The TDNNS and Hellan-Herrmann-Johnson method for nonlinear shells

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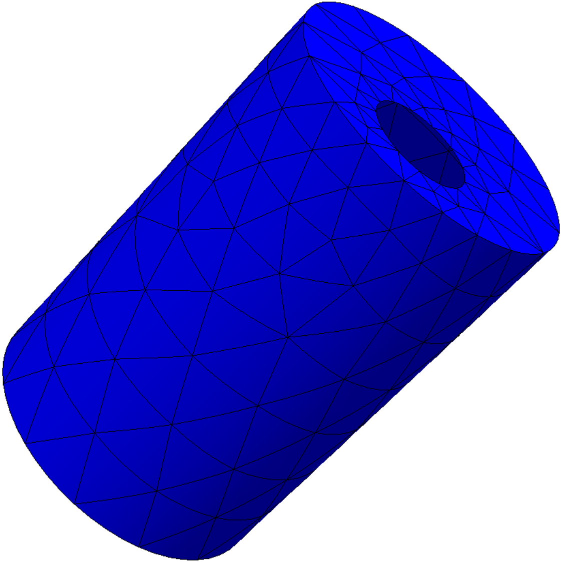
Der Wissenschaftsfonds.

WAND2022, Salzburg, July 13th, 2022

Motivation



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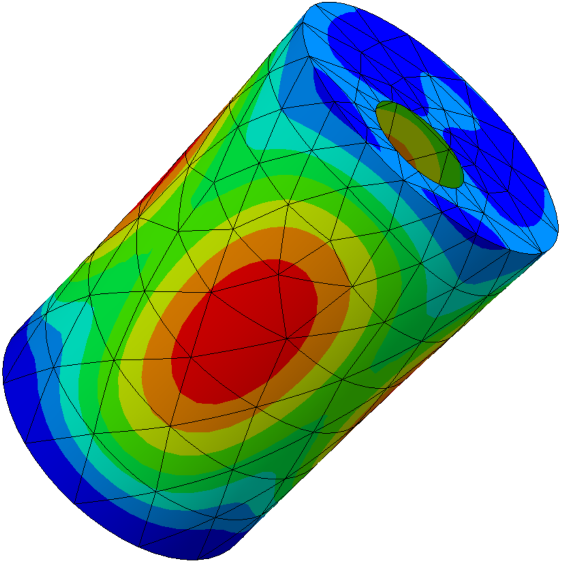

Motivation

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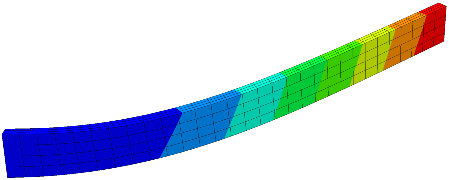
Continuum mechanics

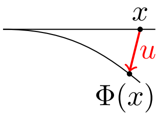
Deformation $\Phi : \Omega \to \mathbb{R}^3$

Displacement $u := \Phi - id$



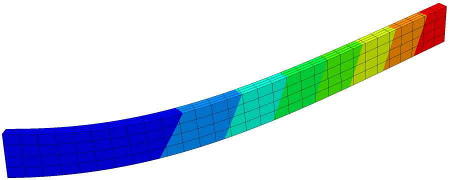
Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla\Phi$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$



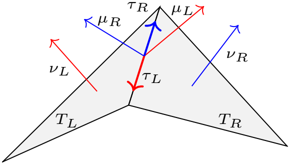
Elasticity

$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|_{\boldsymbol{M}}^2 - \langle f, u\rangle$$



Differential geometry

- Normal vector ν
  Tangent vector τ
  Element normal vector $\mu = \nu \times \tau$

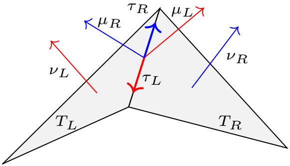


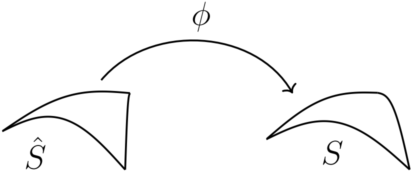
Differential geometry

- Normal vector $\hat{\nu}$

  Tangent vector $\hat{\tau}$

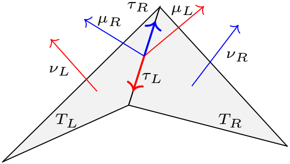
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

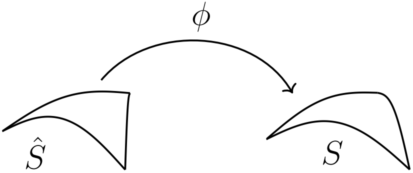




- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$





Differential geometry

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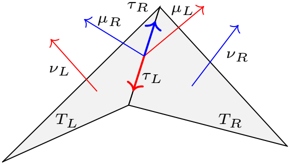
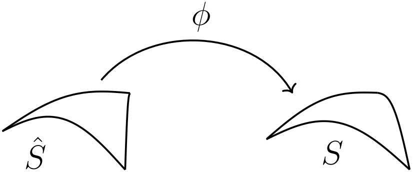


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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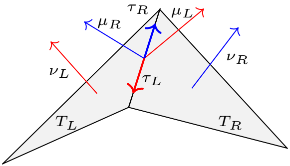
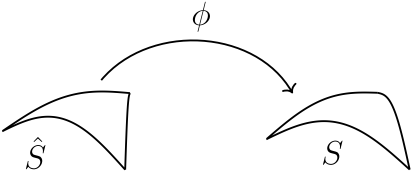
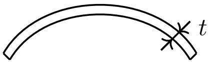


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$

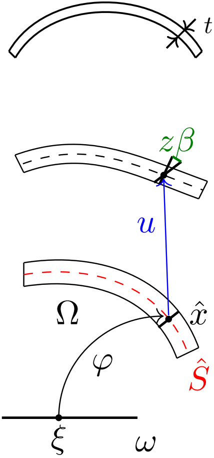
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$

  $$\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi = \frac{(\boldsymbol{F}^\dagger)^\top \hat{\mu}}{\|(\boldsymbol{F}^\dagger)^\top \hat{\mu}\|}$$

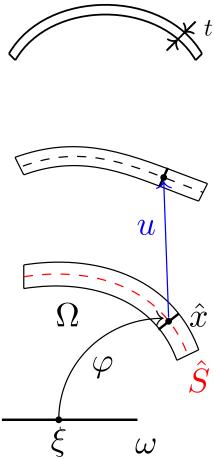




- Model of reduced dimensions



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\,(\nu + \beta) \circ \phi(\hat{x})$



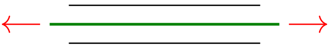
- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\,\nu \circ \phi(\hat{x})$

Shell energy

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

Shell energy

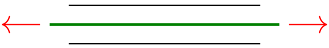
$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\boldsymbol{M}}^2$$

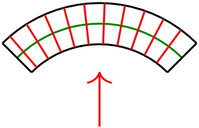


- Membrane energy

Shell energy

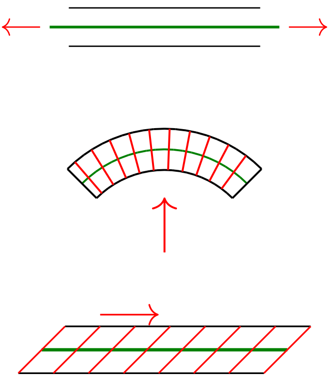
$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$





- Membrane energy
- Bending energy

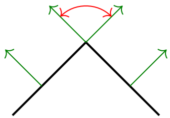
$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla\tilde{\nu}\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{tG\kappa}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2$$



- Membrane energy
- Bending energy
- Shearing energy

Koiter shell model

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$
$$+ \frac{t^3}{24}\sum_{E\in\mathcal{E}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},E}$$



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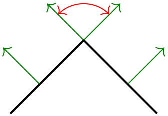

- Measure change of angles

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$
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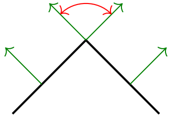

- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle$$
$$+ \sum_{E\in\mathcal{E}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_E$$

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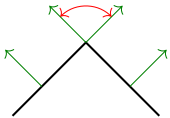

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- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\to$ second order problem

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Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{div}\,\mathrm{div}, \hat{S})$ for

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\, dx \\ &+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\end{aligned}$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

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Shell problem (Hybridization)

Find $u \in [H^1(\hat{S})]^3$, $\boldsymbol{\sigma} \in H(\text{div div}, \hat{S})^{dc}$ and $\alpha \in \Gamma(\hat{S})$ for

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ &+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\{\{\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L}\}\} + \alpha_{\hat{\mu}}[\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\,ds\end{aligned}$$

$$\{\{\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L}\}\} = \tfrac{1}{2}(\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} + \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R}), \qquad [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = \boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} - \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R}$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

The space H(divdiv)

$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

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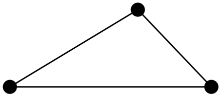
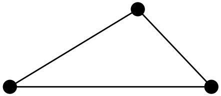
$$V_k := \Pi^k(\mathcal{T}_h) \cap C(\Omega)$$



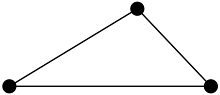
The space H(divdiv)

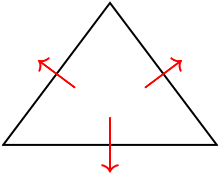
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}(\sigma) \in L^2(\Omega)\}$$



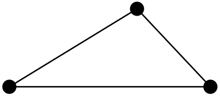
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}(\sigma) \in L^2(\Omega)\}$$

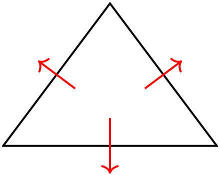
$$BDM_k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^d \,|\, \sigma_n \text{ is continuous over elements}\}$$





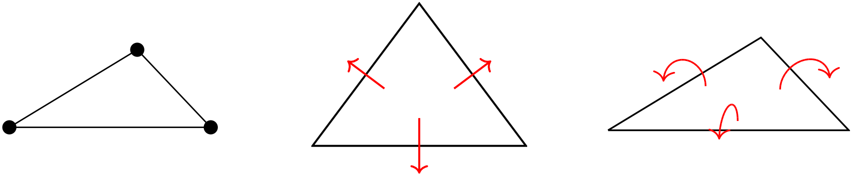
$$H(\text{div div}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{div}(\mathrm{div}(\sigma)) \in H^{-1}(\Omega)\}$$





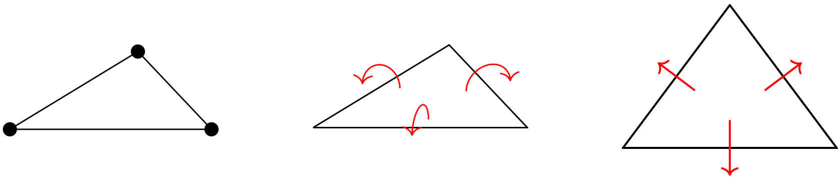
$$H(\text{div div}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \,|\, \mathrm{div}(\mathrm{div}(\sigma)) \in H^{-1}(\Omega)\}$$

$$M_h^k := \{\sigma \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \,|\, n^T\sigma n \text{ is continuous over elements}\}$$

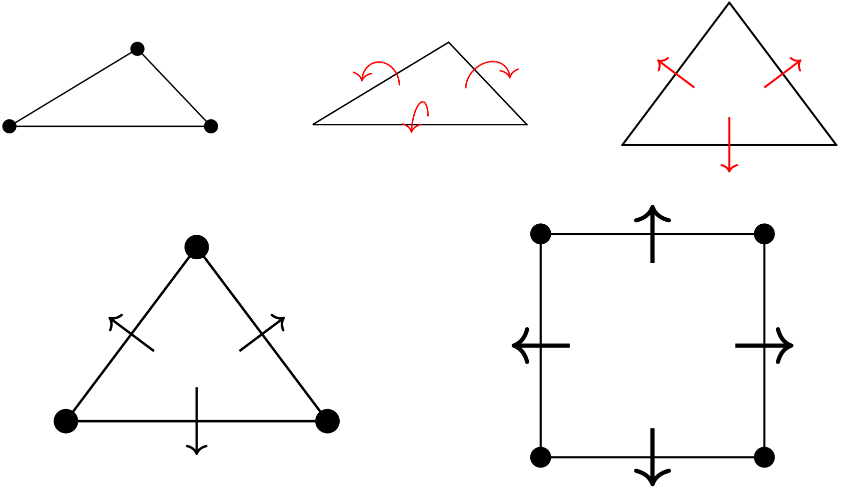


A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Shell element



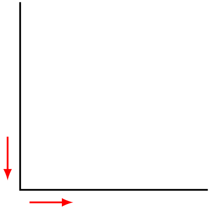
Shell element

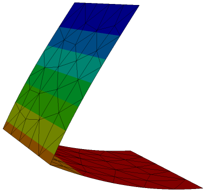


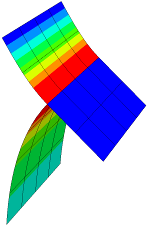
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0$$

$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$



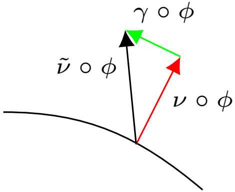




Naghdi shell model

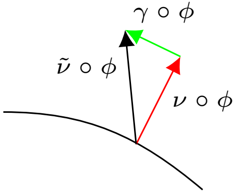
Extension to nonlinear Naghdi shells

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\text{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi = \frac{1}{J}\,\text{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg* (2013) 254, pp. 170–180.

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



$$\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}}$$
$$+ \sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right) : \boldsymbol{\sigma}\,dx$$
$$+ \sum_{E\in\mathcal{E}_h}\int_E \left(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R) + [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$

Linearization

Kirchhoff–Love shell and plate

$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{shell}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|_{\boldsymbol{M}}^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

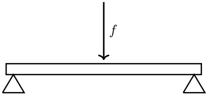
$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{plate}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

Kirchhoff–Love shell and plate

$$\mathcal{L}^{\text{shell}}_{\text{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\operatorname{sym}(\nabla^{\text{cov}}u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

$$\mathcal{L}^{\text{plate}}_{\text{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \sum_{T\in\mathcal{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\right)$$

$$\operatorname{div}(\operatorname{div}(\nabla^2 w)) = f \Leftrightarrow \begin{cases}\boldsymbol{\sigma} = \nabla^2 w,\\ \operatorname{div}(\operatorname{div}(\boldsymbol{\sigma})) = f,\end{cases}$$



M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Reissner–Mindlin shell and plate

$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{shell}}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}}u)\|_{\boldsymbol{M}}^2 + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\Big(\int_T(\boldsymbol{H}_{\hat{\nu}} - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\,dx - \int_{\partial T}((\nabla u^\top\hat{\nu})_{\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{plate}}(w,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathcal{T}_h}\Big(\int_T(\nabla^2 w - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\frac{\partial w}{\partial\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\Big)$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017) 137, pp. 713–740.

Membrane locking

Membrane locking

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$$\mathcal{W}(u) = t\, E_{\mathrm{mem}}(u) + t^3\, E_{\mathrm{bend}}(u) - f \cdot u$$

Membrane locking

$$\mathcal{W}(u) = \frac{1}{t^2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u$$

- Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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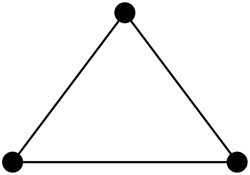

$V_h = \Pi(\mathcal{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

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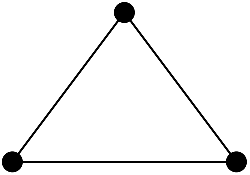
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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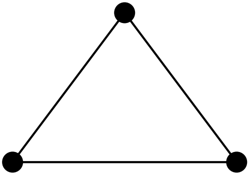
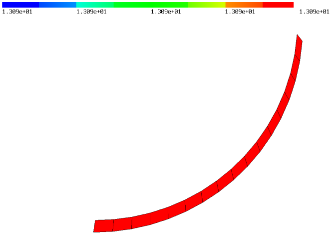
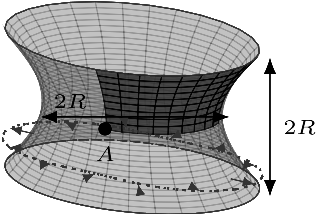
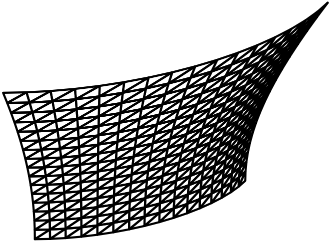


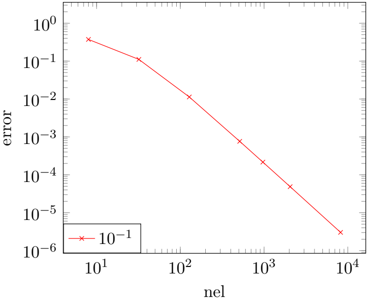


$$V_h = \Pi(\mathcal{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

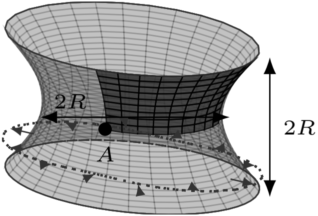
Hyperboloid with free ends

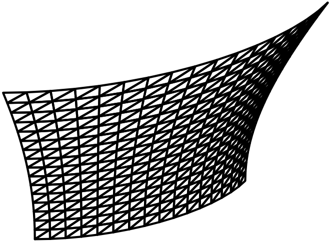


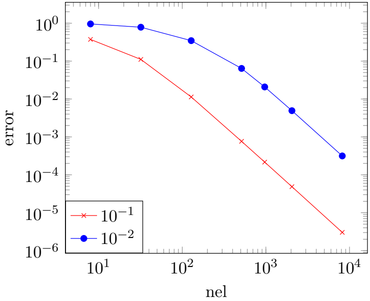




Hyperboloid with free ends

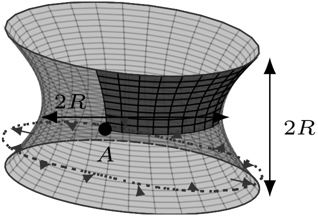


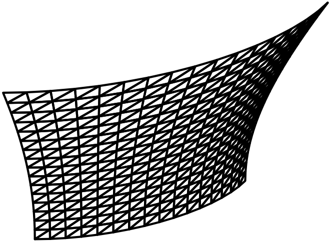


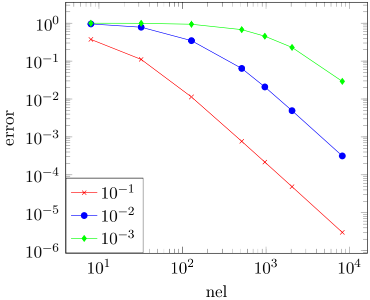


- Pre-asymptotic regime

Hyperboloid with free ends

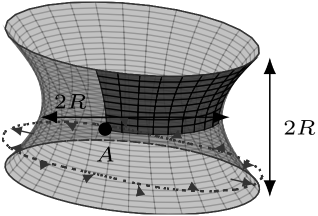


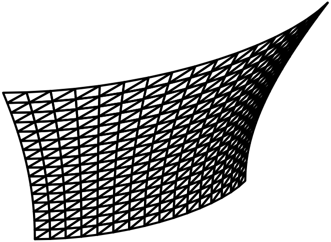


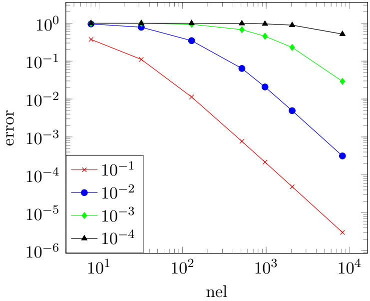


- Pre-asymptotic regime

Hyperboloid with free ends

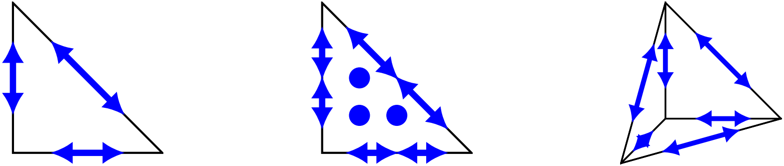






- Pre-asymptotic regime

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma} t \text{ is continuous over elements}\}$$
$$H(\text{curl curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \text{curl } (\text{curl } \boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{2d-3\times 2d-3}\}$$

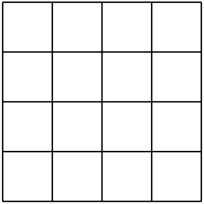


Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

$$\frac{1}{t^2}\| \quad \boldsymbol{E}_{\tau\tau}(u_h)\|^2_{\boldsymbol{M}}$$

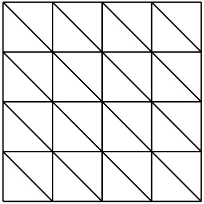
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}_{\tau\tau}(u_h)\|^2_{\boldsymbol{M}}$$



- Reduced integration for quadrilateral meshes

Membrane locking

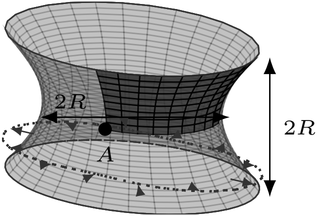
$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}(u_h)\|_{\boldsymbol{M}}^2$$

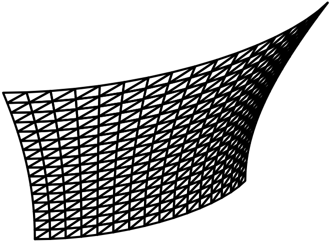


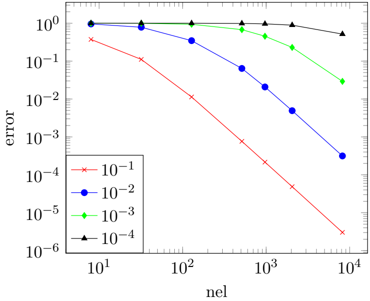
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

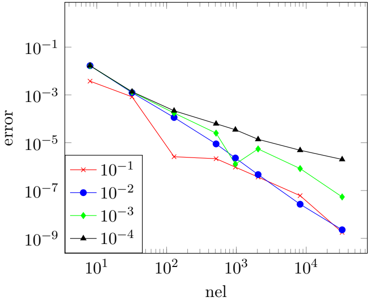
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends

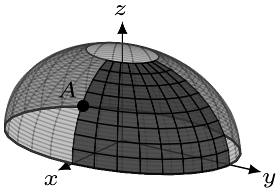


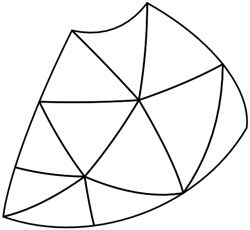


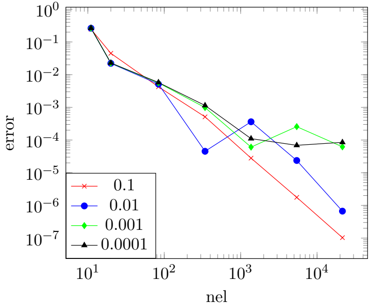


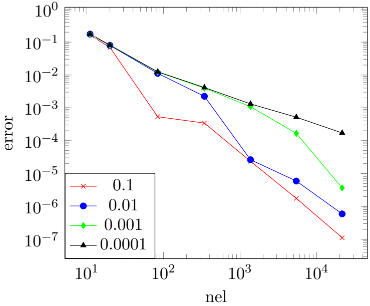


Open hemisphere with clamped ends

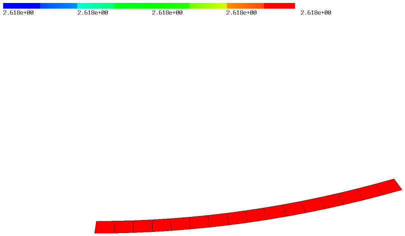


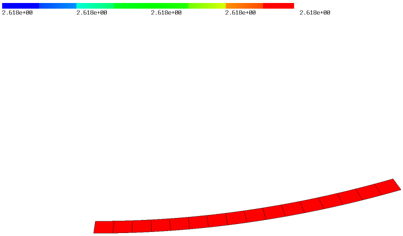






Cantilever subjected to end moment





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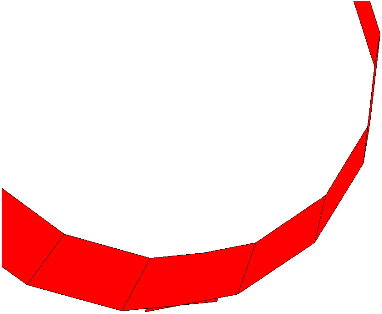

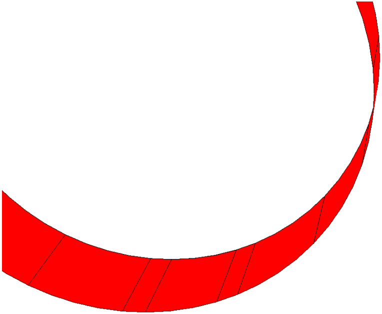

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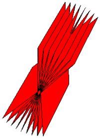

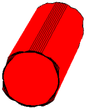

- Hellan–Herrmann–Johnson method for nonlinear Koiter shells
- TDNNS method for nonlinear Naghdi shells
- Regge interpolation avoids membrane locking

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

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N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

Summary and Outlook

- NGSolve Add-on
- Computing high-precision reference values
- Multiphysics

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Thank You for Your attention!

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