Distributional curvatures of Regge metrics

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Problem setup

Let (M,g) be a two- or three-dimensional Riemannian manifold M with metric tensor g. The Riemann curvature tensor $\mathcal{R}$, Gaussian curvature K, curvature operator Q, and second fundamental form $I\!I$ read in coordinates

$$\mathcal{R}_{ijkl} := \partial_i\Gamma_{jkl} - \partial_j\Gamma_{ikl} + \Gamma_{ik}^p\Gamma_{jpl} - \Gamma_{jk}^p\Gamma_{ipl}, \qquad K := \frac{1}{\det g}\mathcal{R}_{1212},$$

$$Q^{ij} := \frac{1}{4\det g}\varepsilon^{ikl}\varepsilon^{jmn}\mathcal{R}_{klmn}, \qquad I\!I_{ij} := \frac{1}{\sqrt{(g^{-1})_{\nu\nu}}}\Gamma_{ij}^k\nu_k,$$

where ν denotes the outer normal vector according to a triangulation $\mathcal{T}$ of M and Christoffel symbols of first and second kind are given by

$$\Gamma_{ijk} := \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \qquad \Gamma_{ij}^k := g^{kp}\Gamma_{ijp}.$$

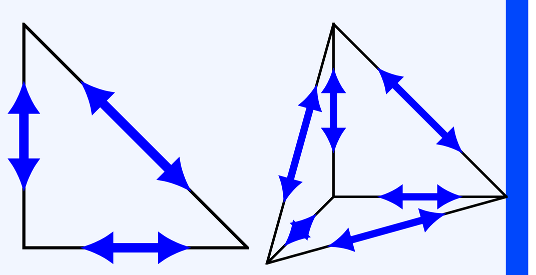
Let g_h be an approximation of g in the Regge FE space [1, 2]

$$\mathrm{Reg}^k := \{\sigma \in \mathcal{P}^k(\mathcal{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, [\![\sigma|_F]\!] = 0 \text{ for all facets } F \in \mathcal{F}\}.$$

The degrees of freedom consist of (tangential-tangential) moments and define the canonical Regge interpolant $\mathcal{I}^k_{\mathrm{Reg}} : C^0(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \to \mathrm{Reg}^k$

$$\int_E \mathcal{I}^k_{\mathrm{Reg}}(\sigma)_{\tau_E\tau_E}\, q\, ds = \int_E \sigma_{\tau_E\tau_E}\, q\, ds \quad q \in \mathcal{P}^k(E),$$

$$\int_T \mathcal{I}^k_{\mathrm{Reg}}(\sigma) : p\, dx = \int_T \sigma : p\, dx \quad p \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}}).$$



Goal: Compute curvature of g_h estimating the exact one.

Distributional curvature

Let $g \in \mathrm{Reg}^k$. The *lifted distributional Gaussian curvature* solves the problem for all $v \in \mathrm{Lag}^{k+1}$ ($\omega_T = \sqrt{\det g}$, $\omega_E = \sqrt{g_{\tau_E\tau_E}}$)

$$\int_\Omega \langle K, v\rangle\, \omega = \langle\!\langle K\omega, v\rangle\!\rangle =: \sum_{T\in\mathcal{T}}\int_T K(g)v\,\omega_T + \sum_{E\in\mathcal{E}}\int_E [\![\kappa_g]\!] v\,\omega_E + \sum_{V\in\mathcal{V}}\Theta_V(g)v,$$

with $\kappa_g := I\!I(\tau_E,\tau_E)$ the geodesic curvature, $\Theta_V(g)$ the angle defect at vertex V, and Lag^k the Lagrangian elements. The *lifted distributional curvature operator* $Q \in \mathrm{Reg}^k$ acts on $V \in \mathrm{Reg}^k$ ($\omega_F = \sqrt{\mathrm{cof}(g)_{\nu\nu}}$)

$$\int_\Omega \langle Q, V\rangle\, \omega = \langle\!\langle Q\omega, V\rangle\!\rangle =: \sum_{T\in\mathcal{T}}\int_T \langle Q, V\rangle\, \omega_T$$

$$+ \sum_{F\in\mathcal{F}}\int_F \langle [\![I\!I]\!], (\nu_g\otimes\nu_g)\times V\rangle\,\omega_F + \sum_{E\in\mathcal{E}}\int_E \Theta_E(g) V_{\tau_{E,g}\tau_{E,g}}\,\omega_E.$$

Analysis

Idea: Extend the formula for evolving metrics $g(t)$

$$\frac{d}{dt}(K\,\omega)|_{t=0} = -\frac{1}{2}\mathrm{inc}_{g(t)}(\sigma)\,\omega(g(t)), \qquad \sigma = g'(0)$$

to distributional setting, $\mathrm{inc}_{g(t)}(\sigma) := \mathrm{curl}_{g(t)}(\mathrm{curl}_{g(t)}(\sigma))$ denotes the covariant incompatibility operator. With $G(t) = g_h + t(g - g_h)$, $\sigma = G'(t)$ [3]

$$\langle\!\langle (K\omega)(g), v\rangle\!\rangle - \langle\!\langle (K\omega)(g_h), v\rangle\!\rangle = -\frac{1}{2}\int_0^1 \langle\!\langle \mathrm{inc}_{G(t)}(\sigma)\,\omega(G(t)), v\rangle\!\rangle\, dt,$$

$$\langle\!\langle \mathrm{inc}_g(\sigma)\,\omega, v\rangle\!\rangle := \sum_{T\in\mathcal{T}}\int_T \mathrm{inc}_g(\sigma) v\,\omega_T - \sum_{E\in\mathcal{E}}\int_E [\![(\mathrm{curl}_g(\sigma) + d(\sigma_{\nu_g\tau_g}))_{\tau_g}]\!]\, v\,\omega_E$$

$$- \sum_{V\in\mathcal{V}}\sum_{T\supset V}[\![\sigma_{\nu_g\tau_g}]\!]_V^T v(V), \qquad d\ldots\text{exterior derivative.}$$

Theorem (convergence)

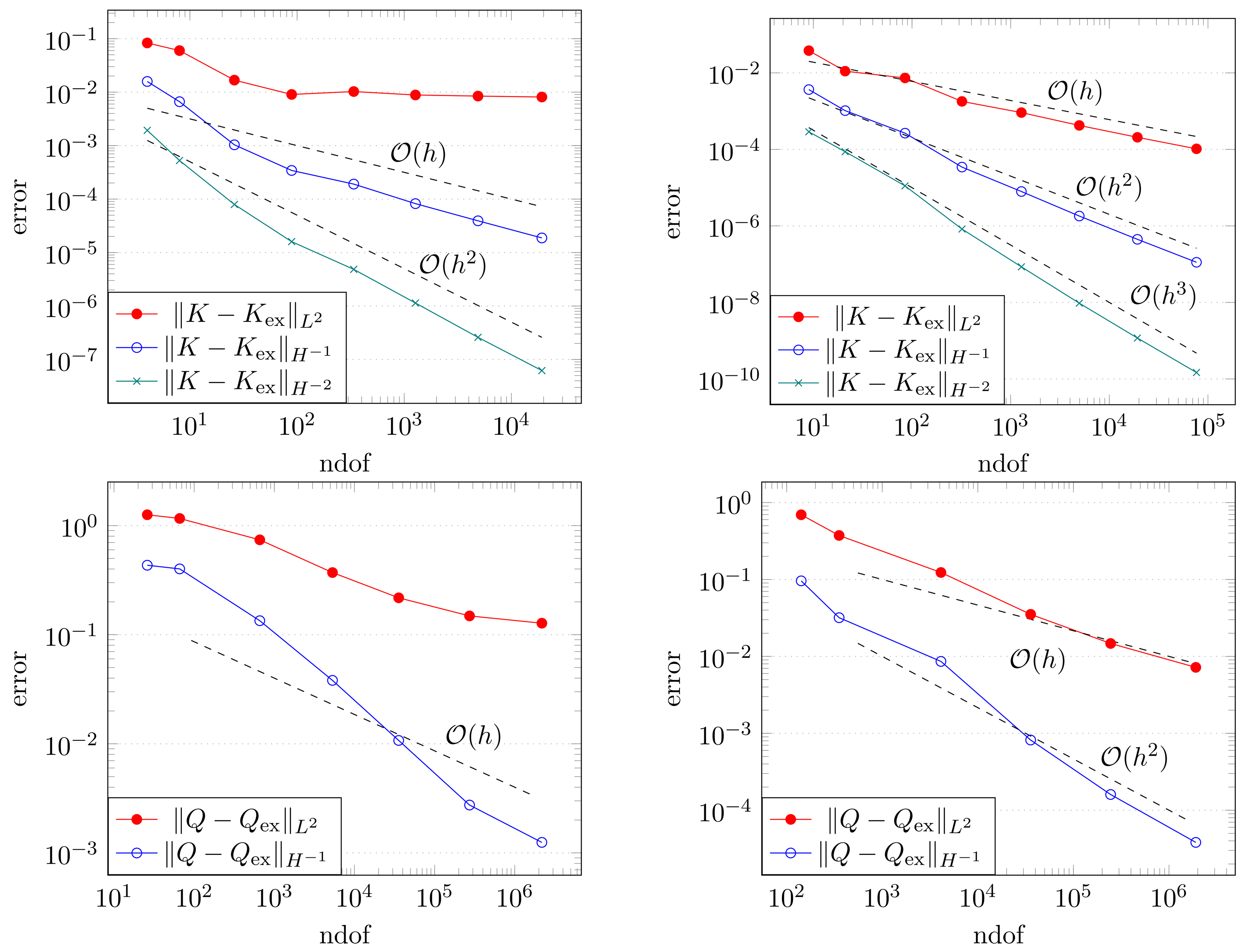
Let $g \in W^{k+1,\infty}(\Omega)$, $g_h = \mathcal{I}^k_{\mathrm{Reg}}g$, and $K_h(g_h) \in \mathcal{V}_h^{k+1}$. There exists $C = C(\Omega, \mathcal{T}, \|g\|_{W^{1,\infty}}, \|g^{-1}\|_{L^\infty})$, $h_0 > 0$ such that for all $h < h_0$, $0 \le l \le k$,

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \le C\, h^{k+1}(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}),$$

$$|K_h(g_h) - K(g)|_{H_h^l} \le C\, h^{k-l}\left(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}\right).$$

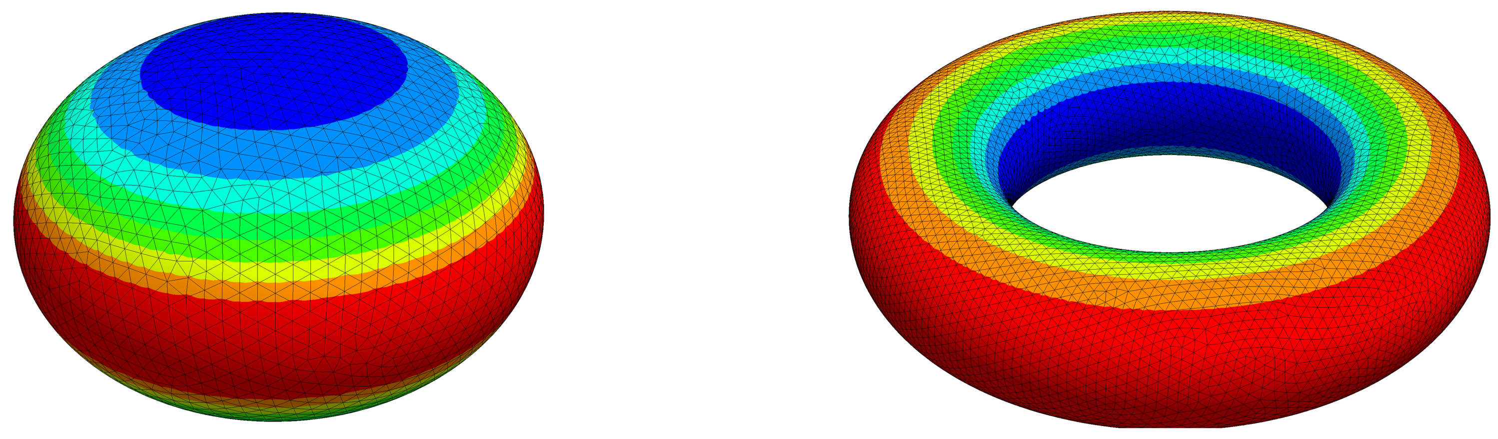
Numerical results

Numerical experiments using NGSolve (www.ngsolve.org) confirm that the analysis is sharp.
top: Gauss curvature K, bottom: curvature operator Q,
left: $g_h \in \mathrm{Reg}^0$, right: $g_h \in \mathrm{Reg}^1$



Curvature approximation of surfaces

With small adaptions (high-order) approximation of Gauss curvature of discrete surfaces is possible.



References

[1] Snorre Christiansen, *On the linearization of Regge calculus*, Numerische Mathematik 119, 4 (2011).

[2] Lizao Li, *Regge Finite Elements with Applications in Solid Mechanics and Relativity*, PhD thesis, University of Minnesota (2018).

[3] Jay Gopalakrishnan, Michael Neunteufel, Joachim Schöberl, Max Wardetzky, *Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics*, arXiv:2206.09343.

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Taming Complexity in Partial Differential Systems