Robust mixed methods for continuum mechanics, plates and shells

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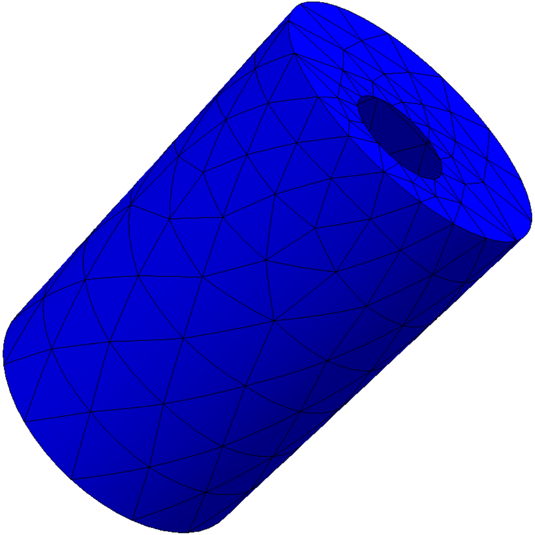
Joachim Schöberl (TU Wien)

Workshop: Geometric Mechanics and Structure Preserving Discretizations of Shell Elasticity
Utica NY, Jul 24th, 2023

Motivation



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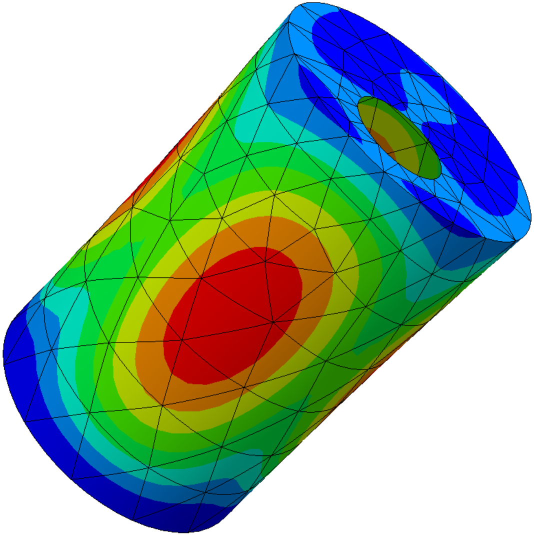

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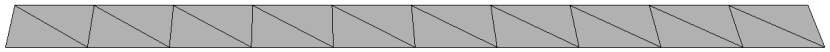

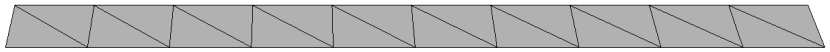

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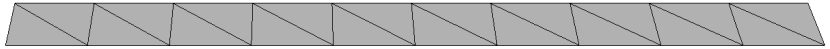
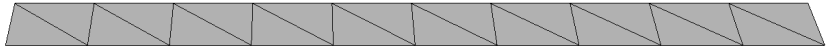




Contents

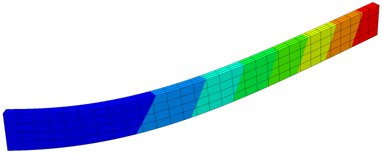
Continuum mechanics

Force balance equation: $-\mathbf{div}(\boldsymbol{\sigma}) = f$

Linear elasticity

$$\int_\Omega \mathbb{C}\boldsymbol{\epsilon}(u) : \boldsymbol{\epsilon}(\delta u)\, dx = \int_\Omega f \cdot \delta u \, dx$$

- u displacement
- $\boldsymbol{\epsilon}(u) = 0.5(\nabla u + \nabla u^T)$
- $\mathbb{C}\boldsymbol{\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I}$
- $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$ stress



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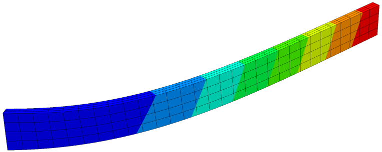

Locking problems:

- $\lambda \to \infty$ (nearly) incompressible material

Linear elasticity

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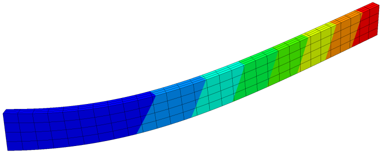

Locking problems:

- $\lambda \to \infty$ (nearly) incompressible material
- Coercivity: $\|\boldsymbol{\epsilon}(u)\|_{L^2}^2 \geq c_K \|\nabla u\|_{L^2}^2, \quad c_K \to 0$ for deteriorating aspect ratio

Nearly incompressible materials

$$\int_\Omega \mathbb{C}\epsilon(u) : \epsilon(\delta u)\, dx = \int_\Omega 2\mu\, \epsilon(u) : \epsilon(\delta u) + \underbrace{\lambda\, \mathrm{div}(u)}_{=:p} \mathrm{div}(\delta u)\, dx$$

- Define pressure $p = \lambda\, \mathrm{div}(u)$ and rewrite as mixed (saddle point) problem
- Well-defined for $\lambda \to \infty$ (need for stable pairing of finite elements)

Find $(u, p) \in U_h \times Q_h$ s.t. for all $(\delta u, \delta p) \in U_h \times Q_h$

$$\begin{aligned} \int_\Omega 2\mu\, \epsilon(u) : \epsilon(\delta u) \quad &+ p\, \mathrm{div}(\delta u)\, dx &&= \int_\Omega f \cdot \delta u\, dx \\ \int_\Omega \mathrm{div}(u)\, \delta p \quad &- \tfrac{1}{\lambda}\, p\, \delta p\, dx &&= 0 \end{aligned}$$

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Quadrilateral elements: $U_h = [\mathcal{L}_h^k(\mathscr{T}_h)]^3$, $Q_h = \mathcal{P}^{k-1}(\mathscr{T}_h)$ (Stokes-stable pairing)

$$\int_\Omega \mathbb{C}\epsilon(u) : \epsilon(\delta u)\,dx = \int_\Omega 2\mu\,\epsilon(u) : \epsilon(\delta u) + \lambda\,\Pi_{L^2}^{k-1}\mathrm{div}(u)\Pi_{L^2}^{k-1}\mathrm{div}(\delta u)\,dx$$

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Mixed method linear elasticity

- $\mathbb{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\epsilon}(u)$

$$\int_\Omega \mathbb{C}\boldsymbol{\epsilon}(u) : \boldsymbol{\epsilon}(\delta u)\, dx = \int_\Omega f \cdot \delta u \, dx \qquad \forall \delta u$$

Mixed method linear elasticity

- $\mathbb{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\epsilon}(u)$

Find $\boldsymbol{\sigma} \in [L^2(\Omega)]^{3\times 3}_{\text{sym}}$ and $u \in [H^1(\Omega)]^3$ s.t.

$$\begin{aligned}
\int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx - \int_\Omega \delta\boldsymbol{\sigma} : \boldsymbol{\epsilon}(u)\, dx &= 0 && \forall \delta\boldsymbol{\sigma} \\
- \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\delta u)\, dx &= -\int_\Omega f \cdot \delta u\, dx && \forall \delta u
\end{aligned}$$

$\langle \mathbf{div}\boldsymbol{\sigma}, u\rangle_{H^{-1}\times H^1}$

- u continuous, $\boldsymbol{\sigma}$ discontinuous

Mixed method linear elasticity

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- $\mathbb{C}^{-1}\boldsymbol{\sigma} = \epsilon(u)$

Find $\boldsymbol{\sigma} \in H(\mathrm{div}, \Omega)^{\mathrm{sym}}$ and $u \in [L^2(\Omega)]^3$ s.t.

$$
\begin{aligned}
\int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx + \int_\Omega \mathbf{div}\delta\boldsymbol{\sigma} \cdot u\, dx &= 0 && \forall \delta\boldsymbol{\sigma} \\
\int_\Omega \mathbf{div}\boldsymbol{\sigma} \cdot \delta u\, dx &= -\int_\Omega f \cdot \delta u\, dx && \forall \delta u
\end{aligned}
$$

$\langle \mathbf{div}\boldsymbol{\sigma}, u\rangle_{H^{-1}\times H^1}$

- u continuous, $\boldsymbol{\sigma}$ discontinuous

$(\mathbf{div}\boldsymbol{\sigma}, u)_{L^2}$

- u discontinuous, $\boldsymbol{\sigma}$ normal continuous, $\boldsymbol{\sigma} n$

Mixed method linear elasticity

- $\mathbb{C}^{-1}\boldsymbol{\sigma} = \epsilon(u)$

Find $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ and $u \in H(\mathrm{curl}, \Omega)$ s.t.

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\int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx + \langle \mathbf{div}\delta\boldsymbol{\sigma}, u\rangle &= 0 && \forall \delta\boldsymbol{\sigma} \\
\langle \mathbf{div}\boldsymbol{\sigma}, \delta u\rangle &= -\int_\Omega f \cdot \delta u\, dx && \forall \delta u
\end{aligned}$$

$\langle \mathbf{div}\boldsymbol{\sigma}, u\rangle_{H^{-1}\times H^1}$	• u continuous,	$\boldsymbol{\sigma}$ discontinuous
$(\mathbf{div}\boldsymbol{\sigma}, u)_{L^2}$	• u discontinuous,	$\boldsymbol{\sigma}$ normal continuous, $\boldsymbol{\sigma} n$
$\langle \mathbf{div}\boldsymbol{\sigma}, u\rangle_{H(\mathrm{curl})^*\times H(\mathrm{curl})}$	• u tangential continuous, $u \cdot \tau$,	$\boldsymbol{\sigma}$ normal-normal continuous, $n^T\boldsymbol{\sigma} n$

(Finite element) Spaces

$$H^1(\Omega) = \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

$$\mathcal{L}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega)$$

$$H(\mathrm{curl}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{curl}\sigma \in [L^2(\Omega)]^{2d-3}\}$$

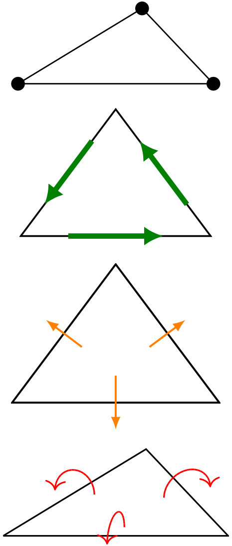
$$\mathcal{N}_{II}^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, [\![\sigma_\tau]\!]_F = 0\}$$

$$H(\mathrm{div}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}\sigma \in L^2(\Omega)\}$$

$$BDM^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, [\![\sigma_n]\!]_F = 0\}$$

$$H(\mathrm{divdiv}, \Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \,|\, \mathrm{div}\mathbf{div}\sigma \in H^{-1}(\Omega)\}$$

$$M_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \,|\, [\![n^T \sigma n]\!]_F = 0\}$$



TDNNS method for linear elasticity

Find $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ and $u \in H(\mathrm{curl}, \Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx + \langle \mathbf{div}\delta\boldsymbol{\sigma}, u\rangle &= 0 && \forall \delta\boldsymbol{\sigma} \\ \langle \mathbf{div}\boldsymbol{\sigma}, u\rangle &= -\int_\Omega f \cdot \delta u\, dx && \forall \delta u \end{aligned}$$

$$\begin{aligned} \langle \mathbf{div}\boldsymbol{\sigma}, u\rangle &:= \sum_{T\in\mathscr{T}_h} \int_T \mathbf{div}\boldsymbol{\sigma} \cdot u\, dx - \sum_{E\in\mathcal{E}_h} \int_E [\![\boldsymbol{\sigma}_{nt}]\!] u_t\, ds \\ &= \sum_{T\in\mathscr{T}_h} -\int_T \boldsymbol{\sigma} : \nabla u\, dx + \sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\sigma}_{nn} [\![u_n]\!]\, ds = -\langle \boldsymbol{\sigma}, \nabla u\rangle \end{aligned}$$

- Robust for $\lambda \to \infty$ (with stabilization)

A. Pechstein, J. Schöberl: Tangential-displacement and normal-normal-stress continuous mixed finite elements for elasticity *(2011)*.

Formulation (TDNNS)

TDNNS method for linear elasticity

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- Robust for $\lambda \to \infty$ (with stabilization)
- Robust in aspect ratio

A. Pechstein, J. Schöberl: Anisotropic mixed finite elements for elasticity *(2012)*.

Formulation (TDNNS)

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- Works for linear material law $\mathbb{C}$
- Problem for nonlinear (not invertible) material law

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TDNNS method for linear elasticity

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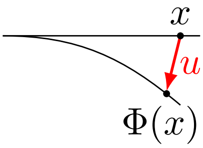
- Works for linear material law $\mathbb{C}$
- Problem for nonlinear (not invertible) material law
- Hu–Washizu three-field principle

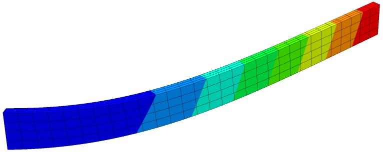
Displacement	$u = \Phi - \mathrm{id}$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$
Energy density	$\mathcal{W}(\boldsymbol{C}) : \mathbb{R}^{3\times 3} \to \mathbb{R}$
Stress tensor	$\boldsymbol{\Sigma} := 2\frac{\partial \mathcal{W}}{\partial C}$



$$\int_\Omega \mathcal{W}(\boldsymbol{C}(u)) - f \cdot u \, dx \to \min!$$

Nonlinear elasticity

$$\int_\Omega 2\frac{\partial \mathcal{W}(\boldsymbol{C})}{\partial C} : \boldsymbol{F}^\top \nabla \delta u \, dx = \int_\Omega f \cdot \delta u \qquad \forall \delta u$$



Continuum mechanics (nonlinear)

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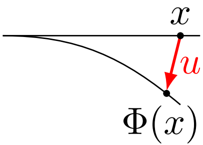

$$\int_\Omega \mathcal{W}(\boldsymbol{C}(u)) - f \cdot u \, dx \to \min!$$

$$-\mathbf{div}(\boldsymbol{F}\boldsymbol{\Sigma}) = f \qquad \text{in } \Omega \qquad + \text{bc}$$

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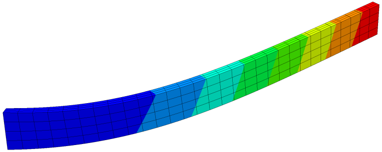

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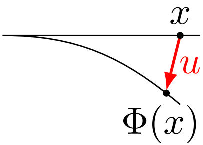

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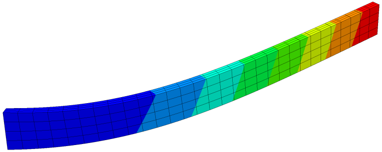

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Hu–Washizu principle

$$\min_{u \in V_h} \int_\Omega \mathcal{W}(\boldsymbol{F}(u)) - f \cdot u \, dx$$

N., Pechstein, Schöberl: Three-field mixed finite element methods for nonlinear elasticity
Comput. Methods Appl. Mech. Engrg 382 (2021)

Hu–Washizu principle

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Hu–Washizu principle

$$\min_{\substack{u\in V_h\\ F=F(u)}} \int_\Omega \mathcal{W}(\boldsymbol{F}) - f\cdot u\,dx$$

$$\mathcal{L}(u,\boldsymbol{F},\boldsymbol{P}) = \int_\Omega \mathcal{W}(\boldsymbol{F}) - f\cdot u\,dx - \langle \boldsymbol{F} - (\nabla u + \boldsymbol{I}), \boldsymbol{P}\rangle$$

- Lifting distribution ∇u to $\boldsymbol{F} \in [L^2]^{3\times 3}$
- 1st Piola–Kirchhoff stress tensor $\boldsymbol{P}\,(=\boldsymbol{F}\boldsymbol{\Sigma})$ as Lagrange multiplier
- $\boldsymbol{P} = \boldsymbol{P}_{\text{sym}} + \boldsymbol{P}_{\text{skew}}, \quad \boldsymbol{P}_{\text{sym}} \in H(\mathrm{divdiv}),\ \boldsymbol{P}_{\text{skew}} \in [L^2]^{3\times 3}_{\text{skew}}$
- $\boldsymbol{F} = \boldsymbol{F}_{\text{sym}} + \boldsymbol{F}_{\text{skew}}, \quad \boldsymbol{F}_{\text{sym}} \in [L^2]^{3\times 3}_{\text{sym}},\ \boldsymbol{F}_{\text{skew}} \in [L^2]^{3\times 3}_{\text{skew}}$

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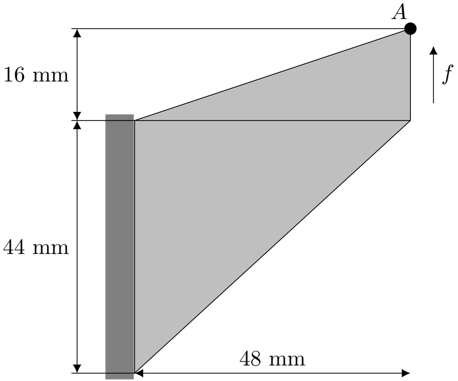
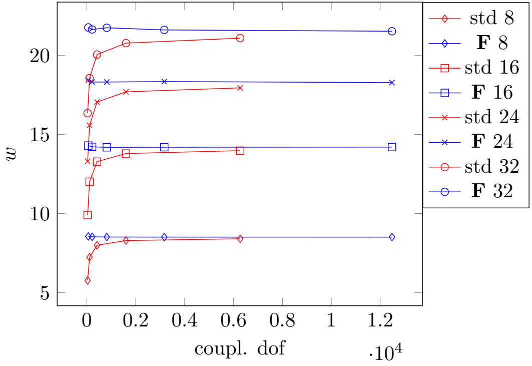
Numerical example: Cook's Membrane





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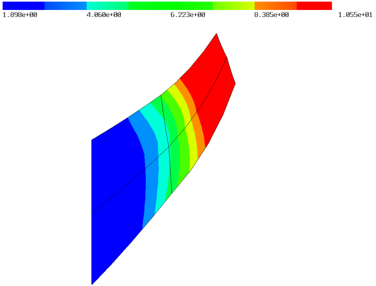

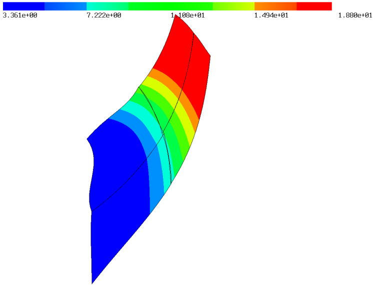

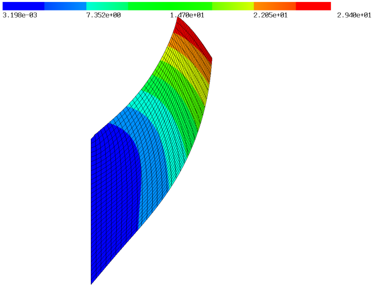

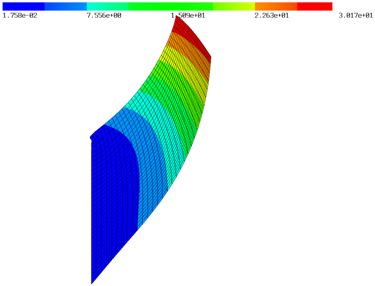

Plates

Kirchhoff–Love plate (Hellan–Herrmann–Johnson)

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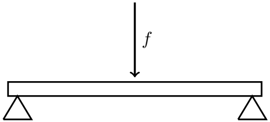
Bi-harmonic plate problem would require C^1-continuous finite elements

$$\mathrm{div}\,\mathbf{div}(\mathbb{C}\nabla^2 w) = f \qquad \Rightarrow w \in H^2(\Omega)$$



Rewrite as mixed method

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbb{C}\nabla^2 w, && \Rightarrow w \in H^1(\Omega) \\ \mathbf{div}\,\mathbf{div}\boldsymbol{\sigma} &= f, && \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

Hellan–Herrmann–Johnson method

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

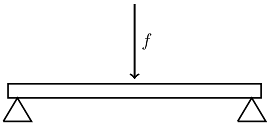
$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2_{\mathbb{C}^{-1}} - \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \mathbf{div}\boldsymbol{\sigma}\, dx + \int_{\partial T} (\nabla w)_\tau \boldsymbol{\sigma}_{\mu\tau}\, ds - \langle f, w \rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Kirchhoff–Love plate (Hellan–Herrmann–Johnson)

Bi-harmonic plate problem would require C^1-continuous finite elements

$$\mathrm{div}\mathbf{div}(\mathbb{C}\nabla^2 w) = f \qquad \Rightarrow w \in H^2(\Omega)$$



Rewrite as mixed method

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbb{C}\nabla^2 w, && \Rightarrow w \in H^1(\Omega) \\ \mathbf{divdiv}\boldsymbol{\sigma} &= f, && \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

Hellan–Herrmann–Johnson method

Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ s.t. for all $(\delta w, \delta\boldsymbol{\sigma}) \in H^1(\Omega) \times H(\mathrm{divdiv}, \Omega)$

$$\mathcal{L}(w, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2_{\mathbb{C}^{-1}} - \langle \mathbf{div}\sigma, \nabla w\rangle - \langle f, w\rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

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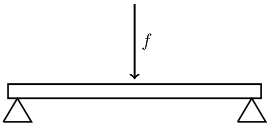

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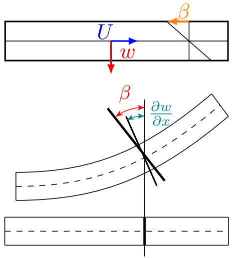
Find $w \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\operatorname{divdiv}, \Omega)$ s.t. for all $(\delta w, \delta\boldsymbol{\sigma}) \in H^1(\Omega) \times H(\operatorname{divdiv}, \Omega)$

$$\begin{aligned} \int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx \quad -\langle \mathbf{div}(\delta\boldsymbol{\sigma}), \nabla w\rangle &= 0 \\ -\langle \mathbf{div}\boldsymbol{\sigma}, \nabla\delta w\rangle &= \int_\Omega f\, \delta w\, dx \end{aligned}$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

Reissner–Mindlin plate (TDNNS)

$$\mathcal{W}(w,\beta) = \frac{1}{2}\int_\Omega \|\epsilon(\beta)\|^2_{\mathbb{C}} + \frac{1}{t^2}\|\nabla w - \beta\|^2\,dx - \int_\Omega f\,w\,dx$$



- thickness t, vertical deflection w, rotation β
- limit $t \to 0 \quad \Rightarrow \quad \beta = \nabla w \quad \Rightarrow \quad$ Kirchhoff–Love plate
- $w \in \mathcal{L}^k_h(\mathscr{T}_h)$ and $\beta \in [\mathcal{L}^k_h(\mathscr{T}_h)]^2$ leads to locking as

$$\nabla w \in H(\mathrm{curl}, \Omega)$$

Reissner–Mindlin plate (TDNNS)

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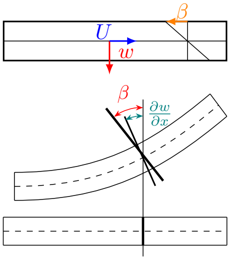

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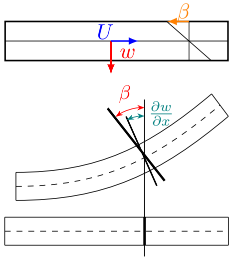

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$$\nabla w \in H(\mathrm{curl},\Omega)$$

- MITC elements: Use interpolant into Nedéléc elements $\mathcal{I}^k_{\mathcal{N}}$
- Use $\beta \in H(\mathrm{curl},\Omega)$ and TDNNS for $\nabla\beta \notin L^2$

TDNNS method for Reissner–Mindlin plate

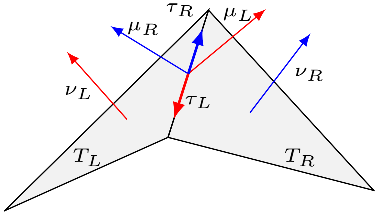
Find $(w,\beta,\boldsymbol{\sigma}) \in \mathcal{L}^k_h(\mathscr{T}_h) \times \mathcal{N}^{k-1}_h \times M^{k-1}_h(\mathscr{T}_h)$ for the Lagrangian

$$\mathcal{L}(w,\beta,\boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2_{\mathbb{C}^{-1}}\, dx - \langle \mathbf{div}\boldsymbol{\sigma}, \beta\rangle + \int_\Omega \frac{1}{t^2}\|\nabla w - \beta\|^2\, dx - \int_\Omega f\, w\, dx$$

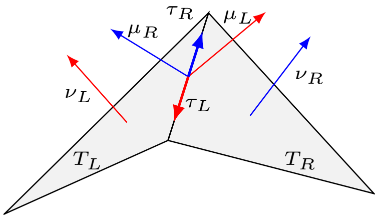
A. Pechstein, J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

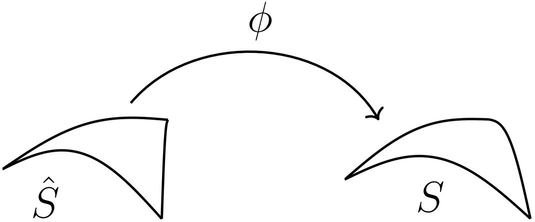
Nonlinear shells

- Normal vector ν
  Tangent vector τ
  Element normal vector $\mu = \nu \times \tau$



- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$





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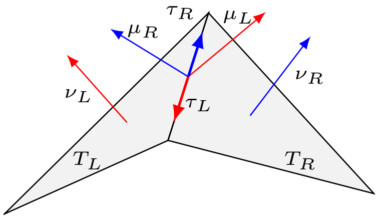
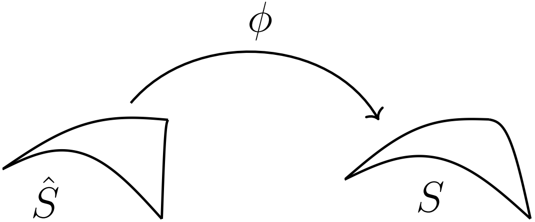
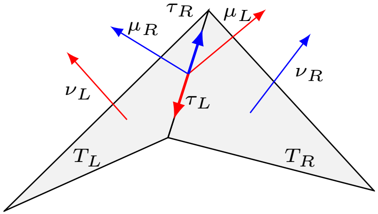


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$

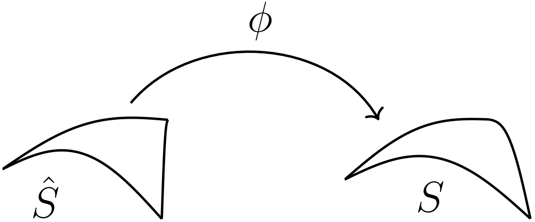


Differential geometry

- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
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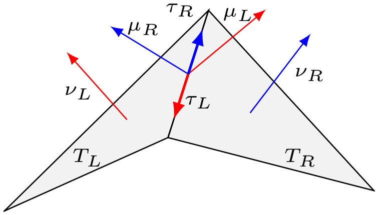
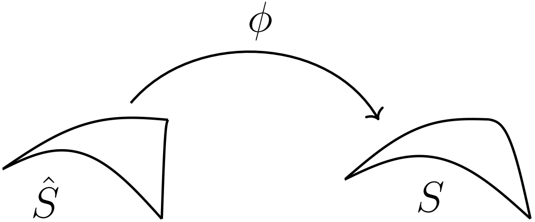


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$



Koiter/Kirchhoff–Love shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$

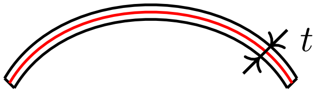
u … displacement of mid-surface

t … thickness

$\mathbb{M}$ … material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



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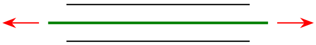

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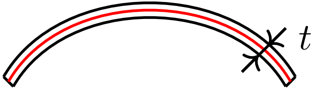

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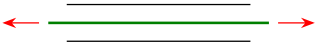

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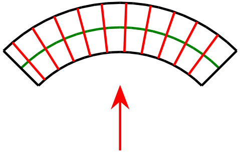
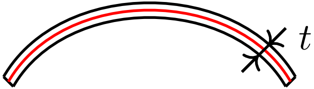




Naghdi/Reissner–Mindlin shell

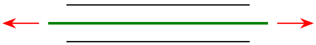
$$\mathcal{W}(u,\gamma) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla(\tilde{\nu}\circ\phi)) - \nabla\hat{\nu}\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2$$

γ ... shearing

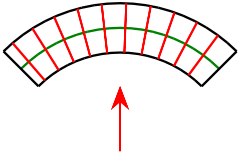
$\tilde{\nu} = \dfrac{\nu+\gamma}{\|\nu+\gamma\|}$ ... director

G ... shearing modulus

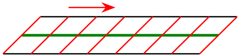
$\kappa = 5/6$ ... shear correction factor



membrane energy

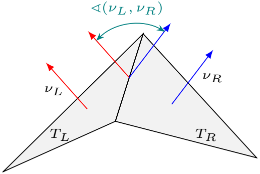


bending energy



shearing energy

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation



$$\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathbb{M}} - \langle f,u\rangle + \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma}\in M_h^{k-1}(\mathscr{T}_h)$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

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Shell problem

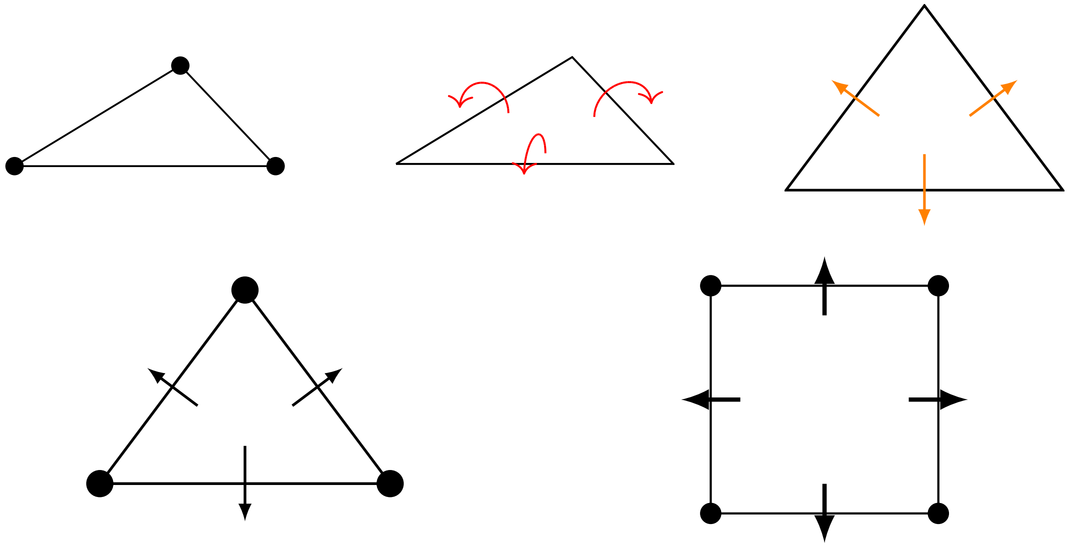
Find $u \in [\mathcal{L}_h^k(\mathscr{T}_h)]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}(\mathscr{T}_h)$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ &+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\end{aligned}$$

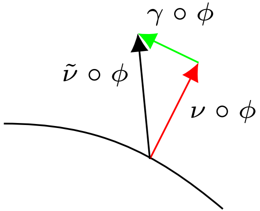
Use hybridization to eliminate $\boldsymbol{\sigma}$ $\to$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Shell element (Koiter)

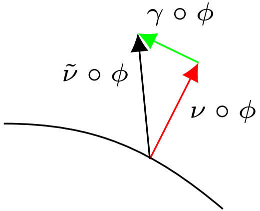


- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu} \circ \phi = \frac{\nu \circ \phi + \gamma \circ \phi}{\|\nu \circ \phi + \gamma \circ \phi\|}$
- Free of shear locking



Extension to nonlinear Naghdi shells

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg* (2013) 254, pp. 170–180.

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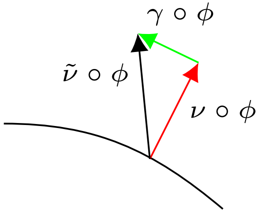
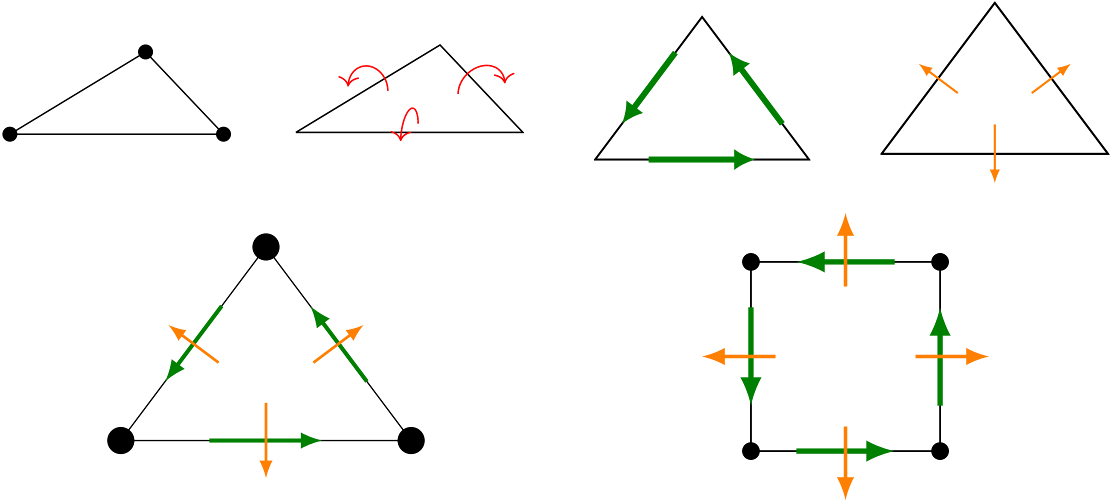


$$\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right):\boldsymbol{\sigma}\,dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E \left(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R) + [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$

Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg* (2013) 254, pp. 170–180.

Shell element (Naghdi)



Linearization

- Linearize to get Reissner–Mindlin and Kirchhoff–Love shell method
- For plates: Recover TDNNS for Reissner-Mindlin plate and Hellan–Herrmann–Johnson method for Kirchhoff–Love plate

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

Membrane locking

$$\mathcal{W}(u) = t\, E_{\mathrm{mem}}(u) + t^3\, E_{\mathrm{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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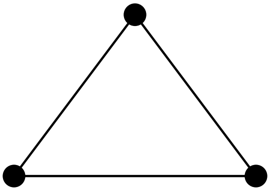

$$\mathcal{L}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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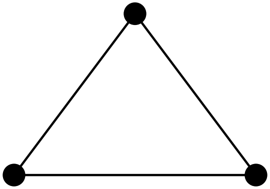
Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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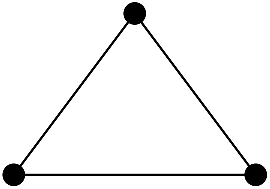

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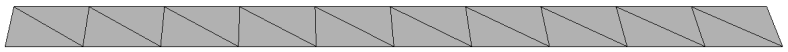
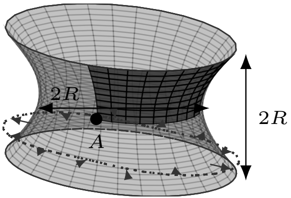
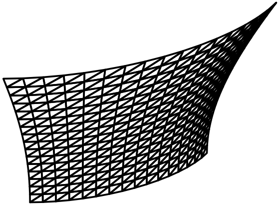
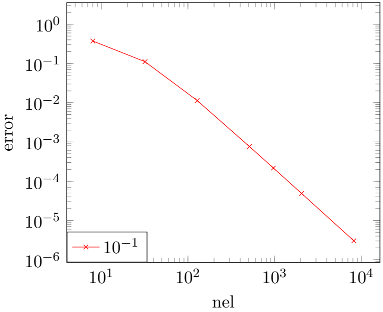


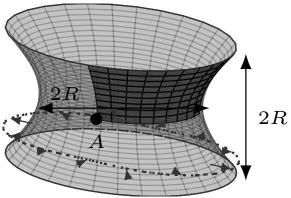
Hyperboloid with free ends

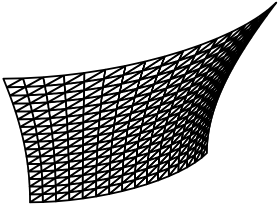


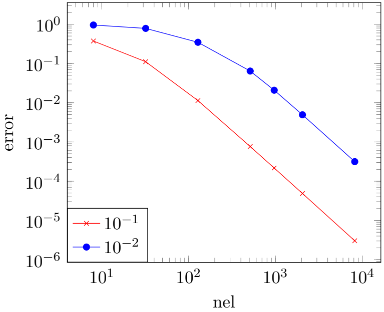




Hyperboloid with free ends

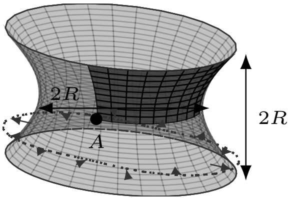


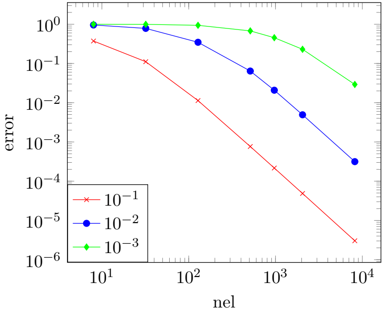


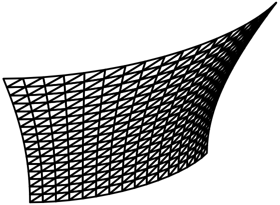


- Pre-asymptotic regime

Hyperboloid with free ends







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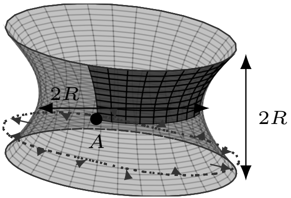
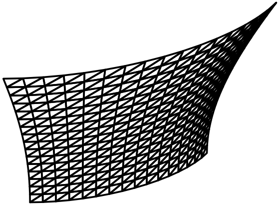
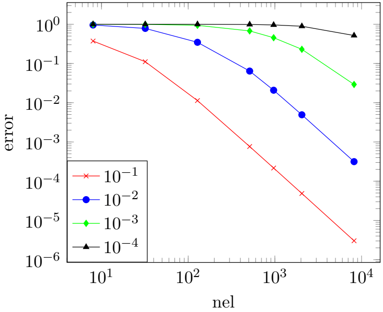




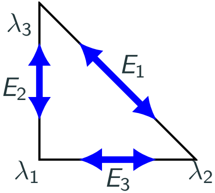


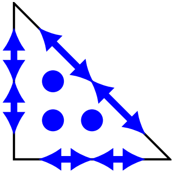
- Pre-asymptotic regime

Regge elements

$$H(\operatorname{curl}\operatorname{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{\mathrm{sym}} \mid \operatorname{curl}\operatorname{curl}\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

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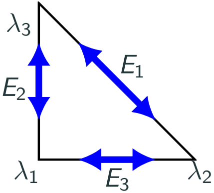

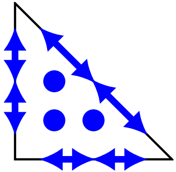

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$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

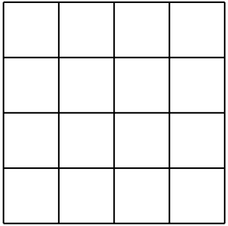
$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking

$$\frac{1}{t^2}\|\quad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

Membrane locking

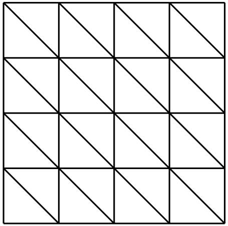
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

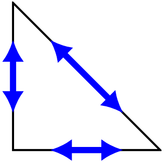


- Reduced integration for quadrilateral meshes

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

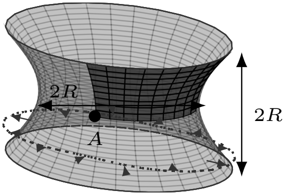


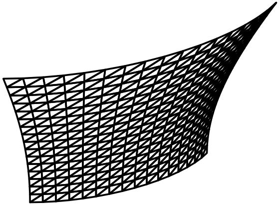


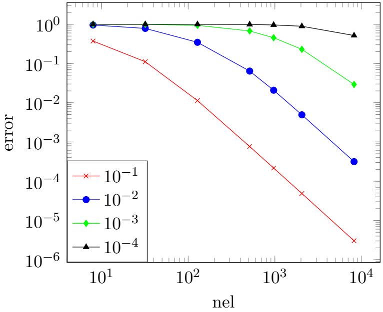
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

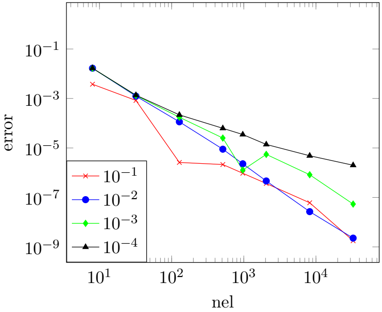
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends

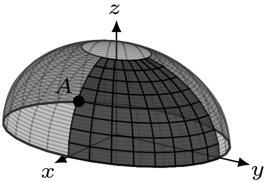


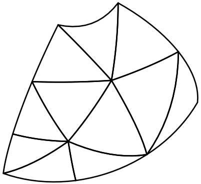


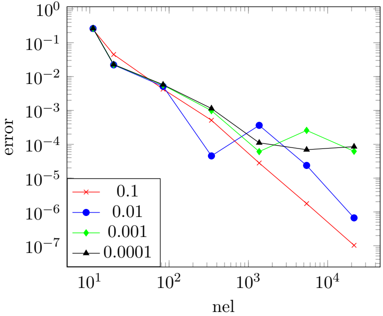


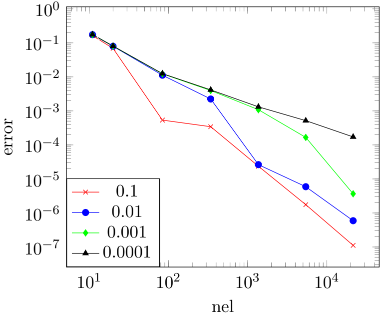


Open hemisphere with clamped ends





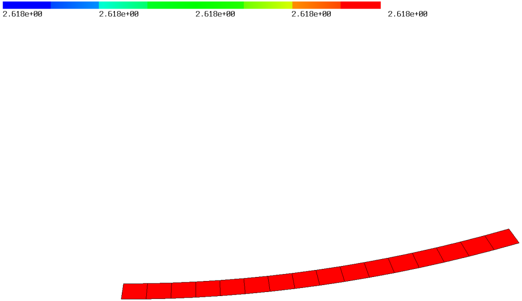


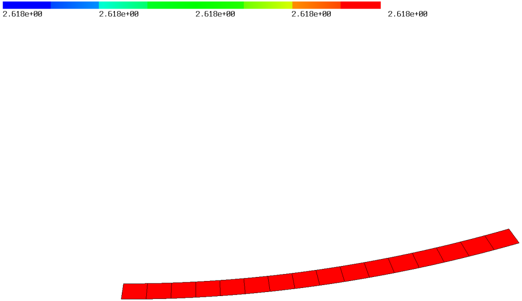


Cantilever subjected to end moment

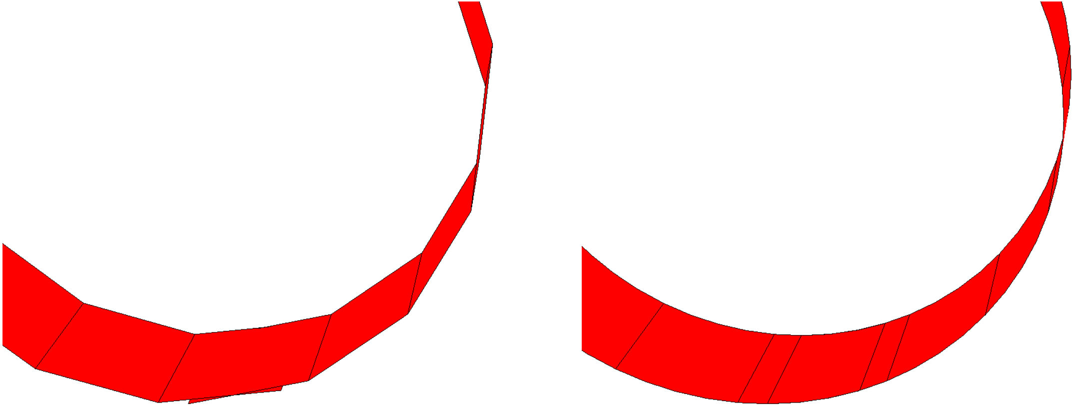
TECHNISCHE UNIVERSITÄT WIEN

NGSolve





Cantilever subjected to end moment



Summary & outlook

- Robust mixed methods for continuum mechanics
- Locking-free mixed methods for (nonlinear) plates & shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

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- Coupling for 3D elasticity (A. Pechstein, M. Krommer; JKU Linz)
- NGSolve Add-On
- Extension to Cosserat elasticity, plates, and shells

Literature

- N., Pechstein, Schöberl: Three-field mixed finite element methods for nonlinear elasticity *Comput. Methods Appl. Mech. Engrg 382 (2021)*
- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*
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Thank You for Your attention!