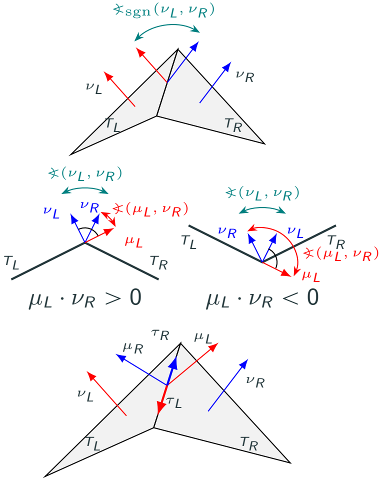
Grinspun, Gingold, Reisman, Zorin Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006)

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\widetilde{\nabla\nu}(\boldsymbol{\Psi}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$$

- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$



DDG & FEM: Generalized Weingarten tensor

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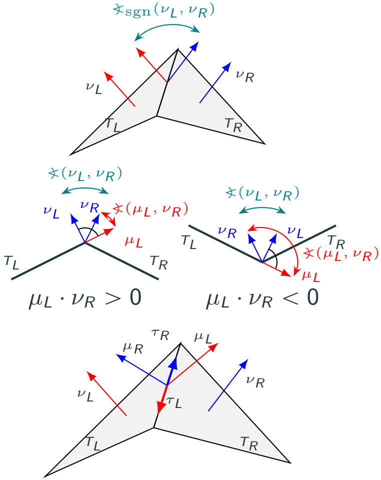
- Signed dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) = \mathrm{sgn}(\nu_L\cdot\mu_R)\sphericalangle(\nu_L,\nu_R)$
- Test function space

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T},\mathbb{R}^{3\times3}_{\mathrm{sym}}) \;:\; (\boldsymbol{\sigma}\nu)|_T = 0,\; (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

- Motivation: TDNNS method: $\nabla H(\mathrm{curl}) \subset H(\mathrm{div}\,\mathrm{div})^*$

  $\Sigma \ldots$ Hellan–Herrmann–Johnson space



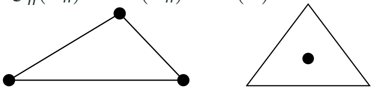
PECHSTEIN, SCHÖBERL: Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity, *M3AS* (2011)

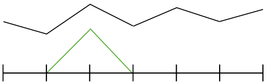
Finite element spaces

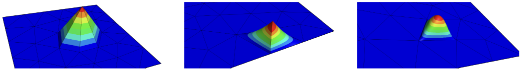
Scalar-valued spaces

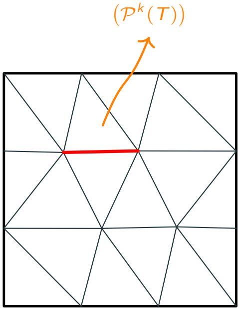
$$
\begin{aligned}
H^1(\Omega) &= \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}, \\
\mathrm{Lag}_h^k(\mathscr{T}_h) &= \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
L^2(\Omega) &= \{u : \Omega \to \mathbb{R} \mid u^2 \text{ integrable}\}, \\
\mathcal{DG}_h^k(\mathscr{T}_h) &= \mathcal{P}^k(\mathscr{T}_h) \subset L^2(\Omega)
\end{aligned}
$$









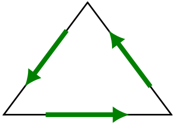
Examples: Density, pressure, temperature
Finite elements: Lagrange (H^1, continuous), discontinuous Galerkin (L^2, discontinuous)

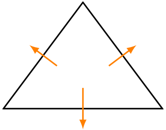
Finite element spaces

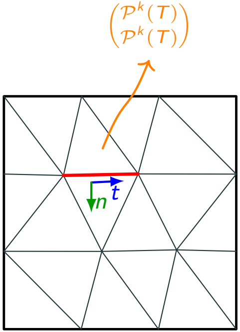
Vector-valued spaces

$$
\begin{aligned}
H(\mathrm{curl},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{curl}\,\sigma \in [L^2(\Omega)]^{2d-3}\},\\
\mathcal{N}^k_{II,h} &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_t]\!]_F = 0\} \subset H(\mathrm{curl},\Omega),
\end{aligned}
$$

$$
\begin{aligned}
H(\mathrm{div},\Omega) &= \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{div}\sigma \in L^2(\Omega)\},\\
\mathrm{BDM}^k_h &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div},\Omega)
\end{aligned}
$$







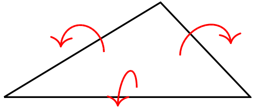
Examples: Deformation, velocity, momentum
Finite elements: Nédélec ($H(\mathrm{curl})$, tangential),
Raviart–Thomas/BDM ($H(\mathrm{div})$, normal)

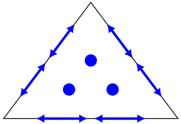
Finite element spaces

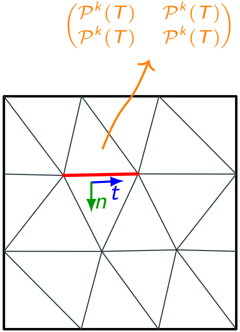
Tensor-valued spaces

$$
\begin{aligned}
H(\mathrm{divdiv},\Omega) &= \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{div}\,\mathrm{div}\,\sigma \in H^{-1}(\Omega)\},\\
\Sigma_h^k(\mathscr{T}_h) &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![n^T\sigma n]\!]_F = 0\},
\end{aligned}
$$

$$
\begin{aligned}
H(\mathrm{curl}\,\mathrm{curl},\Omega) &= \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \mid \mathrm{curl}\,\mathrm{curl}\,\sigma \in H^{-1}(\Omega)\},\\
\mathrm{Reg}_h^k(\mathscr{T}_h) &= \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^T\sigma t]\!]_F = 0\}
\end{aligned}
$$







Examples: Strain, stress, metric
Finite elements: Regge ($H(\mathrm{curl}\,\mathrm{curl})$, tt), Hellan–Herrmann–Johnson ($H(\mathrm{divdiv})$, nn)

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Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \widetilde{\nabla\nu}(\boldsymbol{\sigma}) = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathring{\mathscr{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\, \boldsymbol{\sigma}_{\mu\mu}\, ds.$$

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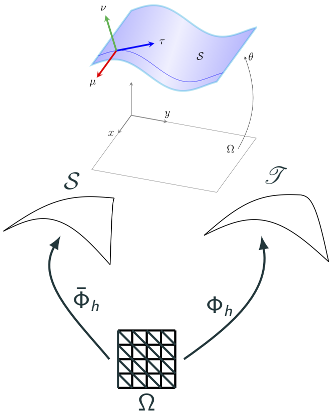
- If $\mathscr{T}\to\mathcal{S}$, does $\boldsymbol{\kappa}\to\nabla\bar{\nu}$?
- Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t)=\bar{\Phi}_h+t(\Phi_h-\bar{\Phi}_h)$ and use integral representation of the error

$$\widetilde{\nabla}\nu(\boldsymbol{\sigma})-\int_{\mathcal{S}}\nabla\nu:\boldsymbol{\sigma}\,dx=\int_0^1\frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma})\,dt$$



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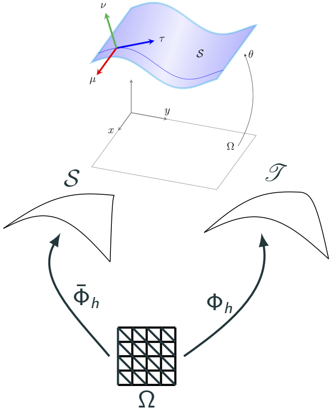
- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ



Lifted curvature & integral representation

Lifting of distributional Weingarten tensor

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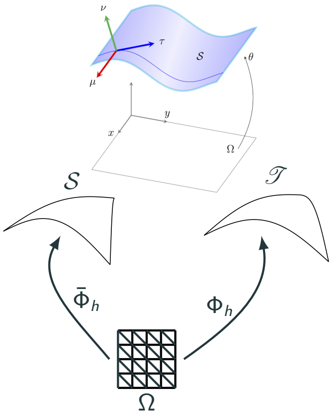
- Solution: Use fixed reference domain (Uhlenbeck trick)
- Then estimate integrand



Linearization

Evolution of quantities in direction $X \circ \Phi = \hat{X} := \dot{\Phi}$ (shape optimization techniques)

Lemma (linearization of geometric quantities)

$$\frac{d}{dt} J = (\mathrm{div}_{\mathcal{S}} X) \circ \Phi \, J \qquad \text{surface determinant}$$

$$\frac{d}{dt}(\nu \circ \Phi) = -((\nabla_{\mathcal{S}} X)^T \nu) \circ \Phi \qquad \text{surface normal}$$

$$\frac{d}{dt}(\boldsymbol{\sigma} \circ \Phi) = (-2\mathrm{div}_{\mathcal{S}}(X)\boldsymbol{\sigma} + 2\mathrm{sym}(\nabla_{\mathcal{S}} X \boldsymbol{\sigma})) \circ \Phi \qquad \text{HHJ test function}$$

$$\frac{d}{dt}(\boldsymbol{\sigma}_{\mu\mu} \circ \Phi) = (-2(\nabla_{\mathcal{S}} X)_{\tau\tau} \boldsymbol{\sigma}_{\mu\mu}) \circ \Phi$$

$$\frac{d}{dt}(\nabla_{\mathcal{S}} \nu \circ \Phi) = \left(\sum_{i=1}^{3} \nu_i \nabla^2_{\mathcal{S}} X_i - (\nabla_{\mathcal{S}} X)^T \nabla_{\mathcal{S}} \nu + \nabla_{\mathcal{S}} \nu ((\nabla_{\mathcal{S}} X^T \nu) \otimes \nu - \nabla_{\mathcal{S}} X) \right) \circ \Phi$$

$$\frac{d}{dt}(\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) \circ \Phi) = [\![(\nabla_{\mathcal{S}} X)_{\nu\mu}]\!] \circ \Phi \qquad \text{dihedral angle}$$

$F = \hat{\nabla} \hat{X}$, $J = \sqrt{\det(F^T F)}$, $\boldsymbol{\sigma} \circ \Phi = J^{-2} F \hat{\boldsymbol{\sigma}} F^T$

Theorem (Gopalakrishnan, N.)

There holds for $\boldsymbol{\sigma} \in \Sigma$ and $X = \dot{\Phi}$

$$\frac{d}{dt}\widetilde{\nabla}\nu(\boldsymbol{\sigma}) = a(\Phi;\boldsymbol{\sigma},X) + b(\Phi;\boldsymbol{\sigma},X),$$

where with $\mathcal{H}_\nu(X) = \sum_{i=1}^3 \mathrm{hesse}(X_i)\nu_i$

$$a(\Phi;\boldsymbol{\sigma},X) = \sum_{T\in\mathscr{T}}\int_T -\mathrm{div}(X)\nabla\nu : \boldsymbol{\sigma} - \sum_{E\in\mathring{\mathscr{E}}}\int_E (\nabla X)_{\tau\tau}\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\boldsymbol{\sigma}_{\mu\mu},$$

$$b(\Phi;\boldsymbol{\sigma},X) = \sum_{T\in\mathscr{T}}\int_T -\mathcal{H}_\nu(X) : \boldsymbol{\sigma} + \sum_{E\in\mathring{\mathscr{E}}}\int_E [\![(\nabla X)_{\nu\mu}]\!]_E\boldsymbol{\sigma}_{\mu\mu}.$$

Bilinear form $b(\Phi;\boldsymbol{\sigma},X)$ is closely related to the surface Hellan–Herrmann–Johnson method

Walker: The Kirchhoff plate equation on surfaces: the surface Hellan–Herrmann–Johnson method, *IMA J. Numer. Anal.* (2021)

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Perform all estimates on the reference domain: Transform bilinear forms back

Lemma (pull-back)

$$a(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat T\in\hat{\mathscr{T}}}\int_{\hat T} -J^{-1}\mathrm{tr}(\hat\nabla\hat X F^\dagger)\hat S : \hat{\boldsymbol{\sigma}} - \sum_{\hat E\in\mathring{\hat{\mathscr{E}}}}\int_{\hat E} J_{\mathrm{bnd}}^{-3}(\hat\nabla_{\hat\tau}\hat X)\cdot(F\hat\tau)\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\circ\Phi\hat{\boldsymbol{\sigma}}_{\hat\mu\hat\mu},$$

$$b(\Phi;\boldsymbol{\sigma},X) = \sum_{\hat T\in\hat{\mathscr{T}}}\int_{\hat T} -\sum_{i=1}^{3} J^{-1}(F_1\times F_2)\left(\hat\nabla^2\hat X_i - \sum_{\alpha=1}^{2}(\hat\nabla_\alpha \hat X_i)\Gamma^\alpha\right) : \hat{\boldsymbol{\sigma}}$$

$$+\sum_{\hat E\in\mathring{\hat{\mathscr{E}}}}\int_{\hat E}\left[\!\!\left[\sum_{i=1}^{3} J^{-1}(F_1\times F_2)\cdot\left(\hat\nabla_{\hat\mu}\hat X_i - J_{\mathrm{bnd}}^{2}(F^TF)_{\hat\tau\hat\mu}\hat\nabla_{\hat\tau}\hat X_i\right)\right]\!\!\right]_E \hat{\boldsymbol{\sigma}}_{\hat\mu\hat\mu}$$

Christoffel symbols of second kind: $\Gamma^\alpha_{\beta\gamma} = \sum_{\ell=1}^3 F^\dagger_{\alpha\ell}\hat\nabla_\beta F_{\ell\gamma}$

$\hat S = \sum_{i=1}^3 \nu_i\circ\Phi\hat\nabla^2\Phi_i$ pull-back of Weingarten tensor: $\nabla_{\mathcal{S}}\nu\circ\Phi = -F^{\dagger^T}\hat S F^\dagger$

$\boldsymbol{\sigma}\circ\Phi = J^{-2}F\hat{\boldsymbol{\sigma}}F^T, \quad \nu\circ\Phi = J^{-1}F_1\times F_2$

Note: $\hat X = \dot\Phi = \Phi_h - \bar\Phi_h \Rightarrow$ gives convergence

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\|\Phi_h - \bar{\Phi}_h\|_{W^{1,\infty}} \to 0$. Then there holds

$$\|\widetilde{\nabla\nu} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\big(1 + \max_{\hat{T}\in\hat{\mathcal{T}}} h_{\hat{T}}^{-1}\|\Phi_h - \bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}(\hat{T})}\big)\|\Phi_h - \bar{\Phi}_h\|_{H^1} \leqslant C\, h^k.$$

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\,dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\,dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
2. $\frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma}) = a(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t)) + b(\Phi;\boldsymbol{\sigma},\dot{\Phi}(t))$ sum of the bilinear forms a and b
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Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)$ always converges in H^{-1}!

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Error analysis (convergence results)

1. $\widetilde{\nabla\nu}(\boldsymbol{\sigma}) - \int_{\mathcal{S}} \nabla\nu : \boldsymbol{\sigma}\, dx = \int_0^1 \frac{d}{dt}\widetilde{\nabla\nu}(\boldsymbol{\sigma})\, dt$ with $\Phi(t) = \bar{\Phi}_h + t(\Phi_h - \bar{\Phi}_h)$
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3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ $\|\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)\|_{W^{1,\infty}} \leqslant h\|\bar{\Phi}_h\|_{W^{\min\{k,2\},\infty}}$

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Dihedral angle $\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R)$ always converges in H^{-1}!

Theorem (Gopalakrishnan, N.)

Let $(\Phi_h)_{h>0} \in \mathrm{Lag}_h^k$ be a family of embeddings such that $\Phi_h = \mathcal{I}_h^{\mathrm{Lag}^k}\bar{\Phi}_h$ for $k \geqslant 1$. Let $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ be the lifted Weingarten tensor. Then $\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$.

Gopalakrishnan, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Shells

Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

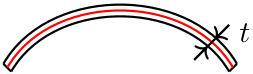
u ... displacement of mid-surface

t ... thickness

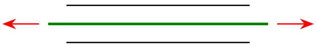
$\mathcal{M}$ ... material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$



$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$



u . . . displacement of mid-surface

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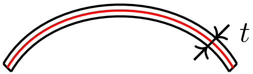

Koiter/Kirchhoff–Love shell

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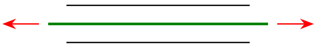

$u \ldots$ displacement of mid-surface

$t \ldots$ thickness

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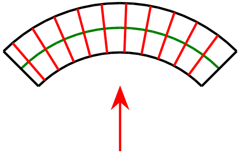

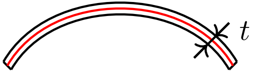

FEM + DDG for Koiter/Kirchhoff–Love shell

Lifted shape operator: $\int_{\mathcal{T}} \boldsymbol{\kappa} : \boldsymbol{\Psi}\, dx = \widetilde{\nabla}\nu(\boldsymbol{\Psi}) := \sum_{T\in\mathcal{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, dx + \sum_{E\in\mathring{\mathcal{E}}} \int_E \sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, ds$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) &= \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f,u\rangle\\ &+ \sum_{T\in\mathcal{T}}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx\\ &- \sum_{E\in\mathcal{E}}\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L,\nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

- Lagrange parameter $\boldsymbol{\sigma}\in\Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

$$\mathcal{W}(u) = t^{-2}E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f}\cdot u, \qquad f = t^3\tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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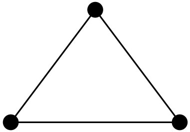
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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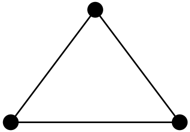

$$\mathrm{Lag}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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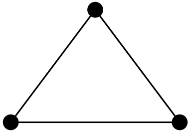

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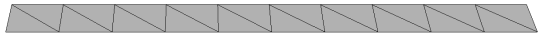
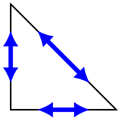


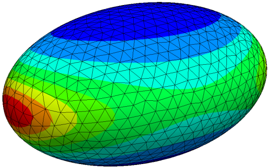


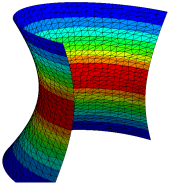
$$\begin{aligned}\mathcal{L}(u, \boldsymbol{\kappa}^{\text{diff}}, \boldsymbol{\sigma}) &= \frac{t}{2} \|\mathcal{I}_{\mathcal{R}}^{k-1} \boldsymbol{E}(u)\|_{\mathcal{M}}^2 + \frac{t^3}{12} \|\boldsymbol{\kappa}^{\text{diff}}\|_{\mathcal{M}}^2 - \langle f, u \rangle \\ &+ \sum_{T \in \mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\text{diff}} - (\boldsymbol{F}^T \nabla(\nu \circ \phi) - \nabla \hat{\nu})\right) : \boldsymbol{\sigma} \, dx \\ &+ \sum_{E \in \mathscr{E}} \int_E (\sphericalangle_{\text{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\text{sgn}}(\hat{\nu}_L, \hat{\nu}_R)) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, ds\end{aligned}$$

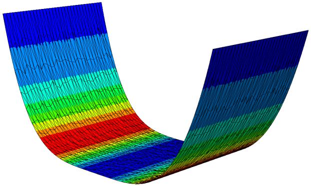
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

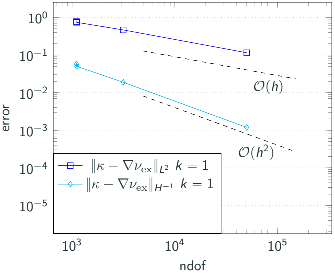
Numerics & Applications

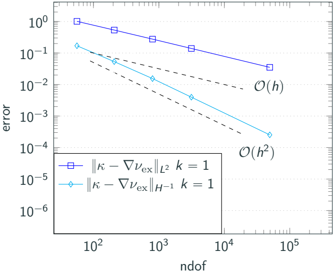
Curvature approximation

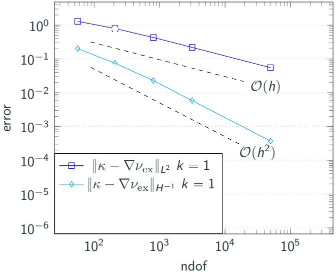






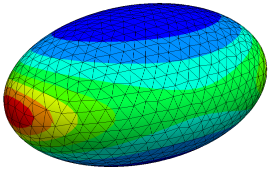


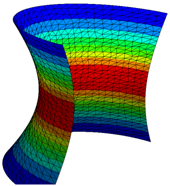


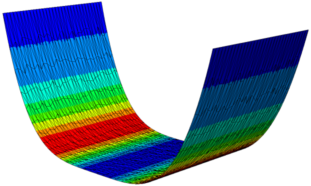


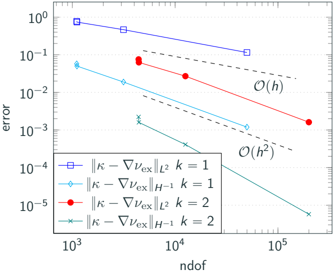
$$\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$$

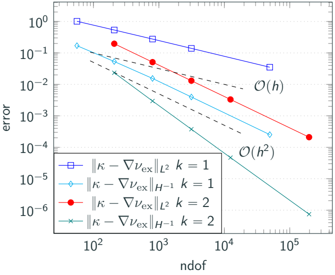
Curvature approximation

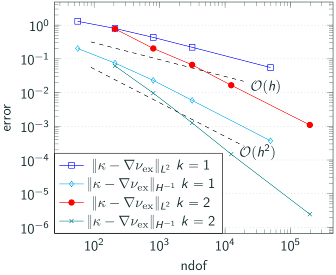






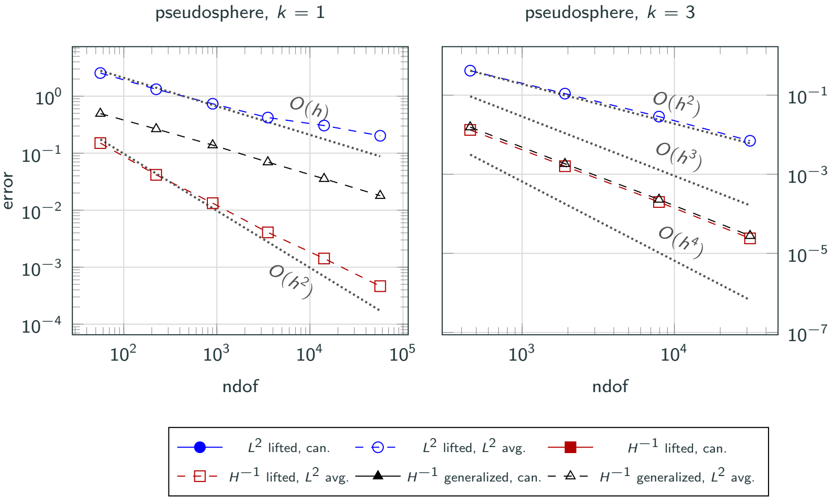


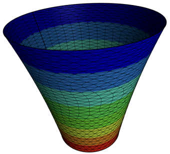




$$\|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$$

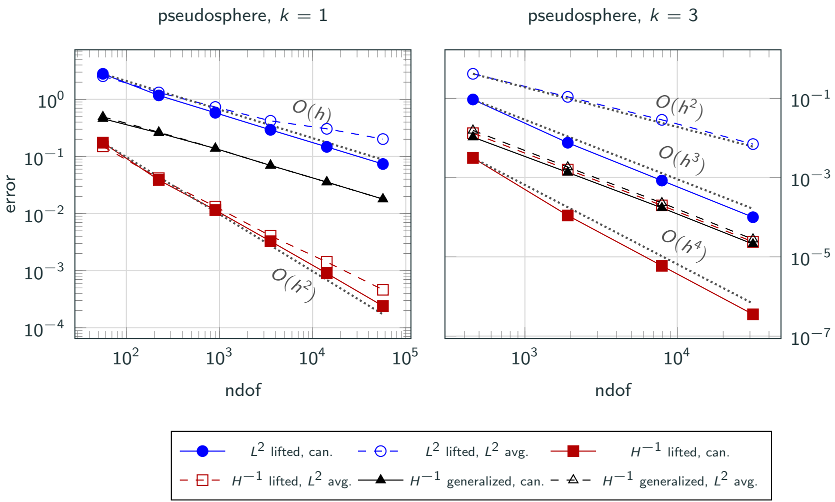
Curvature approximation

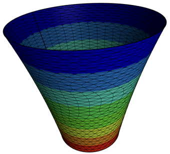




$$\|\widetilde{\nabla}\nu - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\,h^k, \qquad \|\boldsymbol{\kappa} - \nabla\bar{\nu}\|_{H^{-1}} \leqslant C\,h^k$$

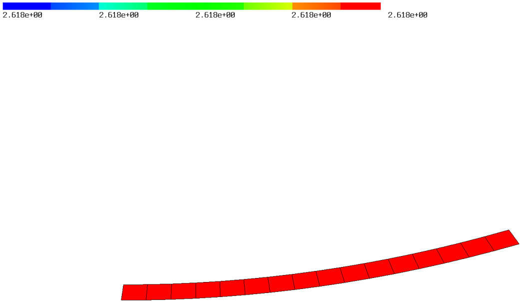
Curvature approximation

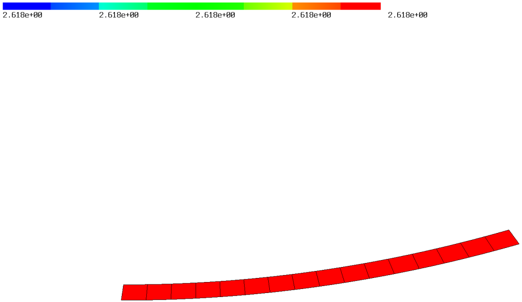




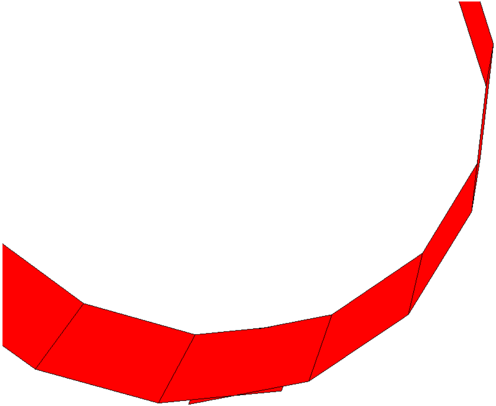
$$\|\widetilde{\nabla \nu} - \nabla \bar{\nu}\|_{H^{-1}} \leqslant C\, h^k, \qquad \|\boldsymbol{\kappa} - \nabla \bar{\nu}\|_{H^{-1}} \leqslant C\, h^{k+1}$$

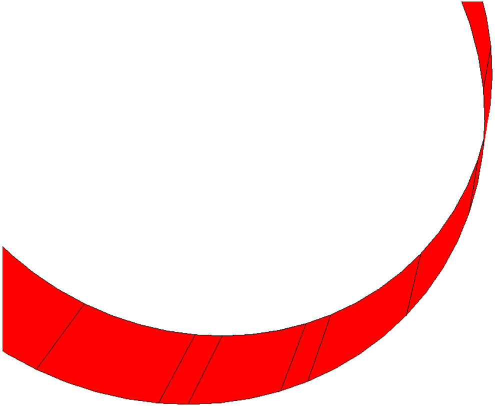
Example (cantilever bending)



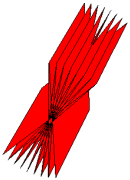


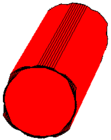
Example (cantilever bending)





Example (cantilever bending)



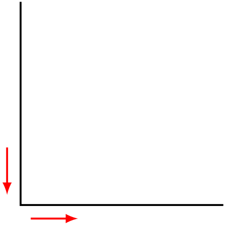


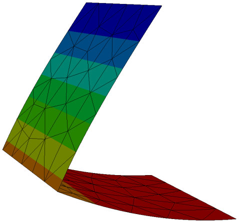
Structures with kinks and branched shells

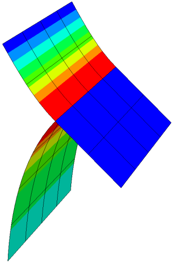
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_E (\sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0$$

$$\Rightarrow \quad \sphericalangle_{\mathrm{sgn}}(\nu_L, \nu_R) - \sphericalangle_{\mathrm{sgn}}(\hat{\nu}_L, \hat{\nu}_R) = 0$$







Application (bending-folding model)

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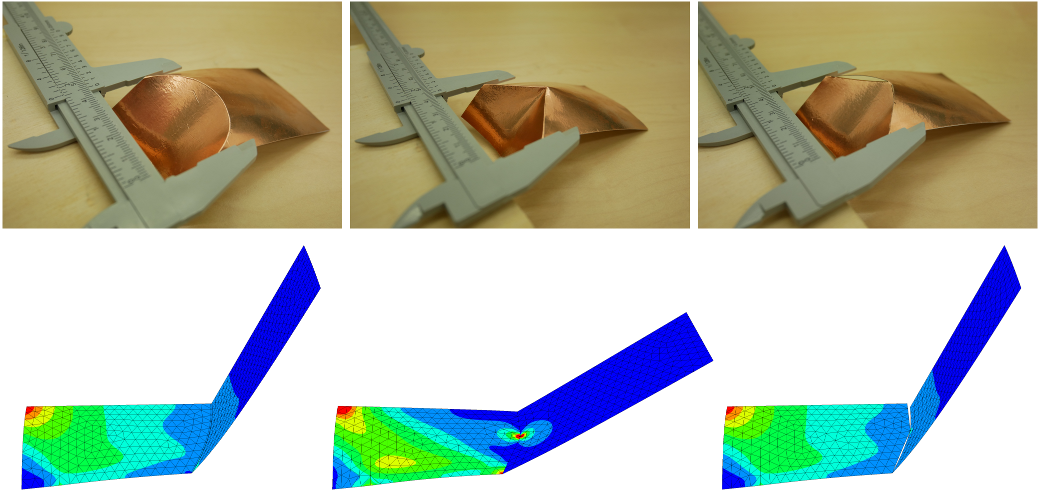


Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces Free Bound.* (2026).

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

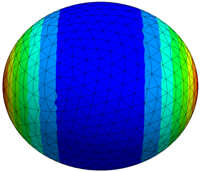
κ_b bending elastic constant

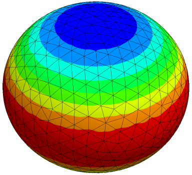
$H = 0.5\,\mathrm{tr}(\nabla\nu)$ mean curvature

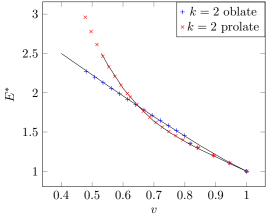
$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

Functional: $\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$







N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

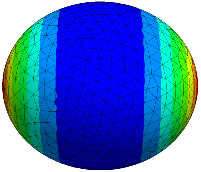
κ_b bending elastic constant

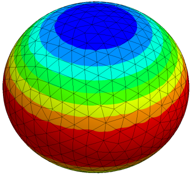
$H = 0.5\,\mathrm{tr}(\nabla\nu)$ mean curvature

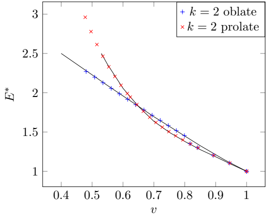
$2H_0$ spontaneous curvature

Constraints: $|\Omega| = V_0, \quad |\mathcal{S}| = A_0$

Functional: $\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$







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Summary & Outlook

TECHNISCHE UNIVERSITÄT WIEN

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- Bending energy for shell model
- Applications (origami, cell membranes)

- Extension to higher dimensional hypersurfaces
- Extension to submanifolds embedded in Riemannian manifolds

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Thank You for Your attention!

Literature

- GOPALAKRISHNAN, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- BARTELS, BONITO, HORNUNG, N., Babuška's paradox in a nonlinear bending-folding model, *Interfaces and Free Boundaries* (2026).
- N., SCHÖBERL: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)
- N., SCHÖBERL, STURM, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)
- GANGL, STURM, N., SCHÖBERL, Fully and Semi-Automated Shape Differentiation in NGSolve, *Multidiscip. Optim.* (2021)
- N., SCHÖBERL: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg (2021).*
- N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis (2021).*
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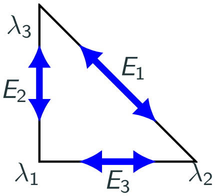
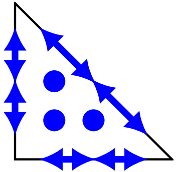
Regge elements & interpolation operator

$$H(\mathrm{curl\,curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl\,curl}\,\boldsymbol{\sigma} \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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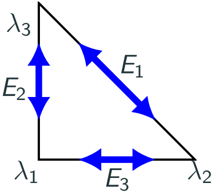

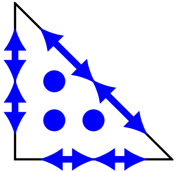

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$$\mathcal{I}_{\mathcal{R}}^k : C^0(\Omega) \to \mathrm{Reg}_h^k$$ canonical interpolant

$$\int_E (g - \mathcal{I}_{\mathcal{R}}^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{I}_{\mathcal{R}}^k g) : Q\, dx = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$