Curvature approximation with Finite Elements

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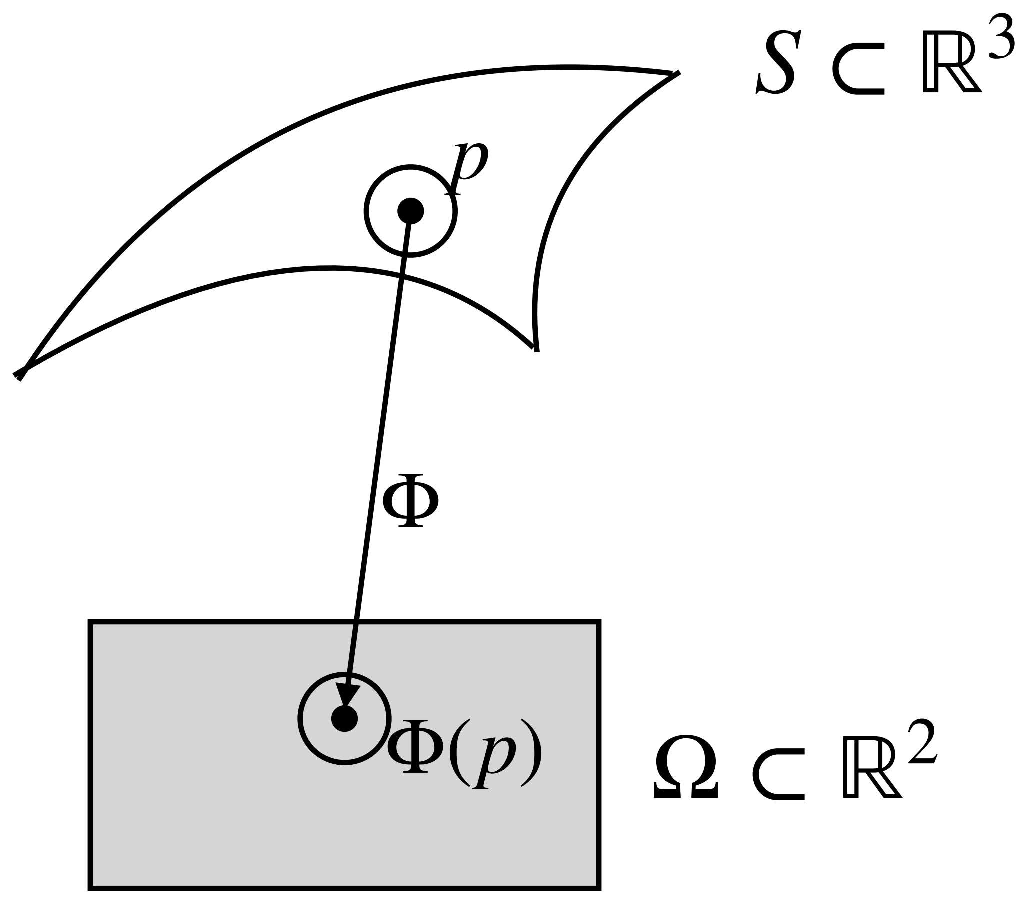
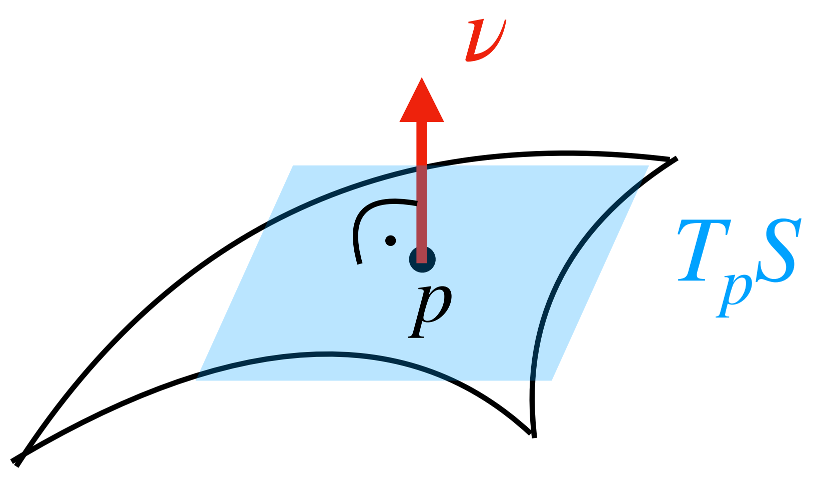
Curvature on surfaces





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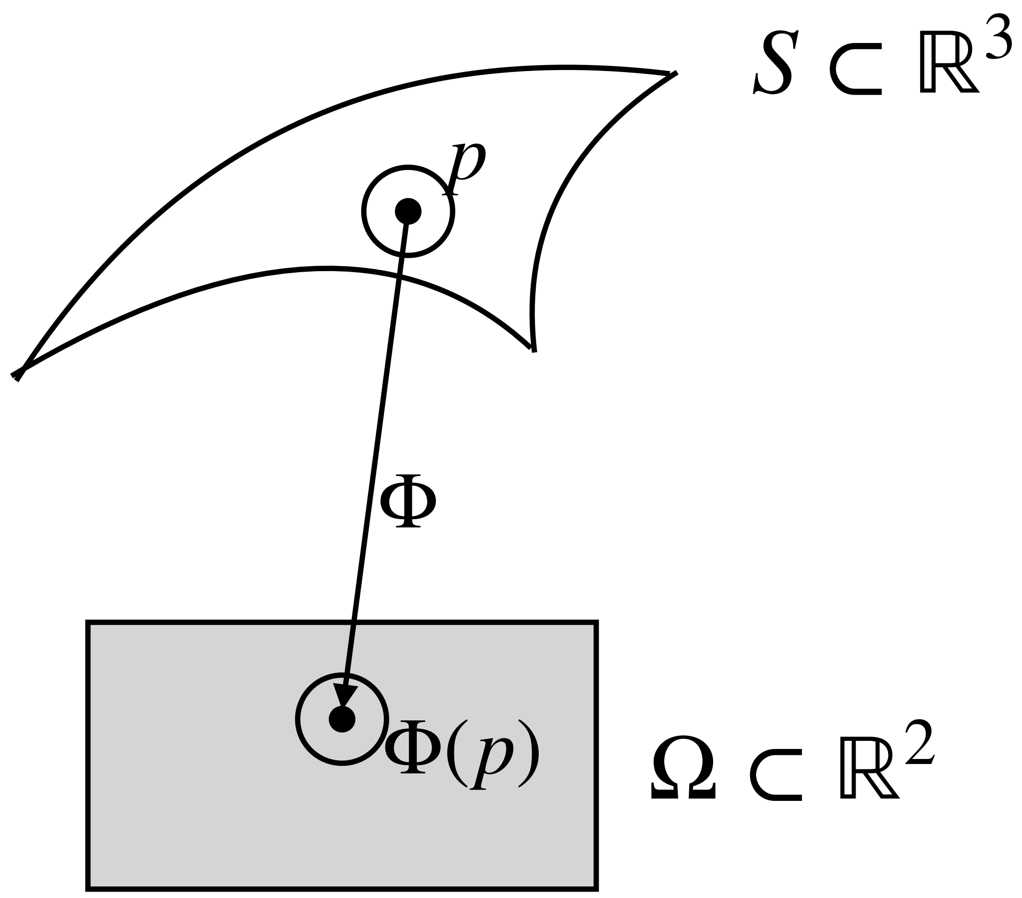
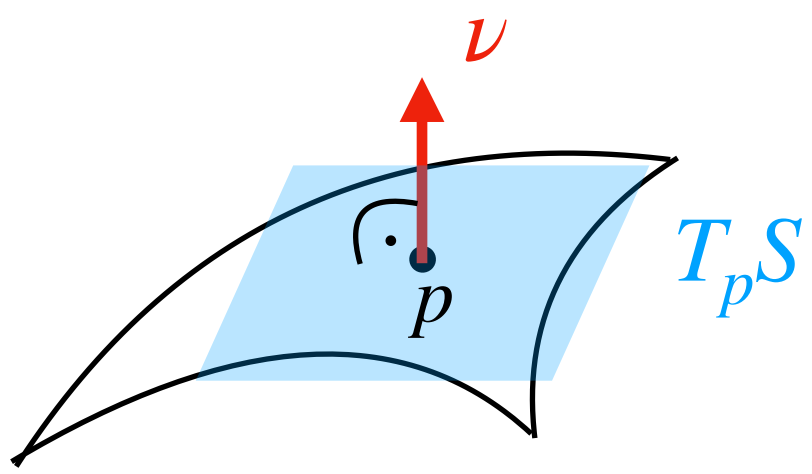
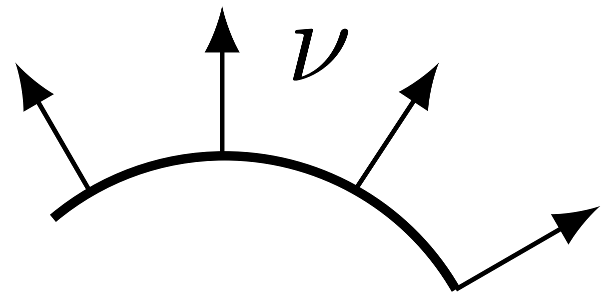


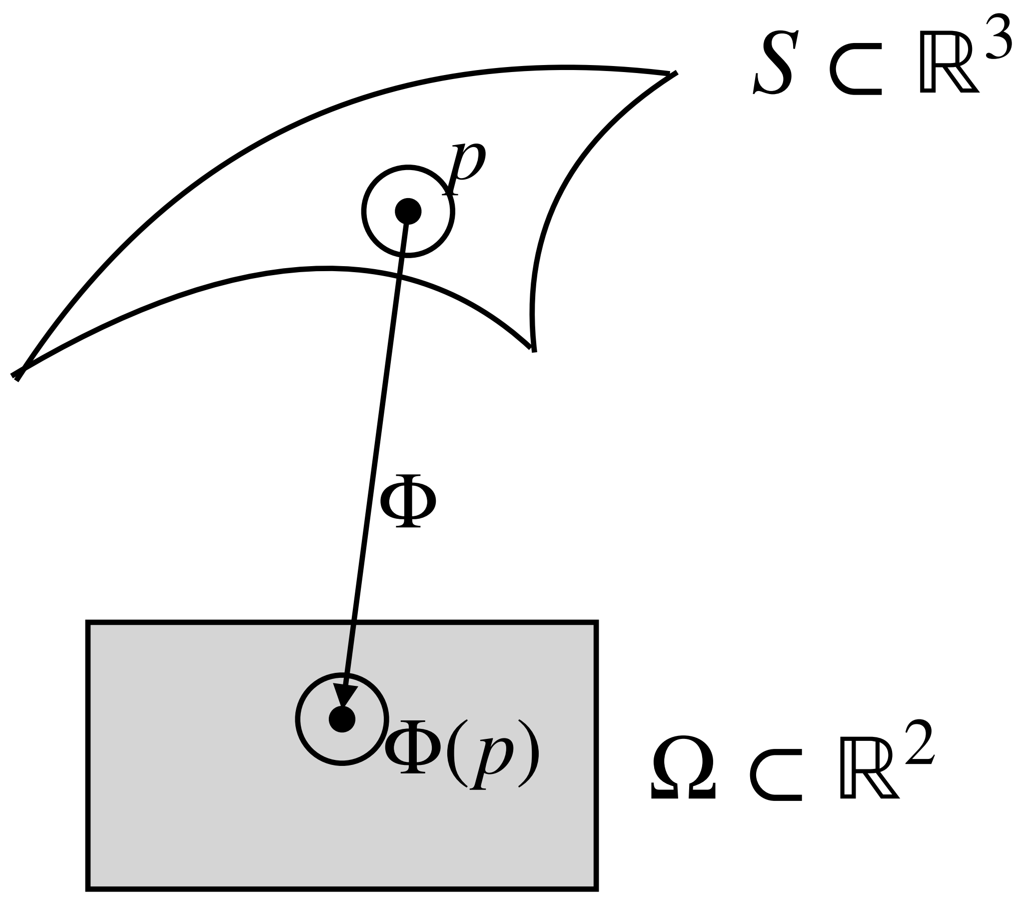


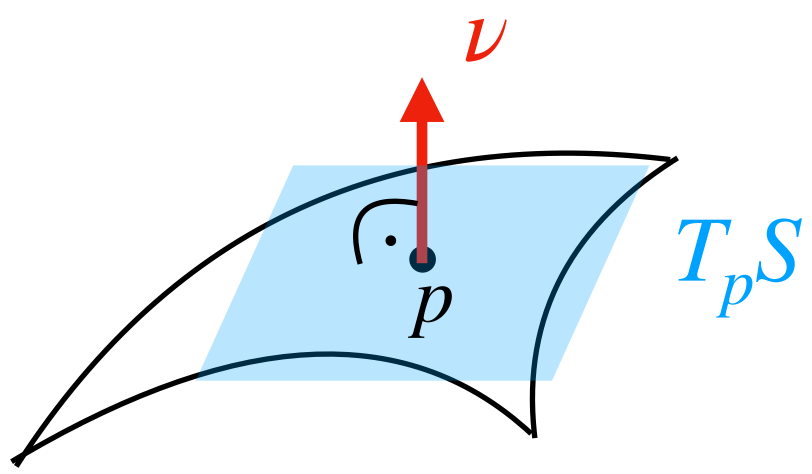
$$\nabla_S \nu = P_S(\nabla \tilde{\nu})$$



- Shape operator, Weingarten tensor, second fundamental form
- Eigenvalues: $0, \kappa_1, \kappa_2,$ principal curvatures

Curvature on surfaces





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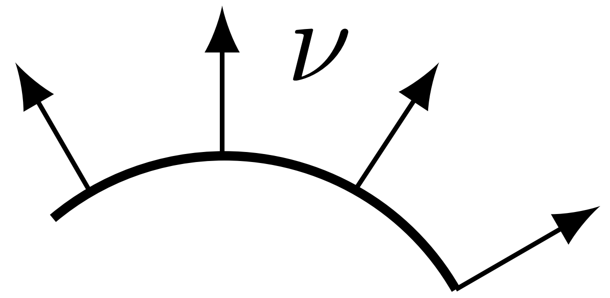

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- Mean curvature $H = 0.5(\kappa_1 + \kappa_2)$ ← extrinsic curvature
- Gauss curvature $K = \kappa_1 \kappa_2$ ← intrinsic curvature

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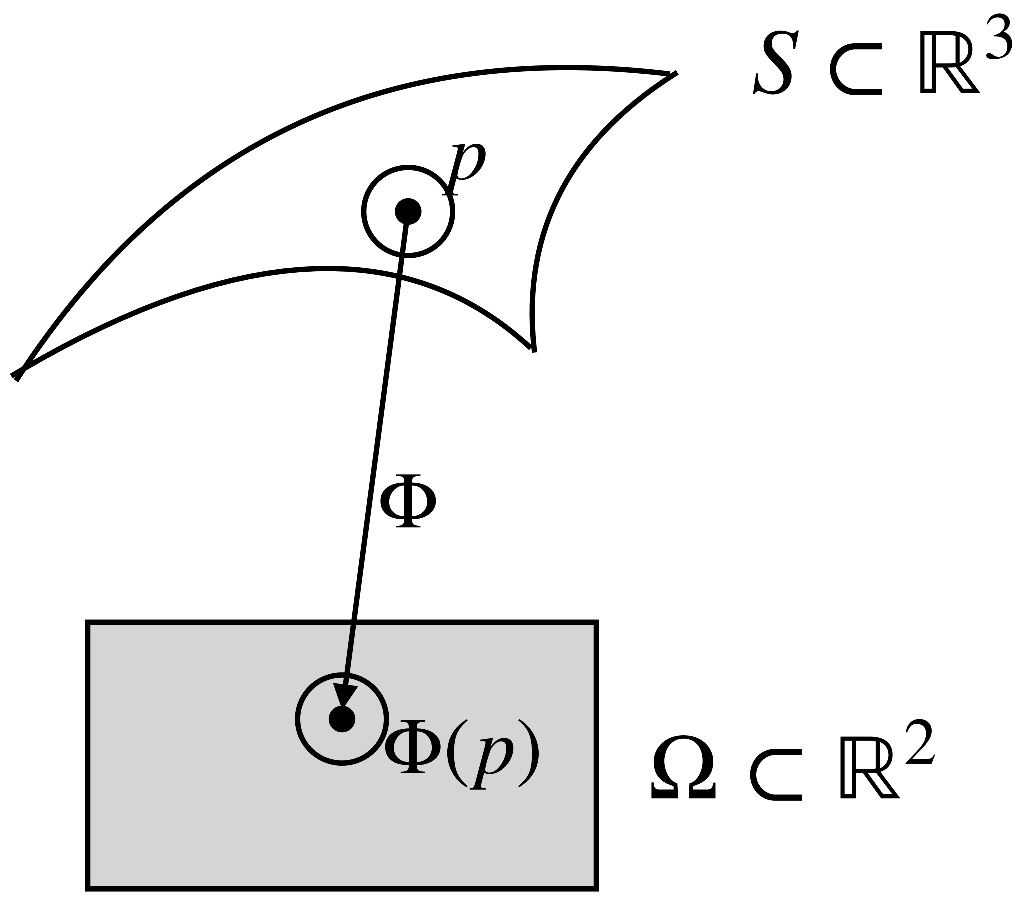

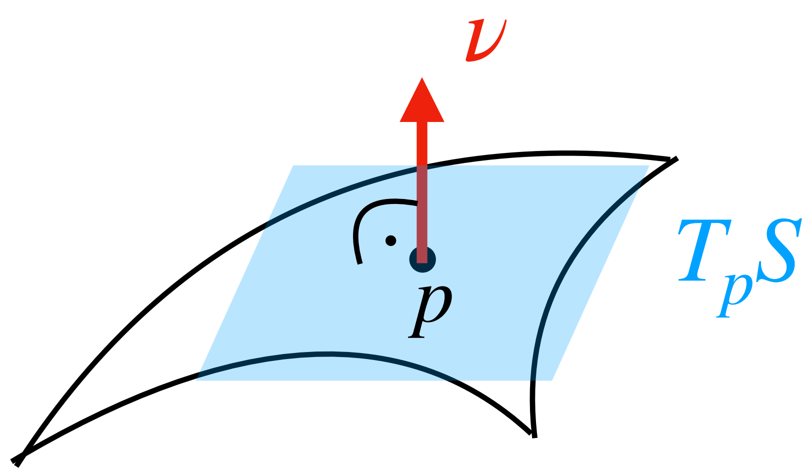

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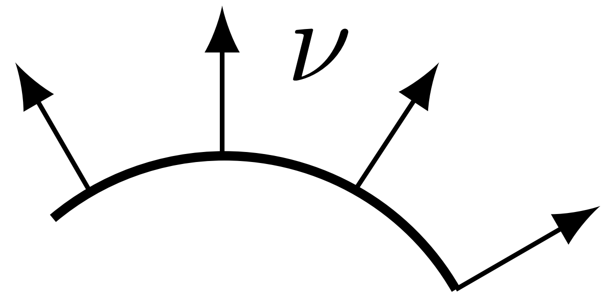

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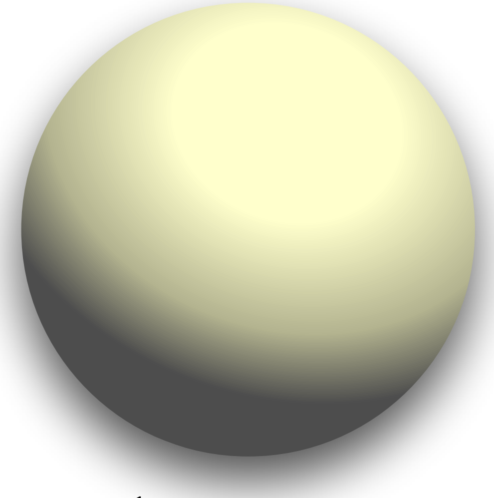
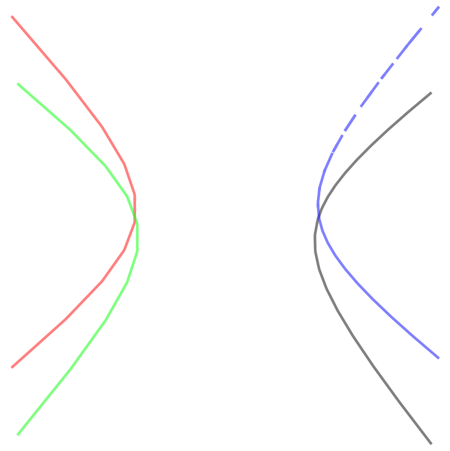


$K = \frac{1}{r^2}, \quad H = \frac{1}{r}$

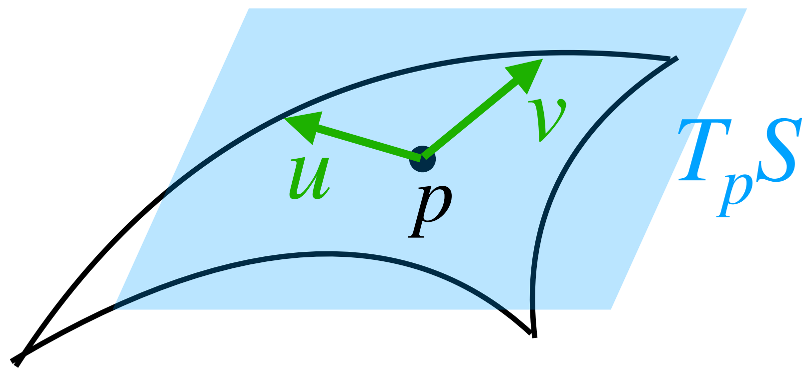
$K = 0, \quad H = \frac{1}{2r}$

2



$K < 0$

Intrinsic curvature on surfaces



- First fundamental form, metric $g(\,\cdot\,,\cdot\,) = I(\,\cdot\,,\cdot\,) := \langle\,\cdot\,,\cdot\,\rangle|_{TS}$
- Compute lengths and angles on the manifold
  $\|v\|_g = \sqrt{g(v,v)}, \quad \cos(\alpha) = g(u,v)/(\|u\|_g\|v\|_g)$

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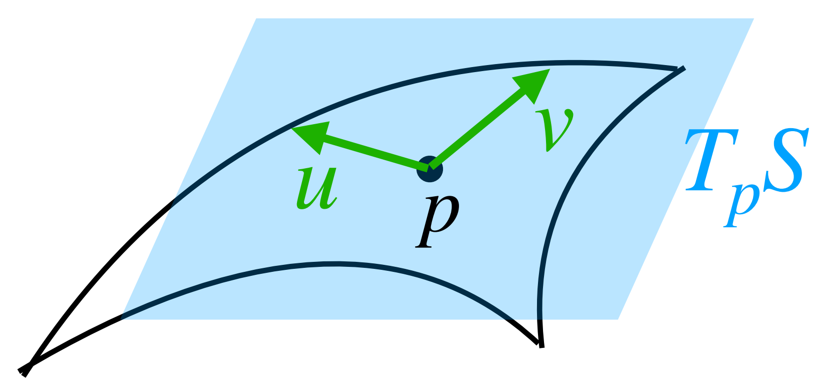

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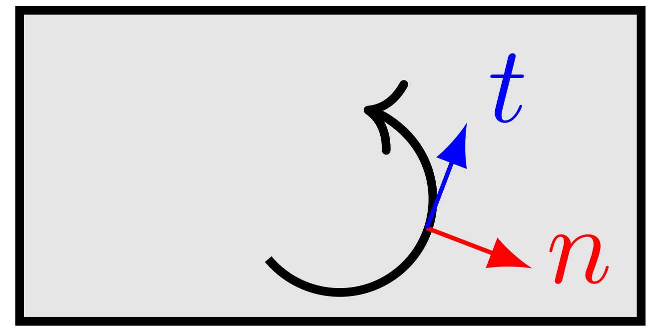

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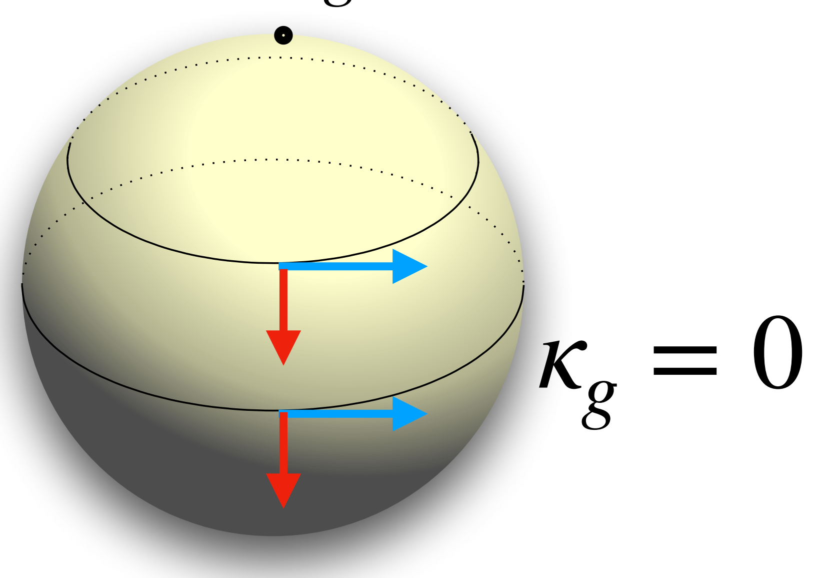

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$\tau, \mu \in TS$

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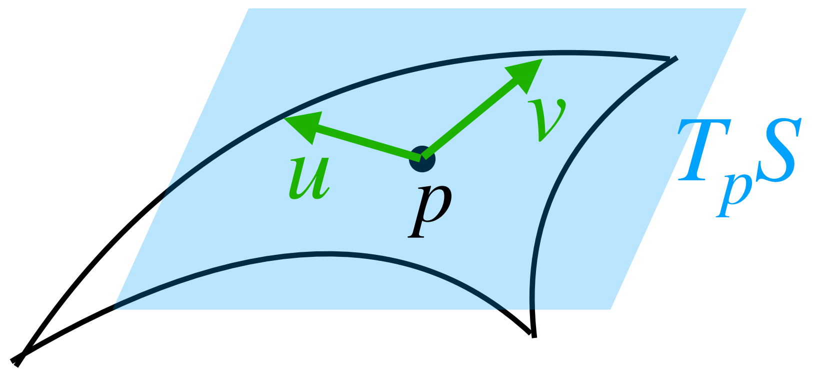

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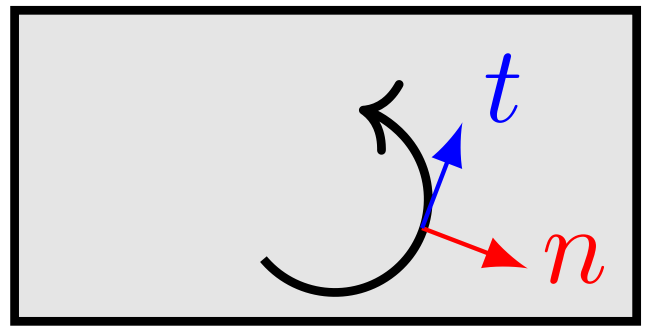

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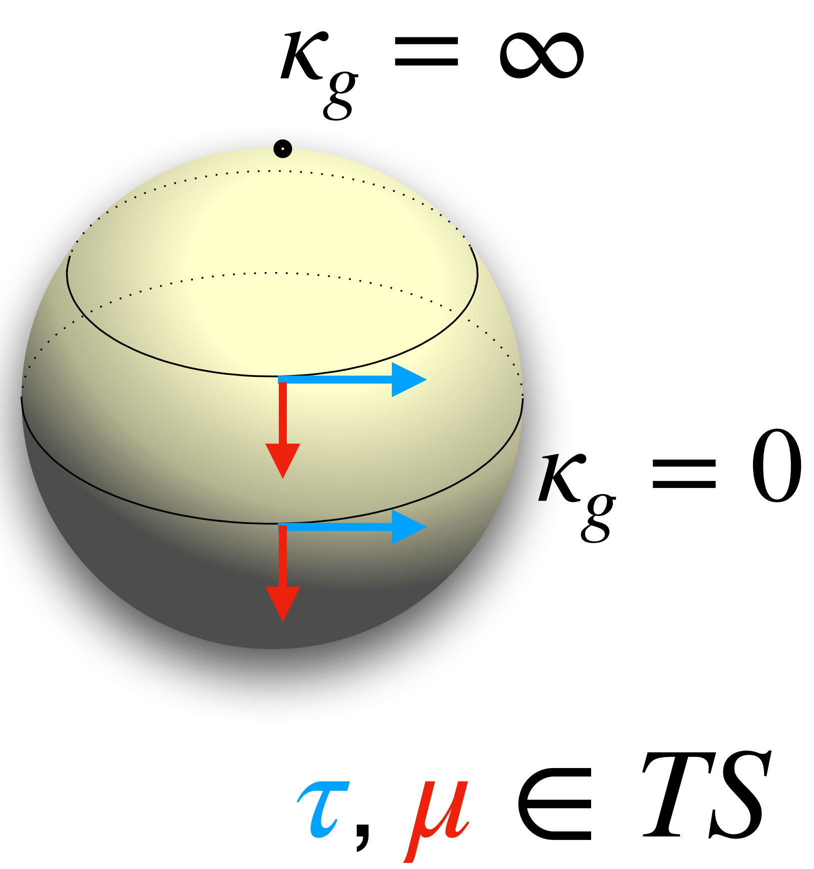

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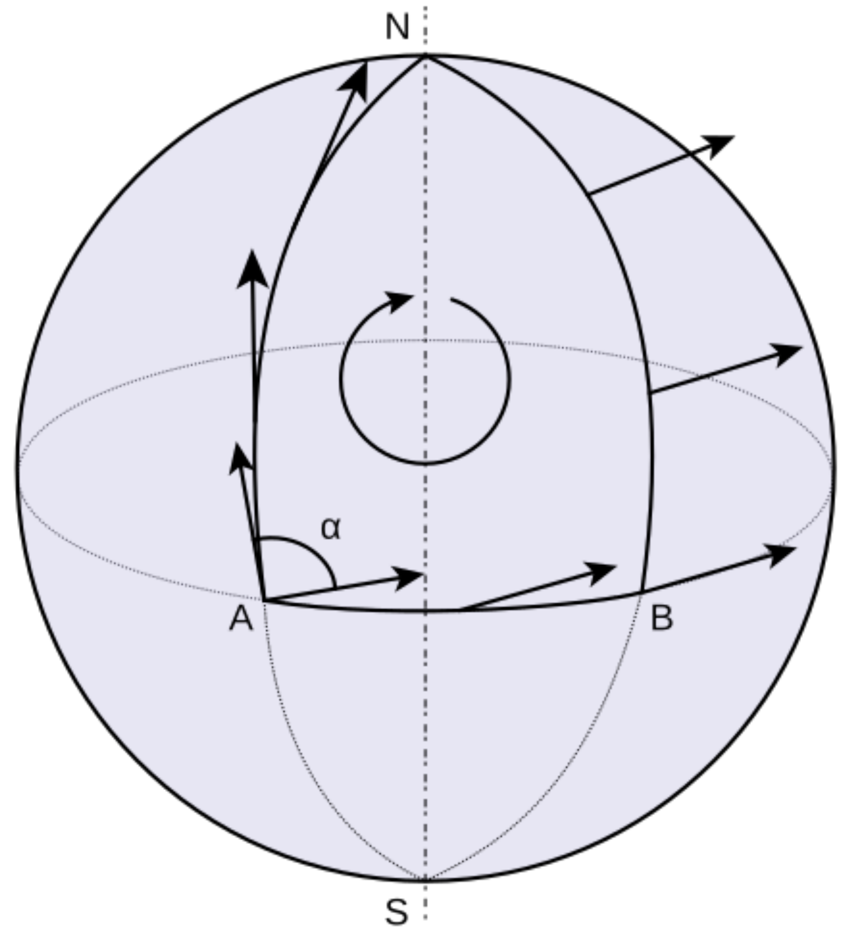
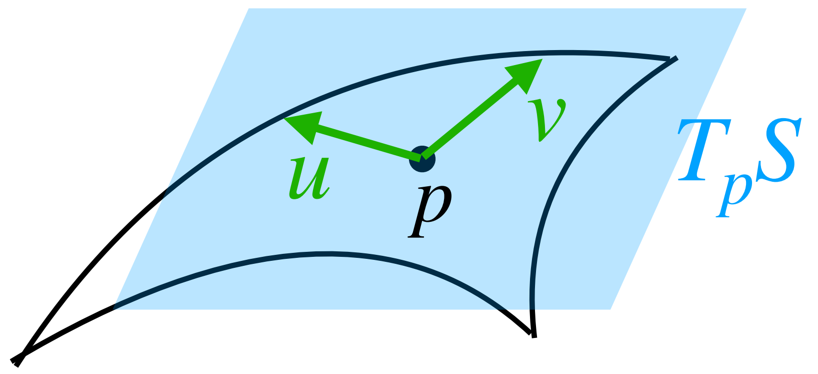


From Wikipedia: Parallel transport

Intrinsic curvature on surfaces



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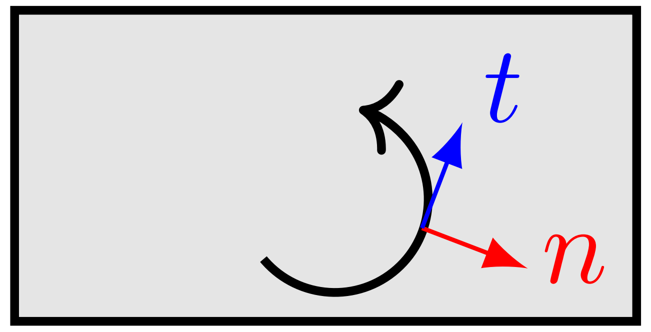

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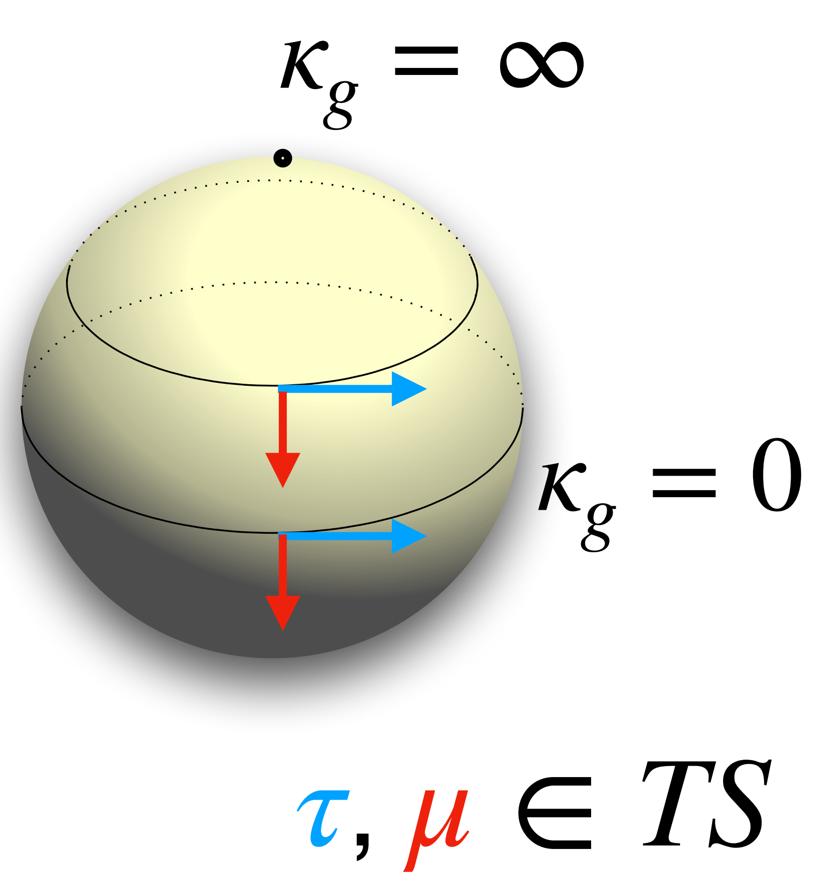

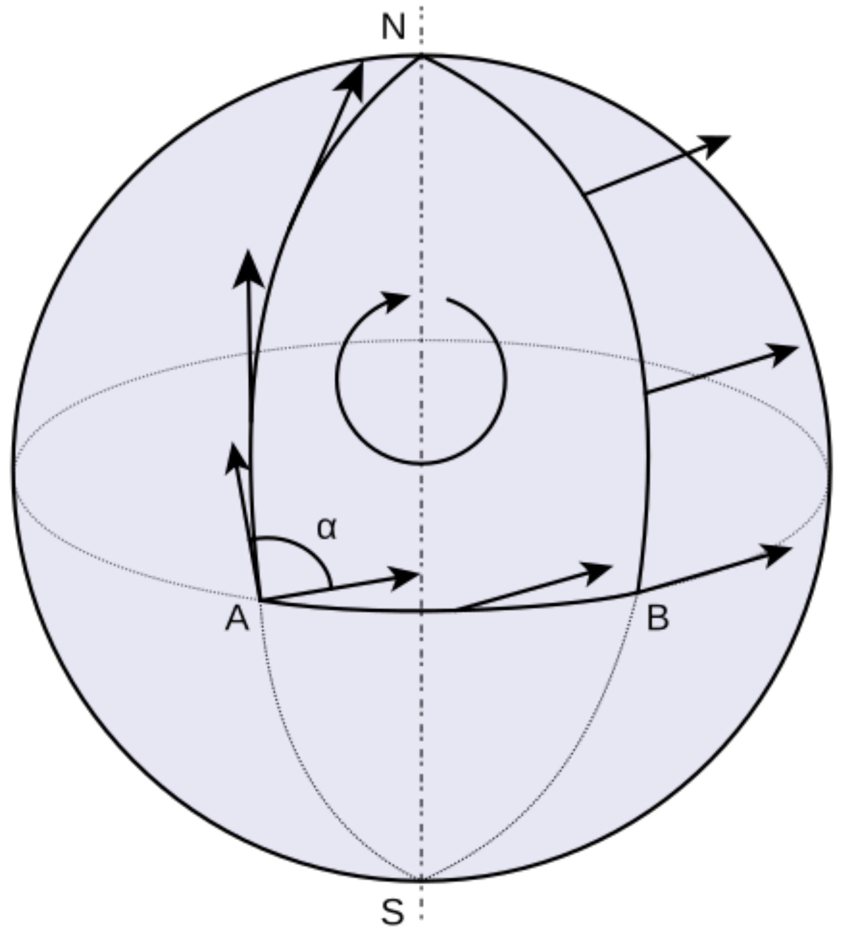

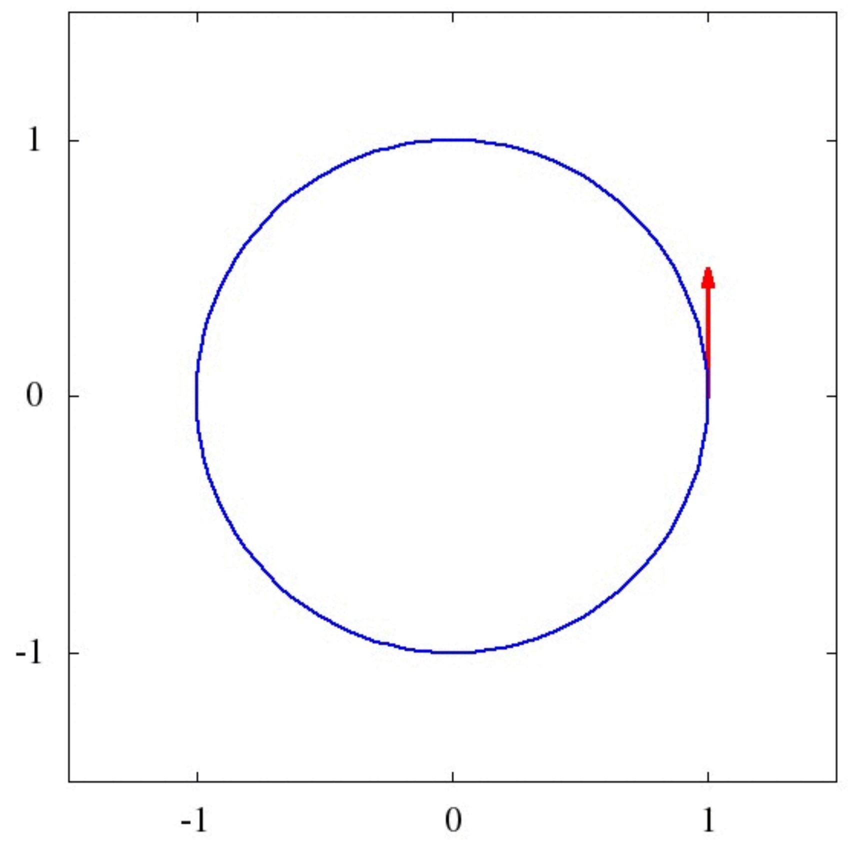

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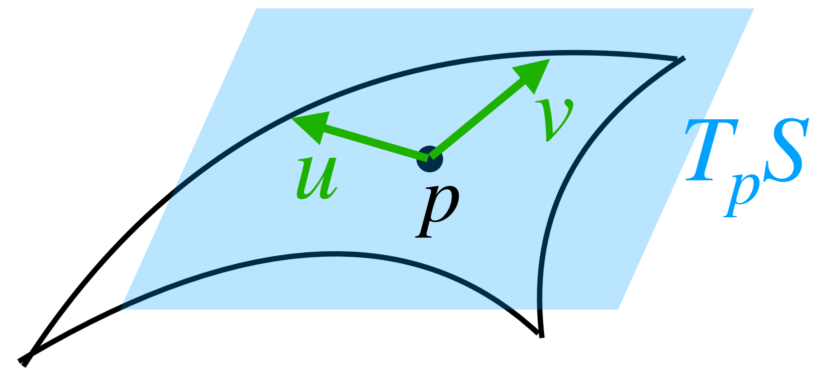

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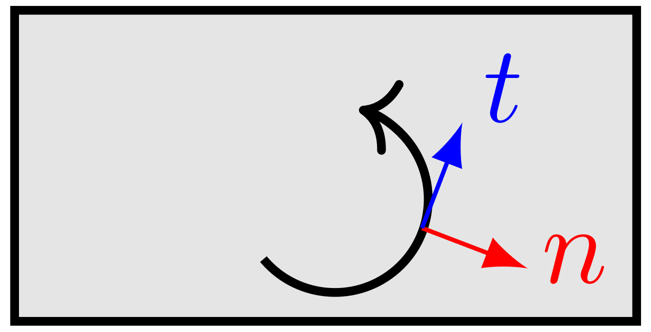

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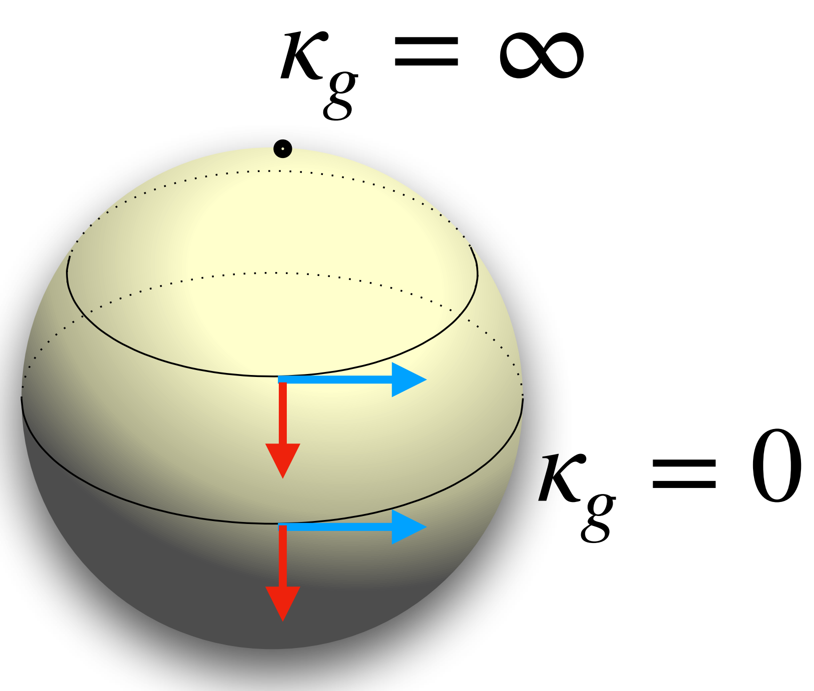

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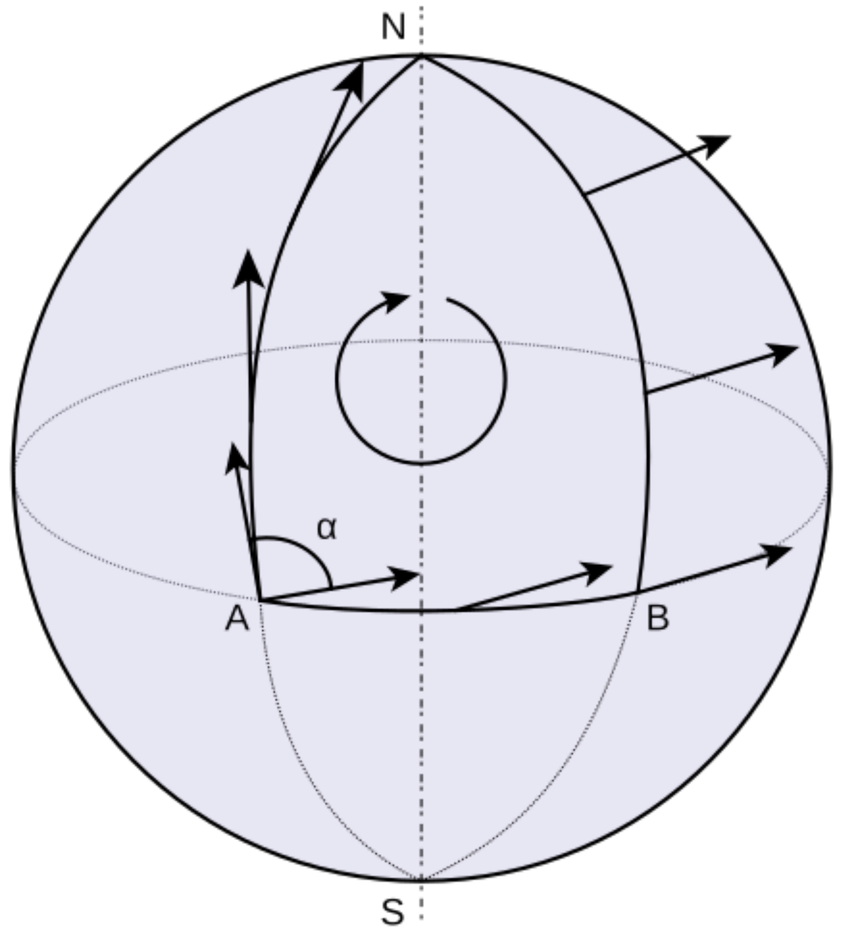
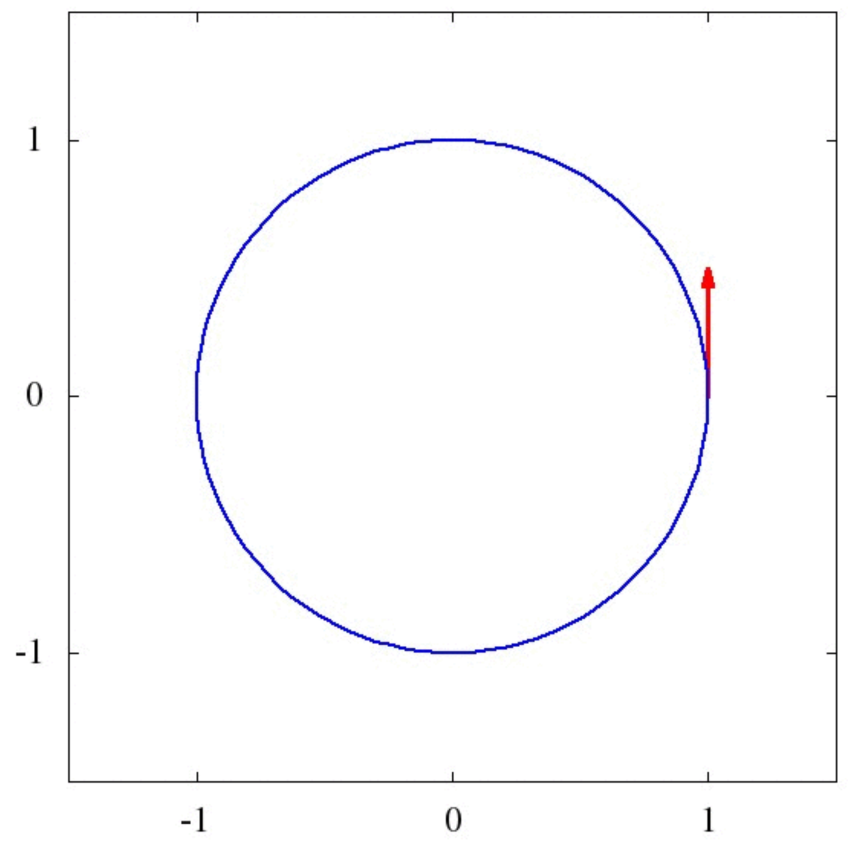
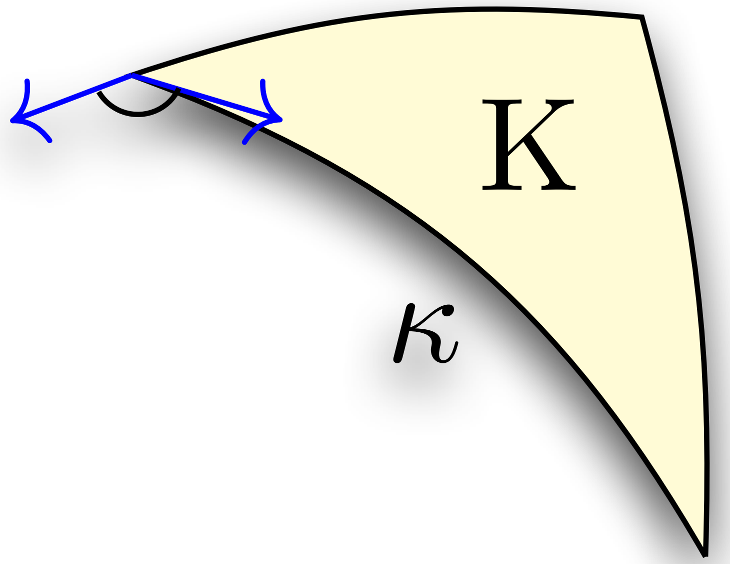




- Gauss-Bonnet theorem

$$\int_T K\,ds + \int_{\partial T} \kappa_g\,ds + \sum_{i=1}^{3} \alpha_i = 2\pi$$

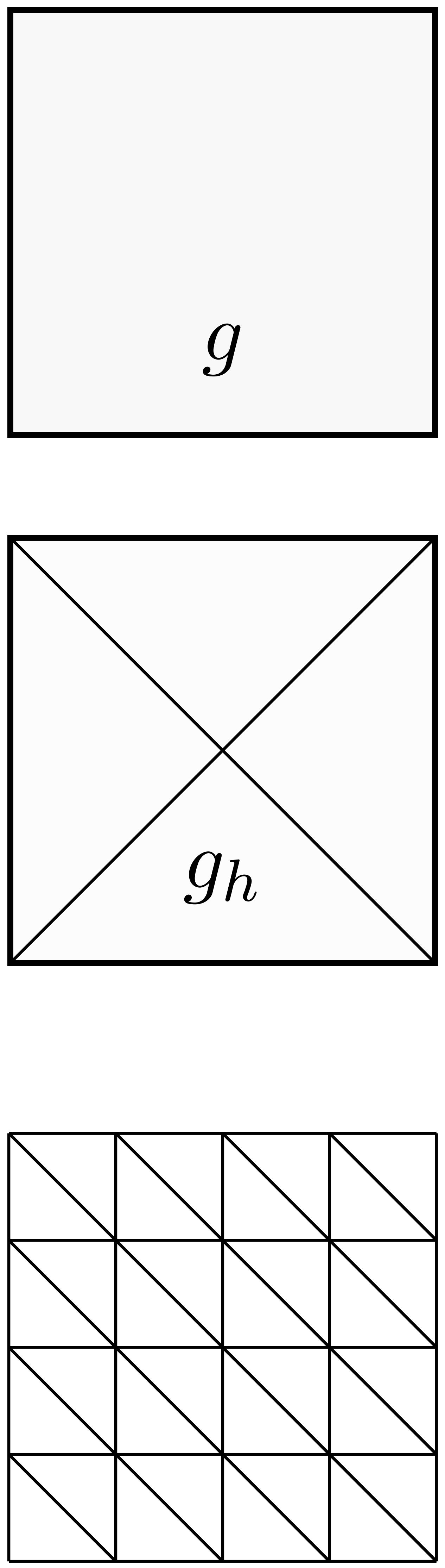
$\tau, \mu \in TS$



From Wikipedia: Parallel transport

Riemannian manifolds

- Riemannian manifold $(\Omega, g), \Omega \subset \mathbb{R}^N, g$ metric tensor
- Approximation g_h of g on a triangulation
- How to approximate g?
- How to compute discrete curvature? Convergence?



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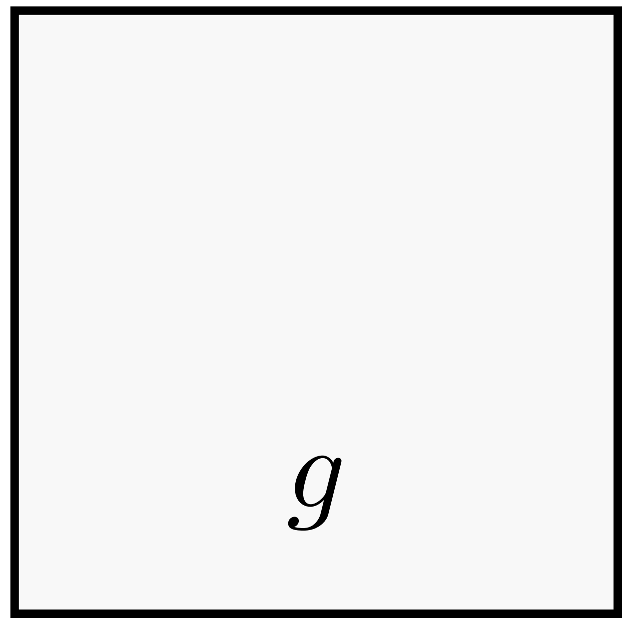

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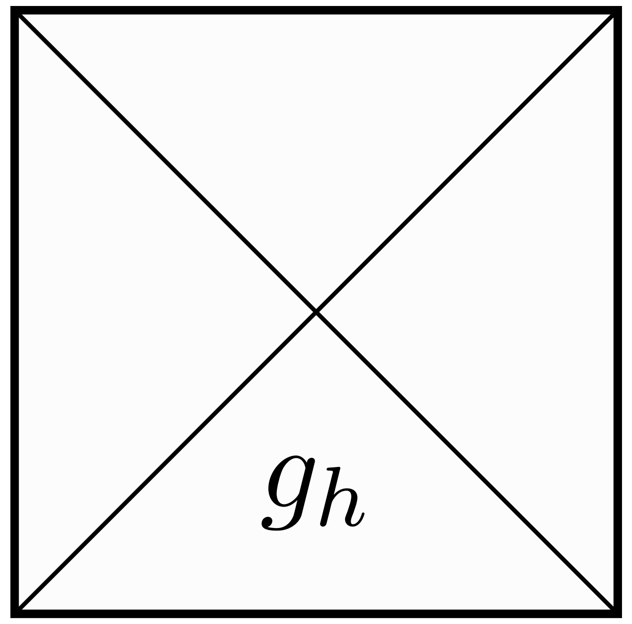
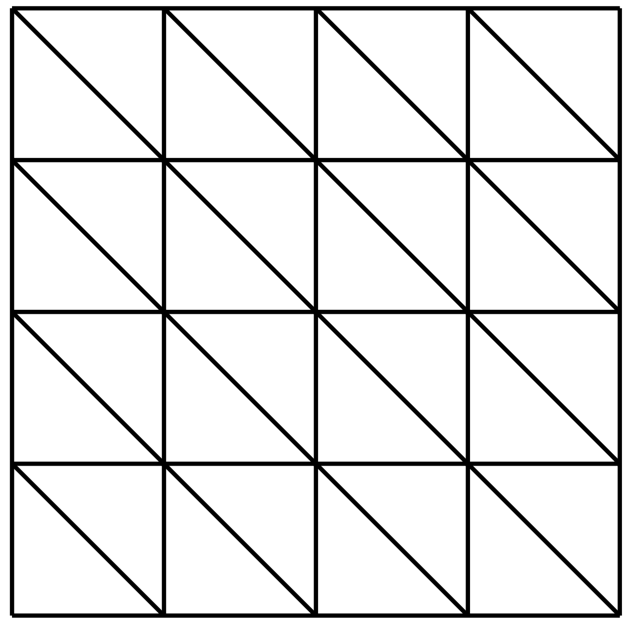
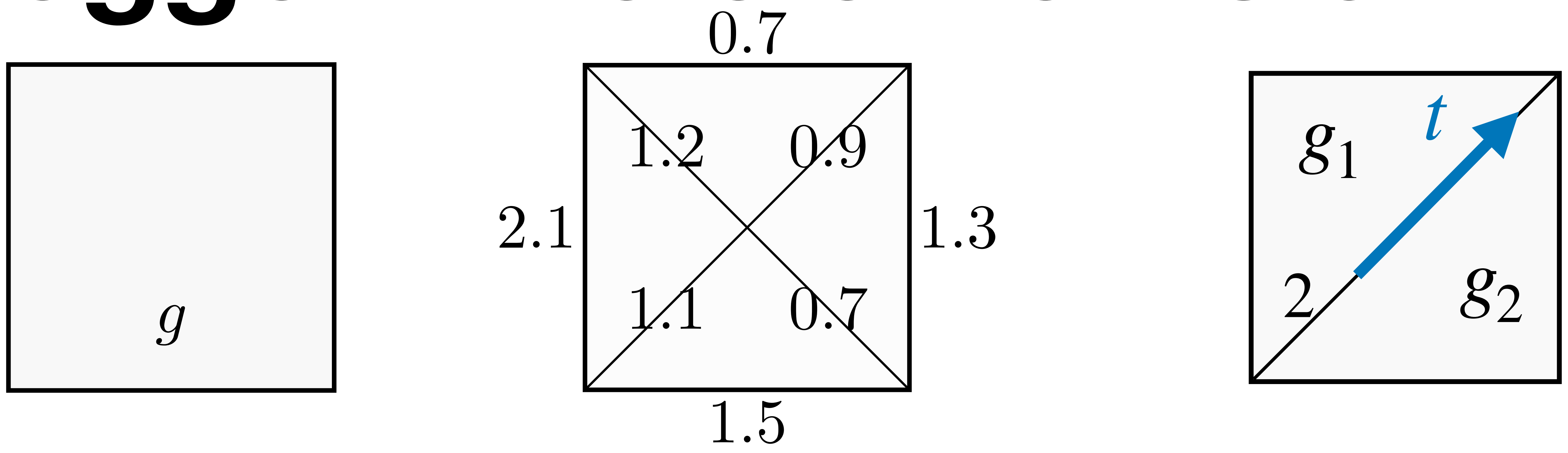


$g_h \in ?$ $\qquad K(g_h) = ?$ $\qquad \|K(g_h) - K(g)\|_? \leq \mathcal{O}(h^?)$



Regge finite elements & metrics



$$\int_E g_1(t,t)\,ds = \int_E g_2(t,t)\,ds = 2$$

$$g_h = g_1 \cup g_2$$

- Element-wise symmetric matrix of polynomials
- Pose interface conditions to connect them

Christiansen: On the linearization of Regge calculus, Numerische Mathematik, 2011.

Li: Finite Elements with Applications in Solid Mechanics and Relativity, PhD thesis, 2018.

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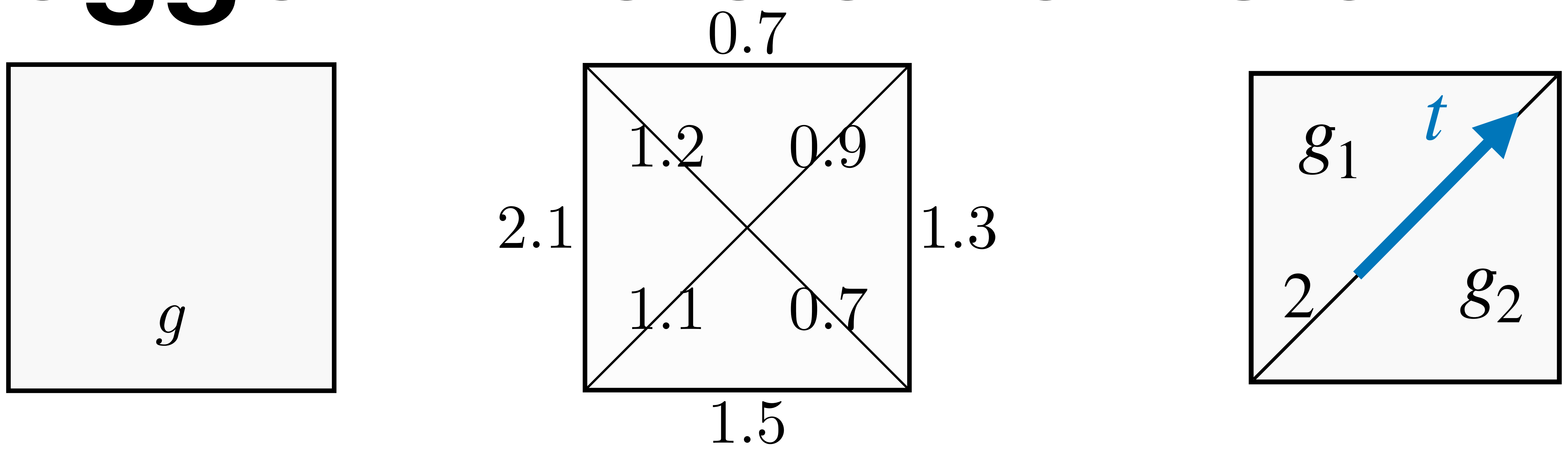

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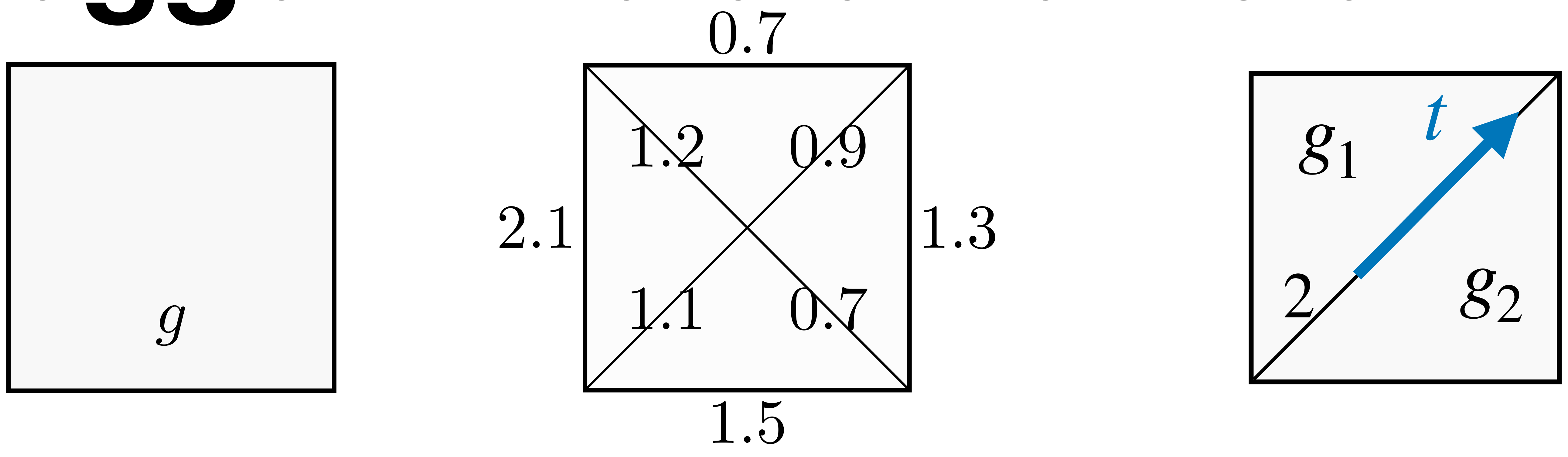

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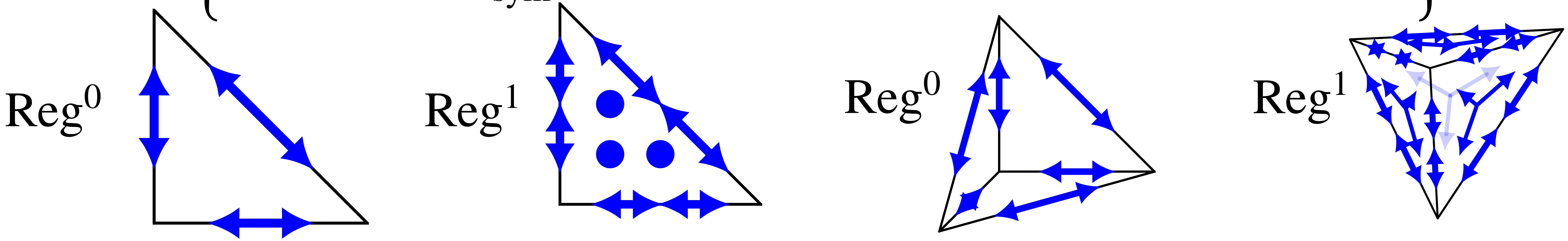
$$\mathrm{Reg}^k := \left\{ \sigma \in \mathscr{P}^k(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \mid \sigma \text{ is tangential-tangential continuous} \right\}$$



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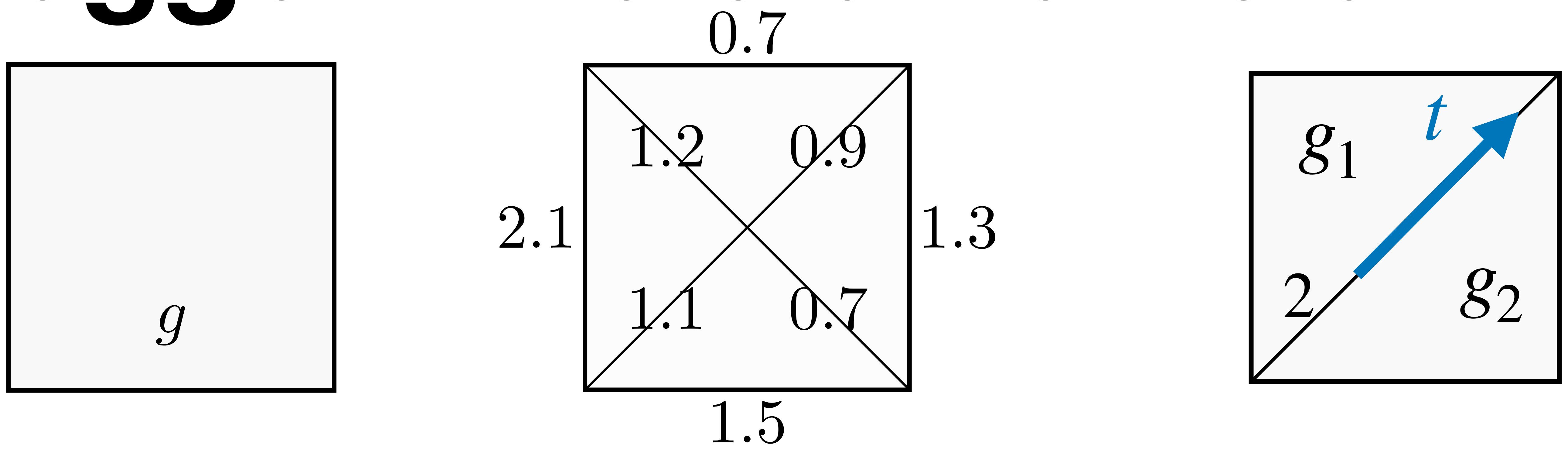

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$$\|g - \mathcal{I}^k_{\text{Reg}} g\|_{L^2} \leq C h^{k+1} \|g\|_{H^{k+1}}$$

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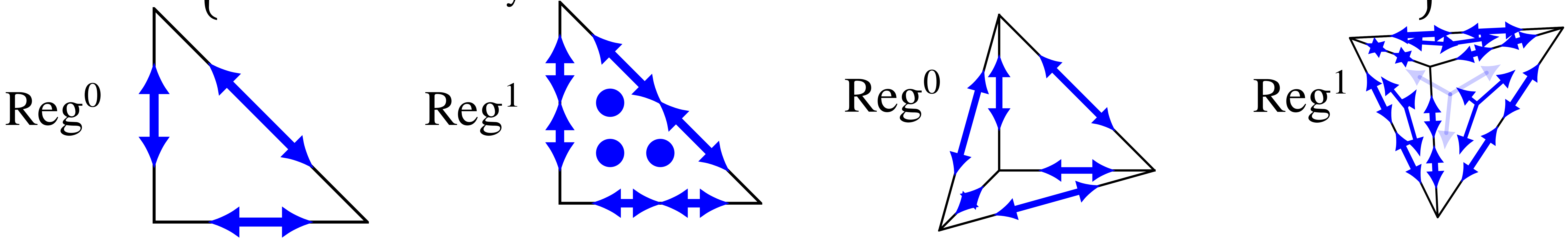

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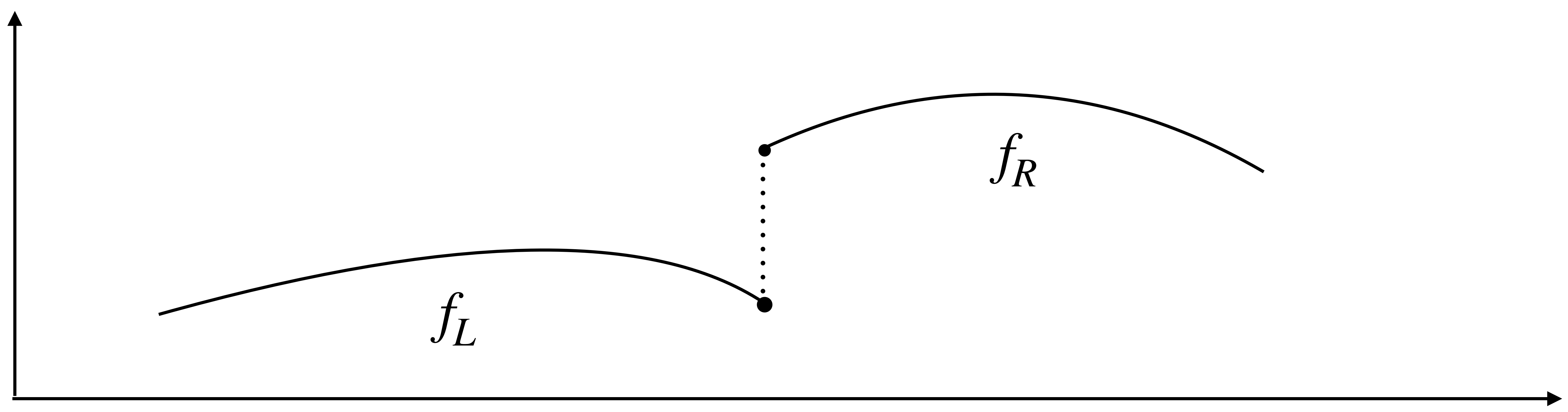
Distributions



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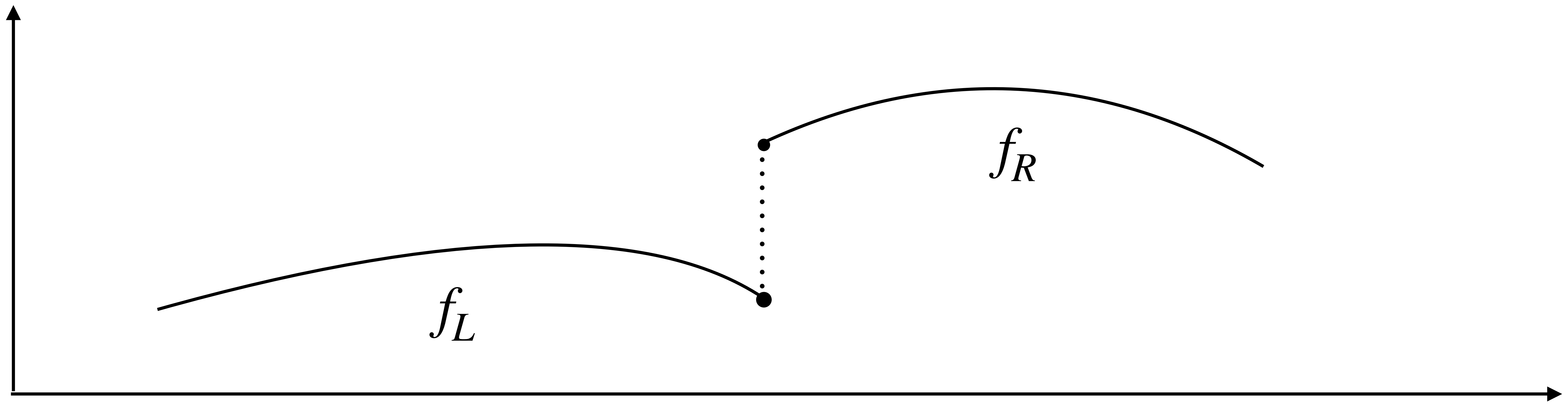

- Dirac deltas at jump $f'(x) = f'_L(x) + f'_R(x) + \underbrace{(f_R(x_0) - f_L(x_0))}_{=[\![f]\!]}\, \delta_{x_0}$

- Theorie of Schwarz distributions
$$(f')_{\text{dist}}(\Psi) := -\int_\Omega f\,\Psi'\,dx = \int_{\Omega_L} f'_L\,\Psi\,dx + \int_{\Omega_R} f'_R\,\Psi\,dx + [\![f]\!]\Psi(x_0), \quad \Psi \in C_0^\infty(\Omega)$$

- Restriction: we cannot multiply distributions $\delta_1 \cdot \delta_2$ (Colombeau algebra)

Curvature of Regge metric

- $K = \mathfrak{R}_{1212} / \det(g) = \mathfrak{R}(X, Y, Y, X) / \|X \wedge Y\|_g$ (sectional curvature)
- Riemann curvature tensor
- $\mathfrak{R}(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$

Levi-Civita connection $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

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$$\mathfrak{R}_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} - \Gamma_{ilp}\Gamma^p_{jk} + \Gamma_{jlp}\Gamma^p_{ik}$$

$$\Gamma^p_{jk} = g^{pq}\Gamma_{jkq}, \quad g^{pq} = (g^{-1})_{pq}$$

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_k g_{ij})$$

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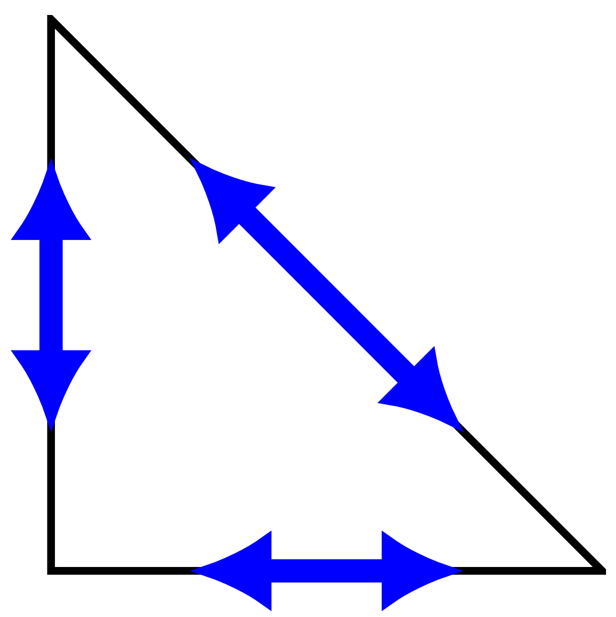

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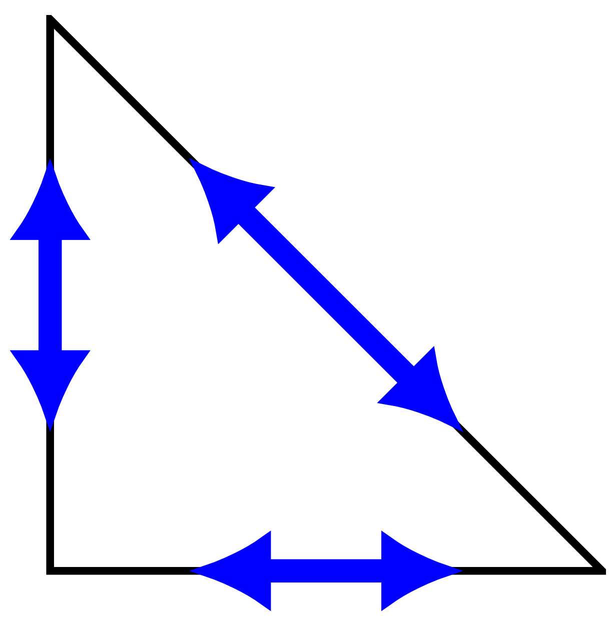

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Linear second-order derivative

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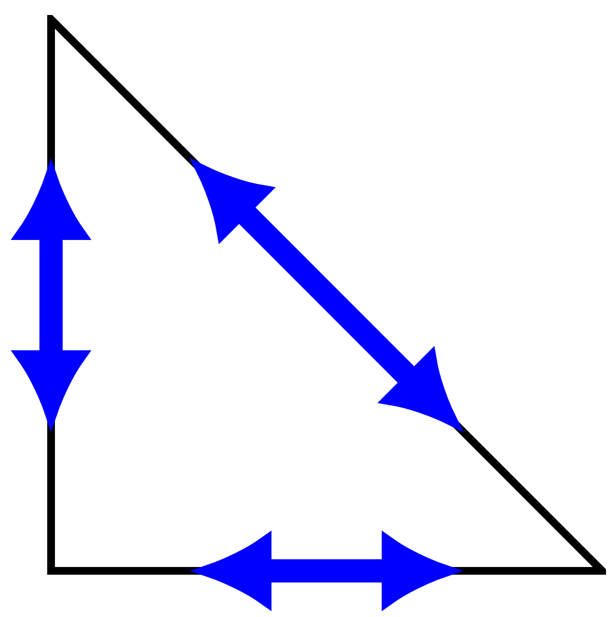
- $\mathfrak{R}(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$



Levi-Civita connection $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

$$\mathfrak{R}_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} - \Gamma_{ilp}\Gamma^p_{jk} + \Gamma_{jlp}\Gamma^p_{ik}$$

$$\Gamma^p_{jk} = g^{pq}\Gamma_{jkq}, \quad g^{pq} = (g^{-1})_{pq}$$

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_k g_{ij})$$

g_h is only tt-continuous

Linear second-order derivative

Multiplication of first-order distributional derivatives

Curvature of Regge metric

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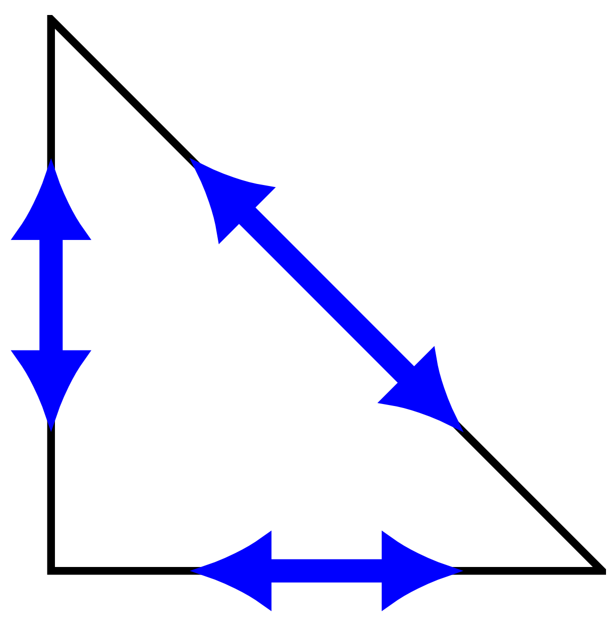

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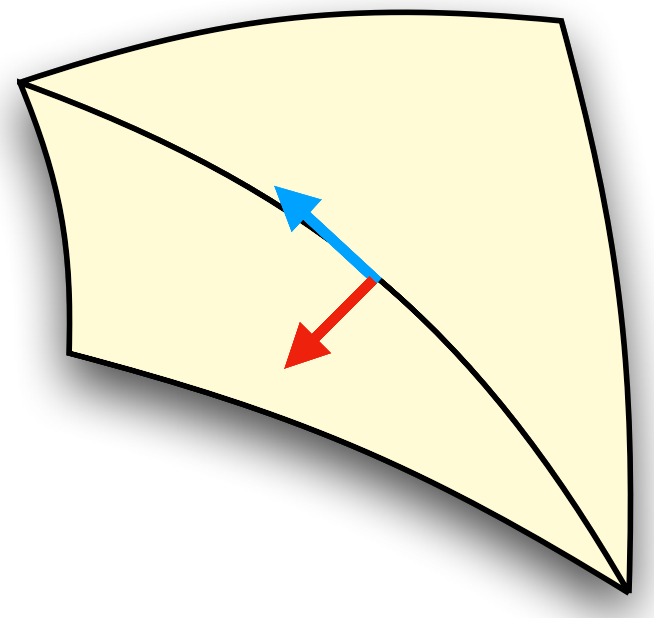
Multiplication of first-order distributional derivatives

$\mathfrak{R}(g_h)$ is a nonlinear distribution!

Distributional Gauss curvature Part 1

- Compute the geodesic curvature from both elements $\kappa_g = g(\nabla_\tau \tau, \mu)$
- μ changes sign
- g smooth $\Rightarrow [\![\kappa_g]\!] = 0$



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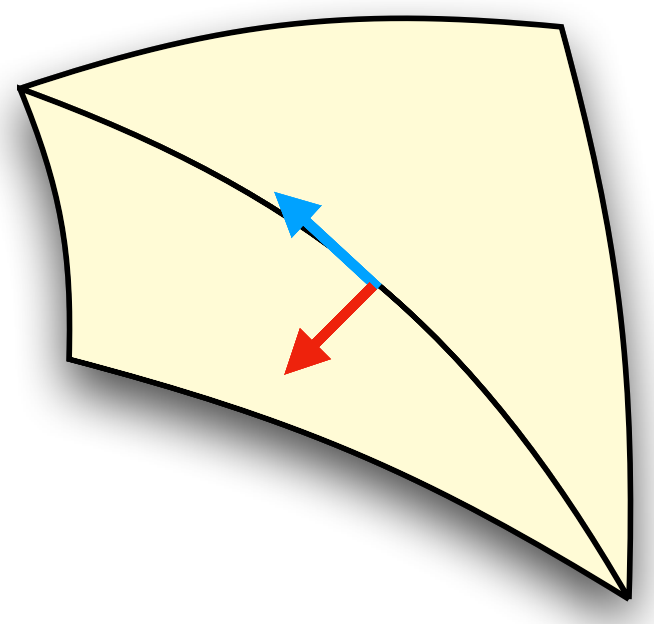

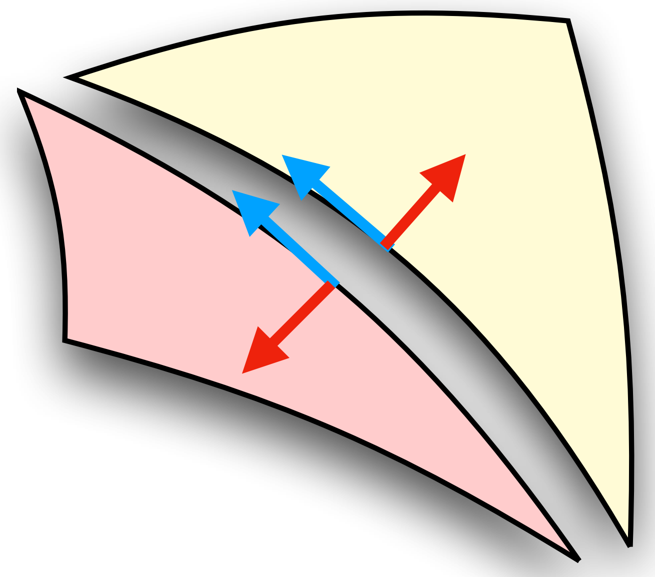

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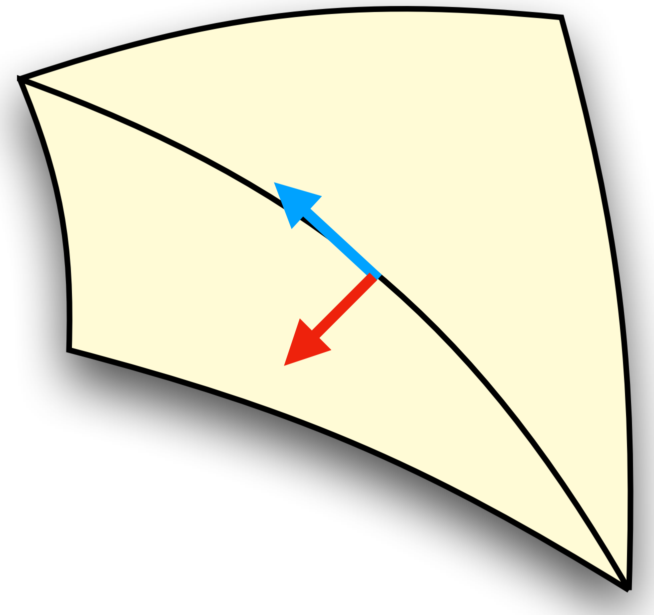
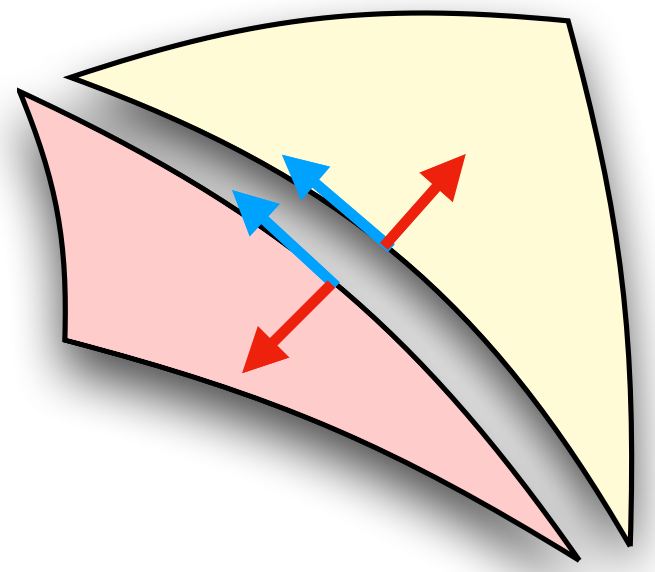
- $\omega_T = \sqrt{\det g}, \omega_E = \sqrt{g(\tau,\tau)}$

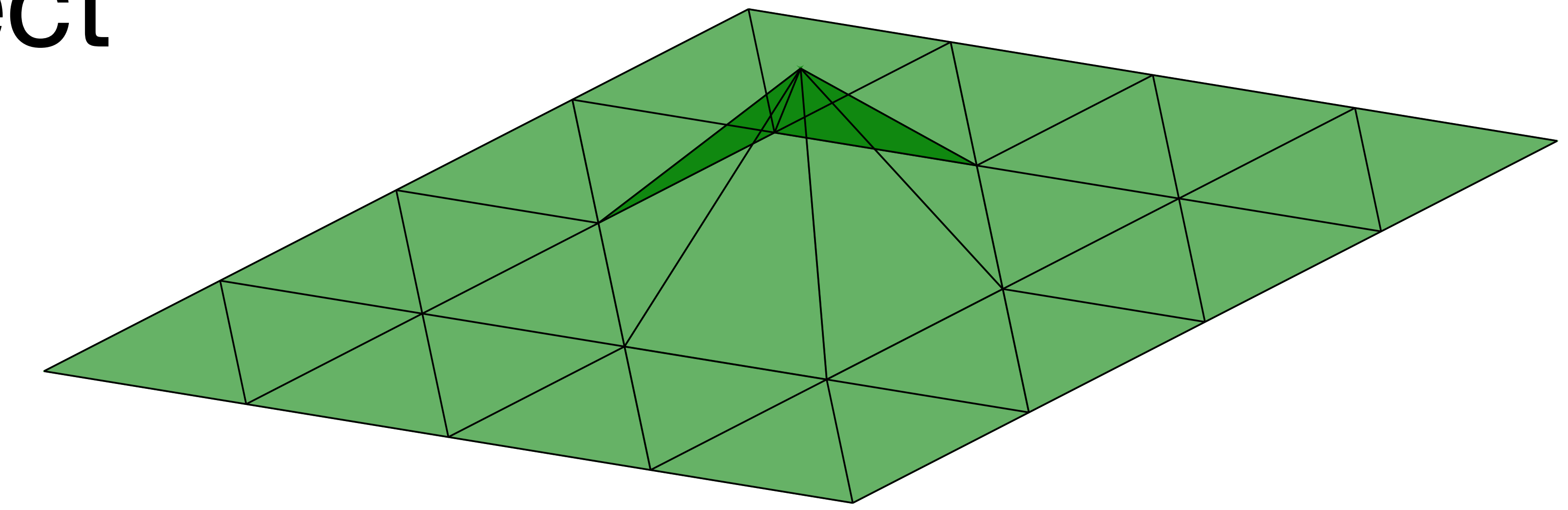


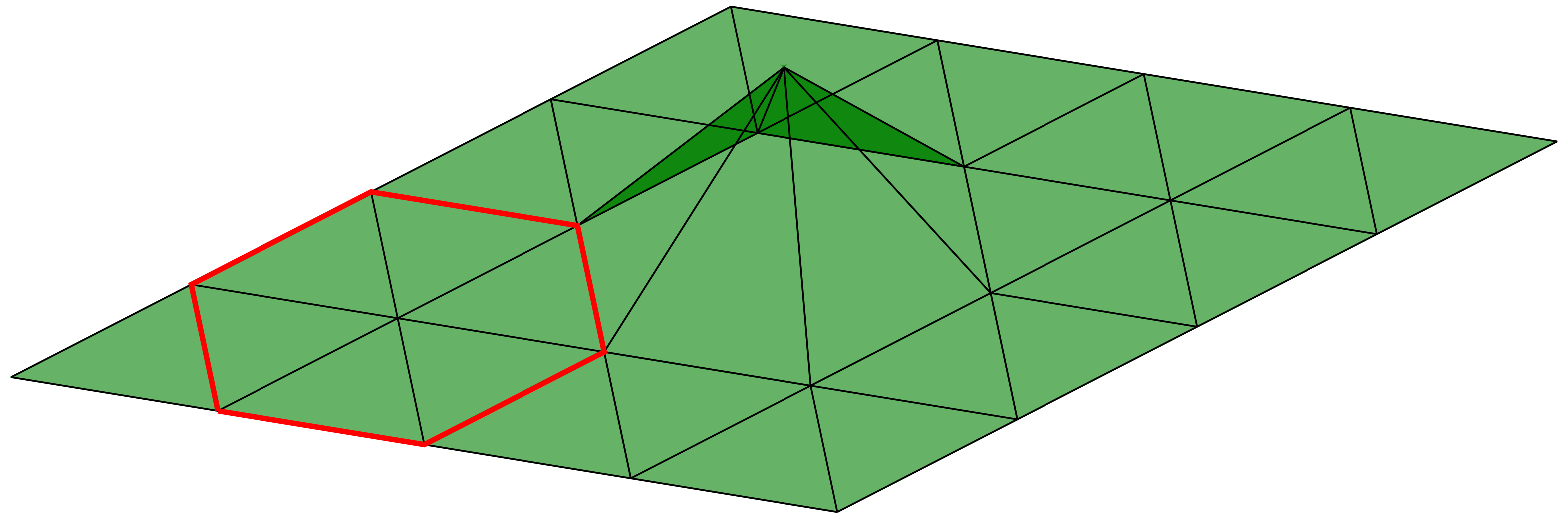


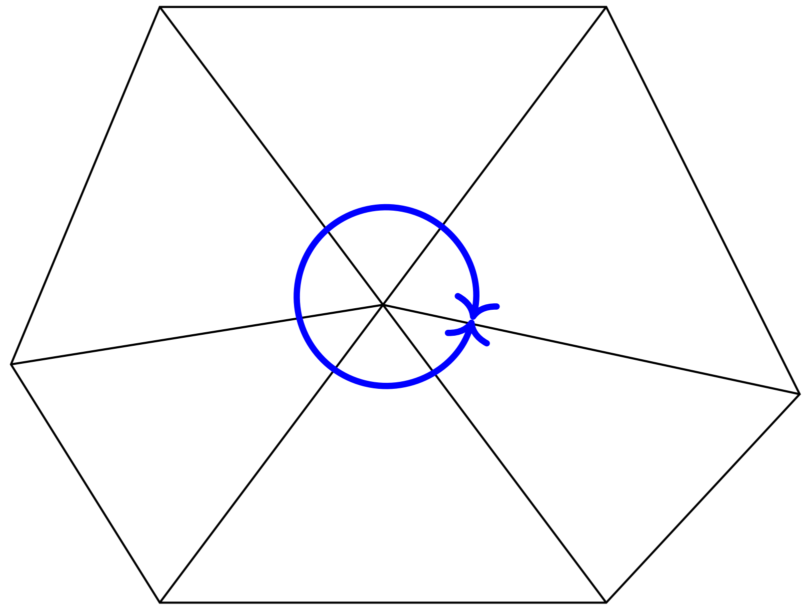
$$(K\omega)_{\text{dist}} = \sum_T K_T \omega_T + \sum_E [\![\kappa_g]\!] \omega_E \delta_E$$

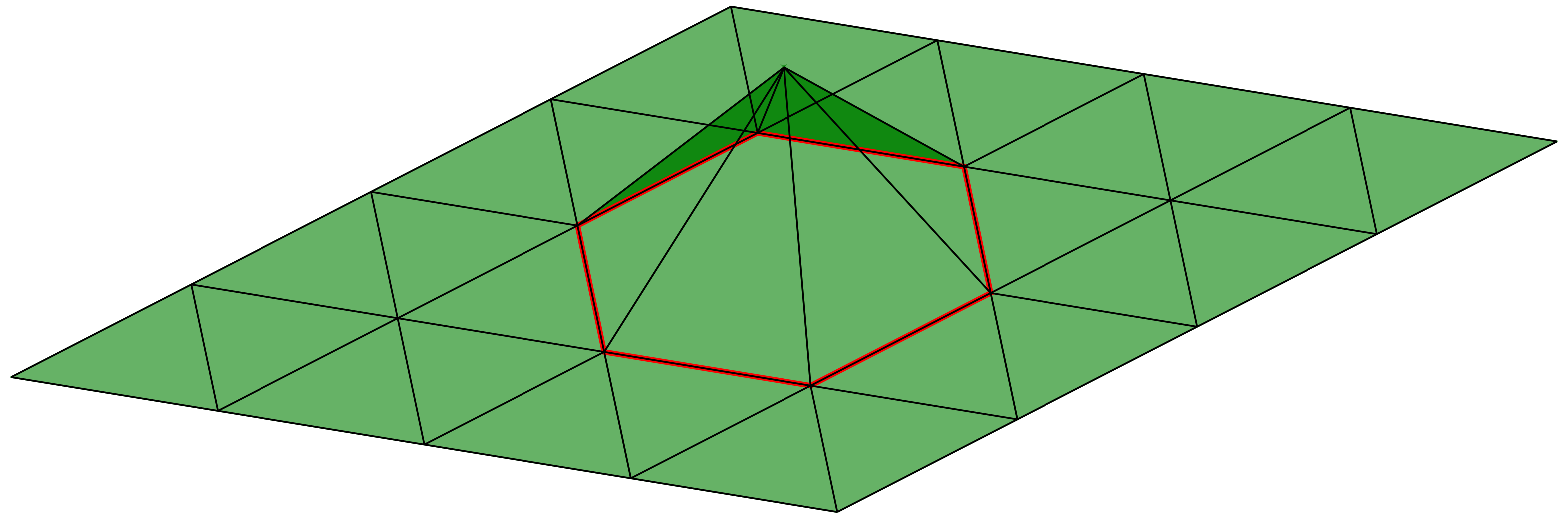
Distributional Gauss curvature Part 2

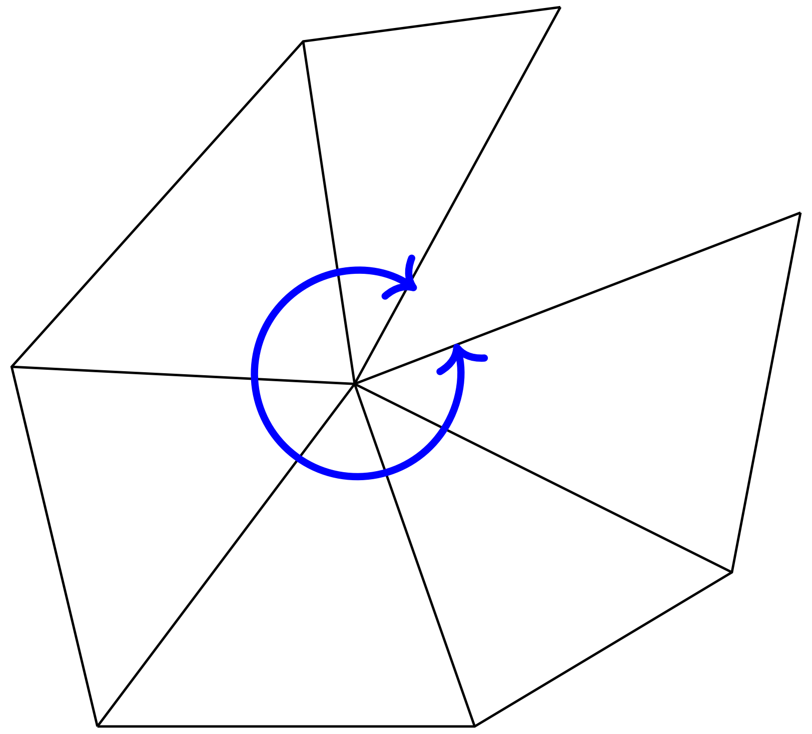
- Angle defect











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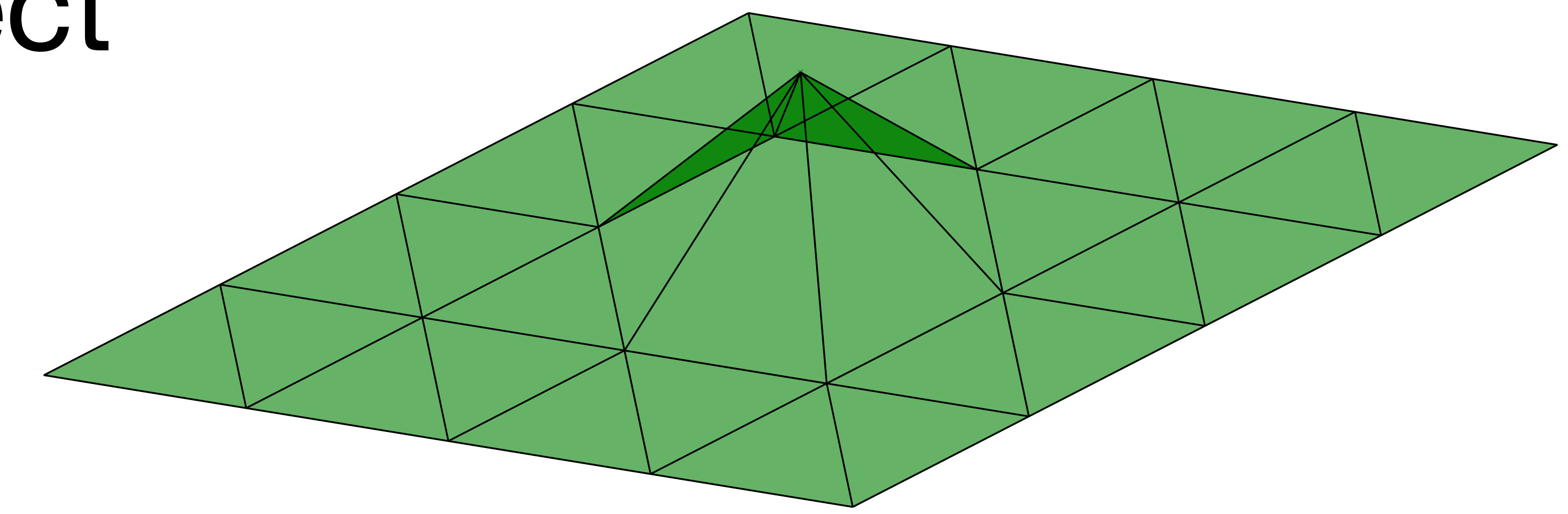

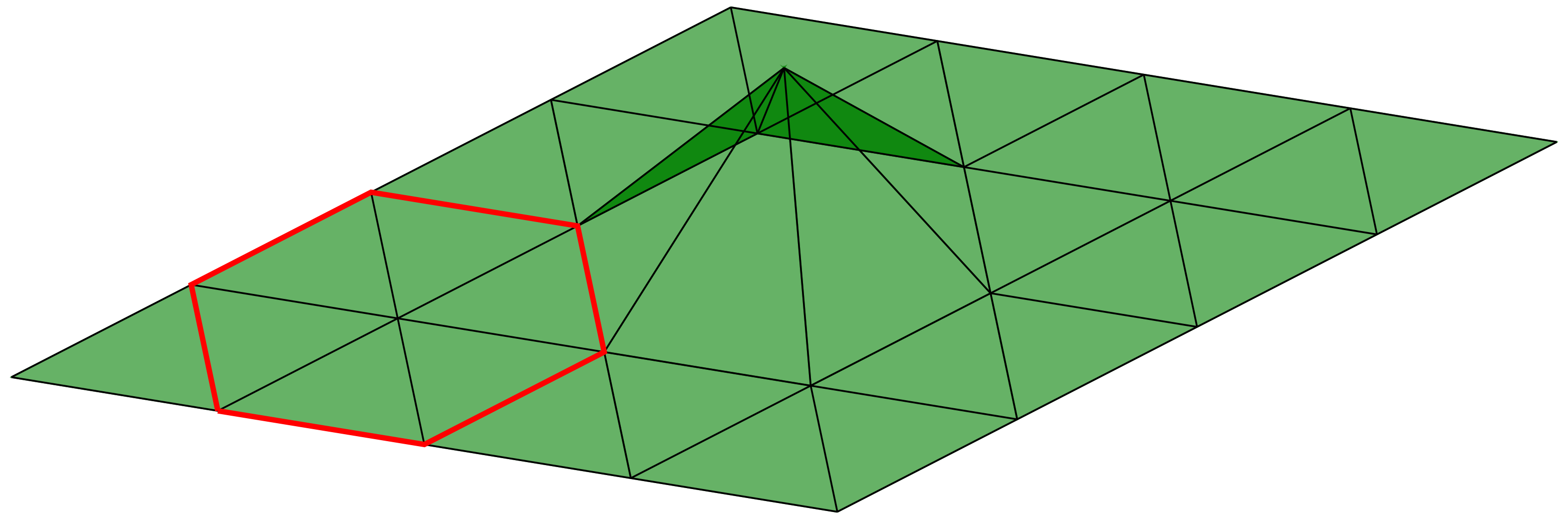

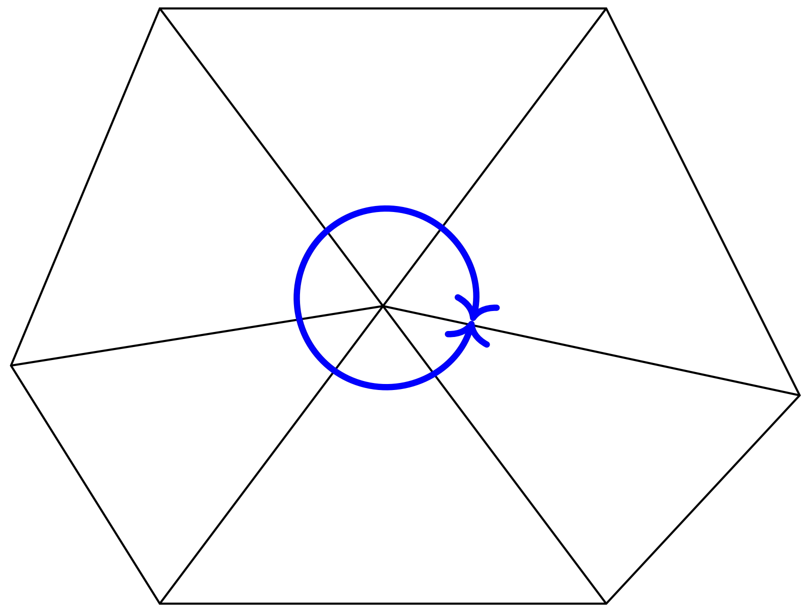

- Sum inner angles $\neq 2\pi \Rightarrow$ curvature
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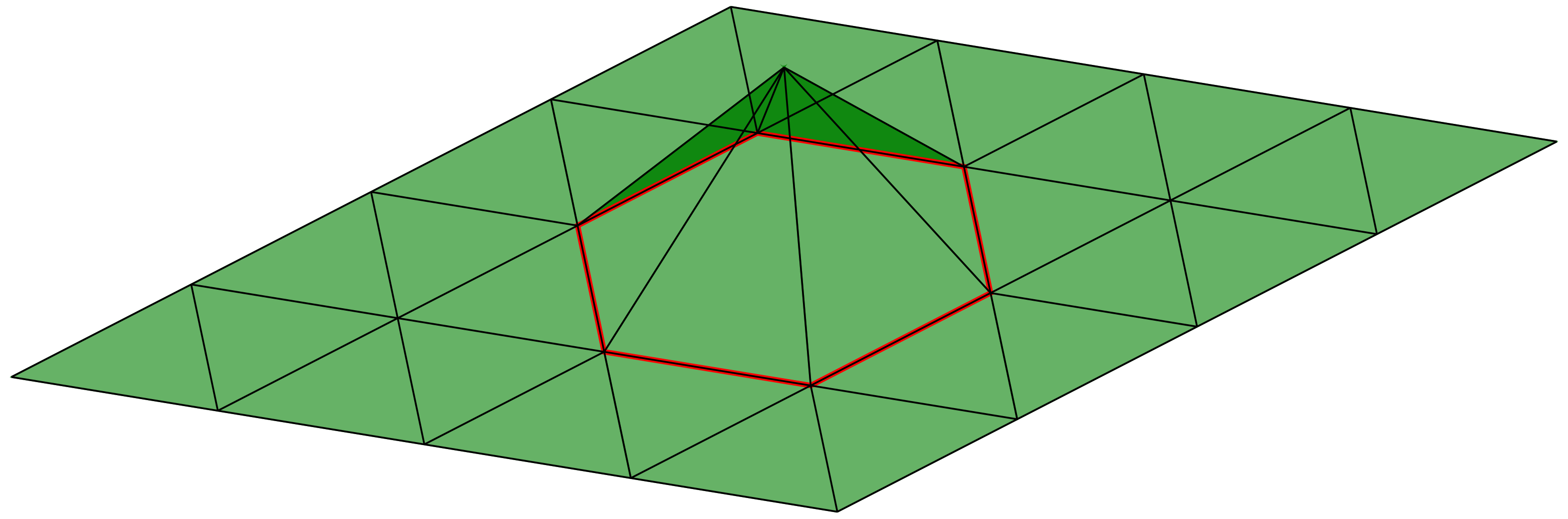

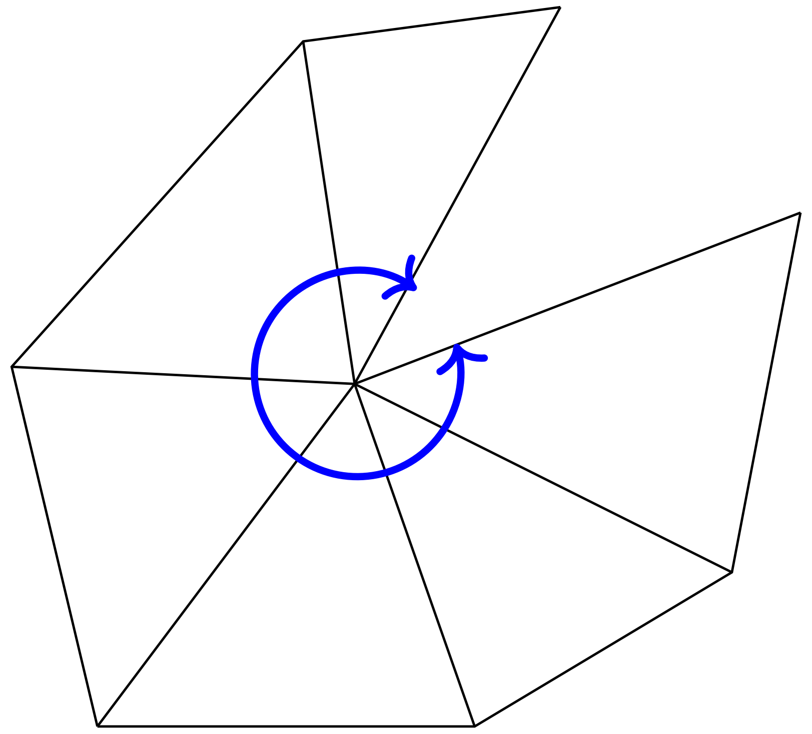

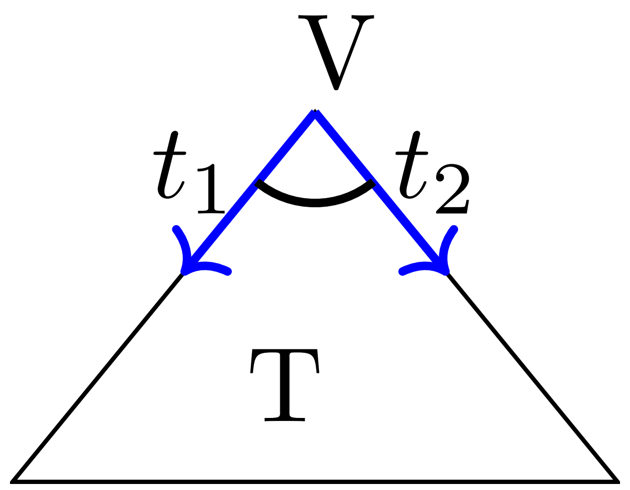

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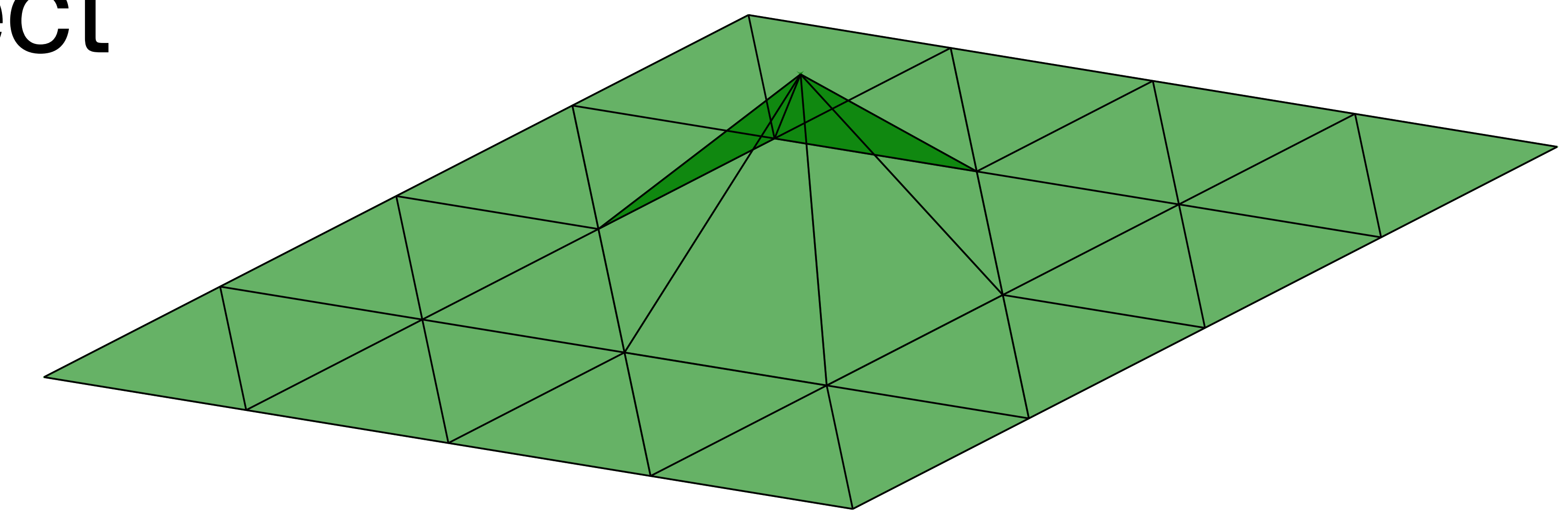

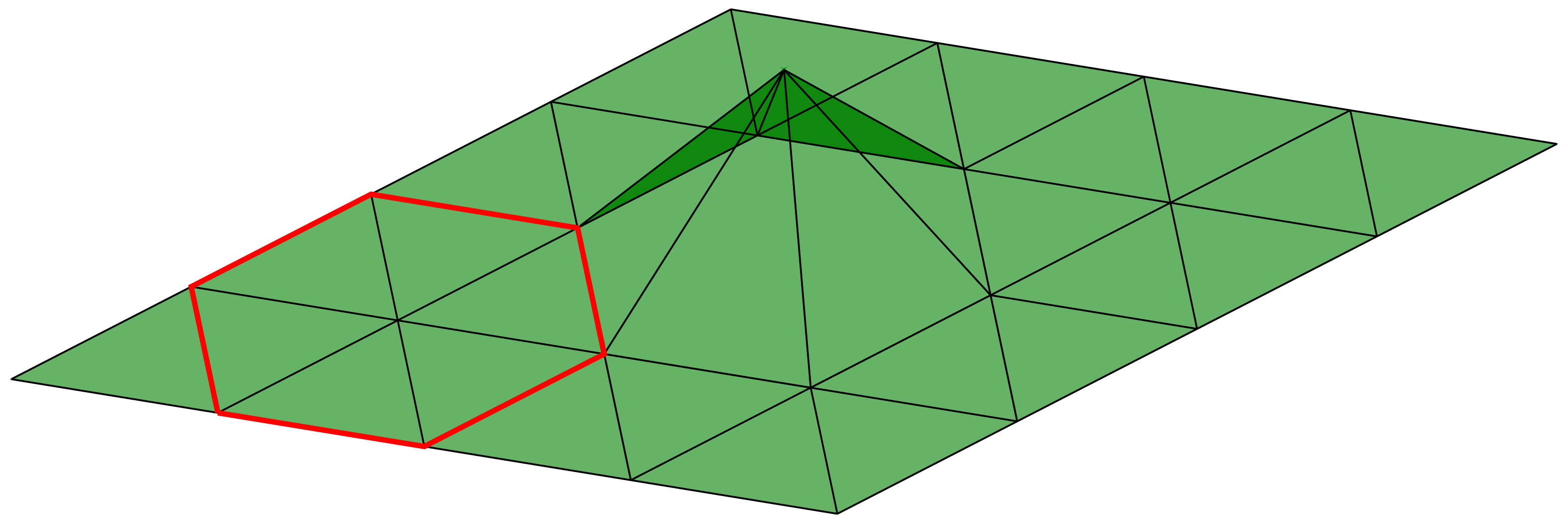

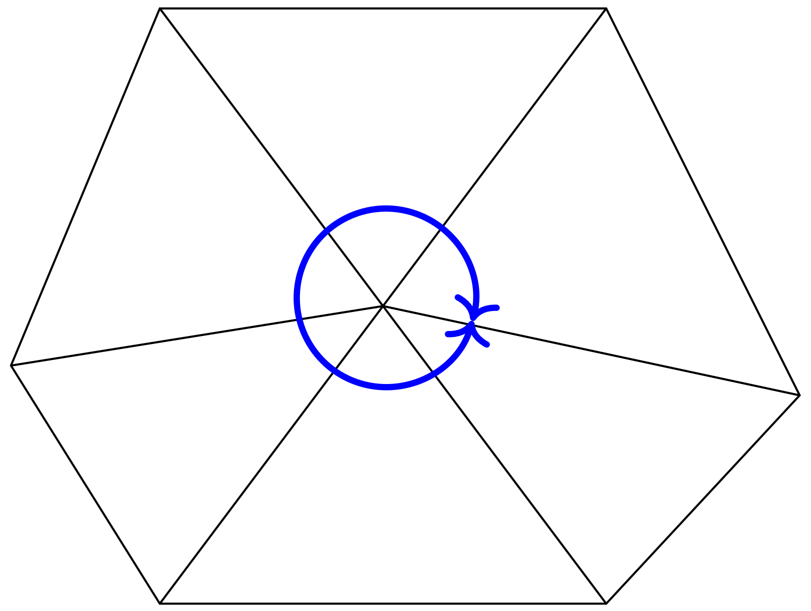

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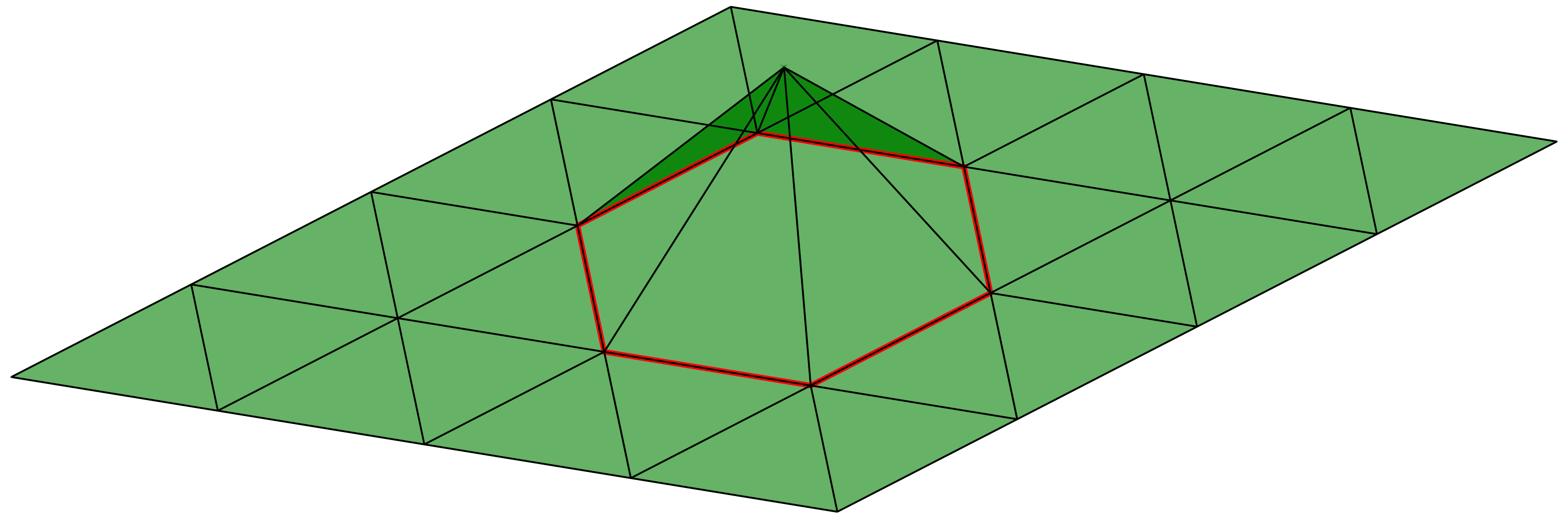

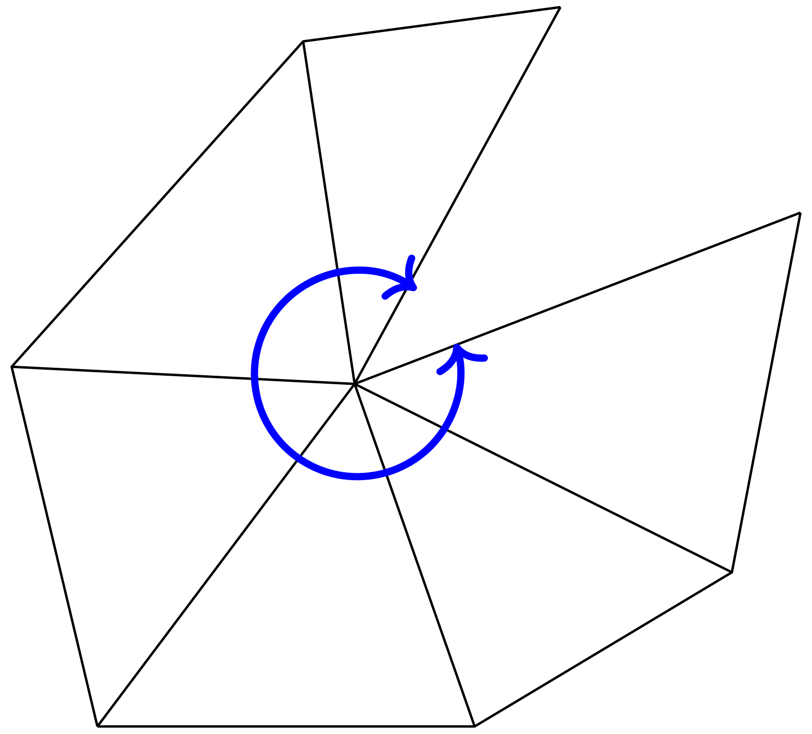

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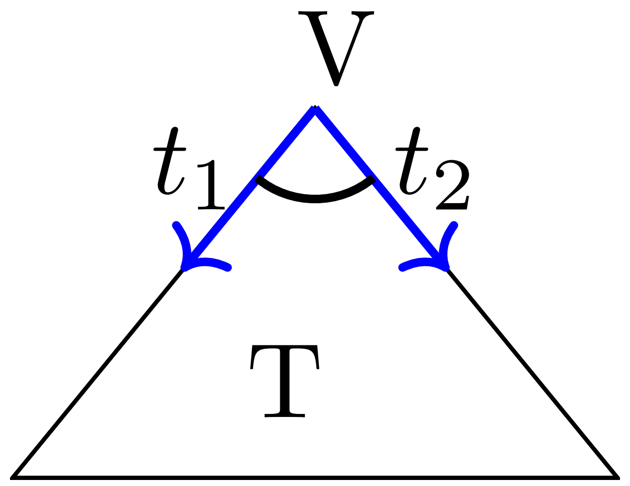

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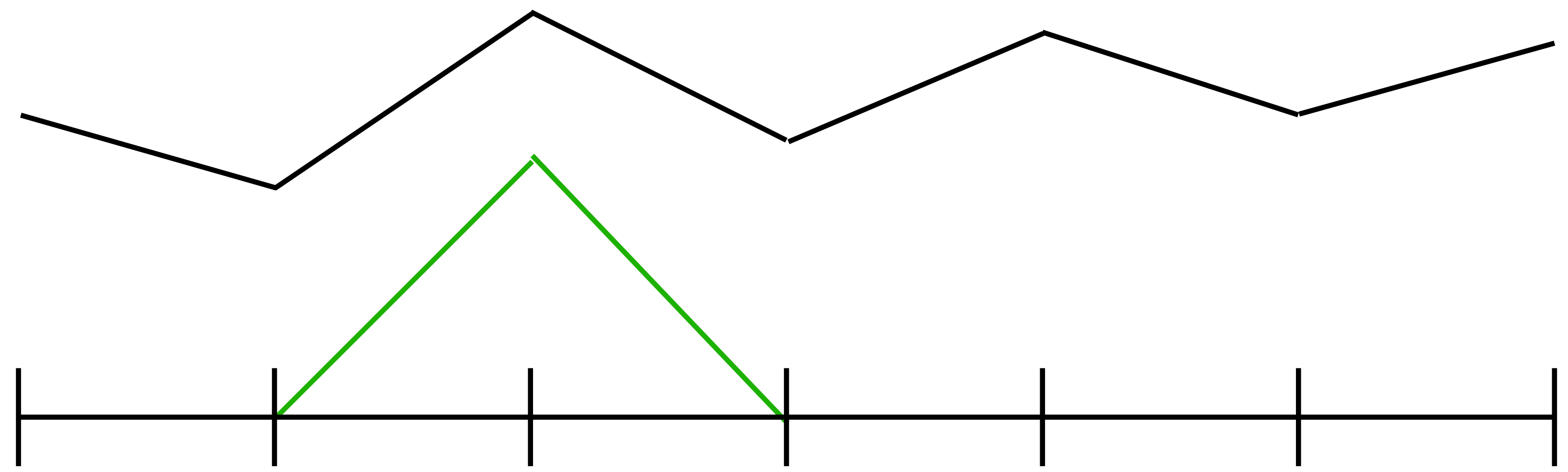
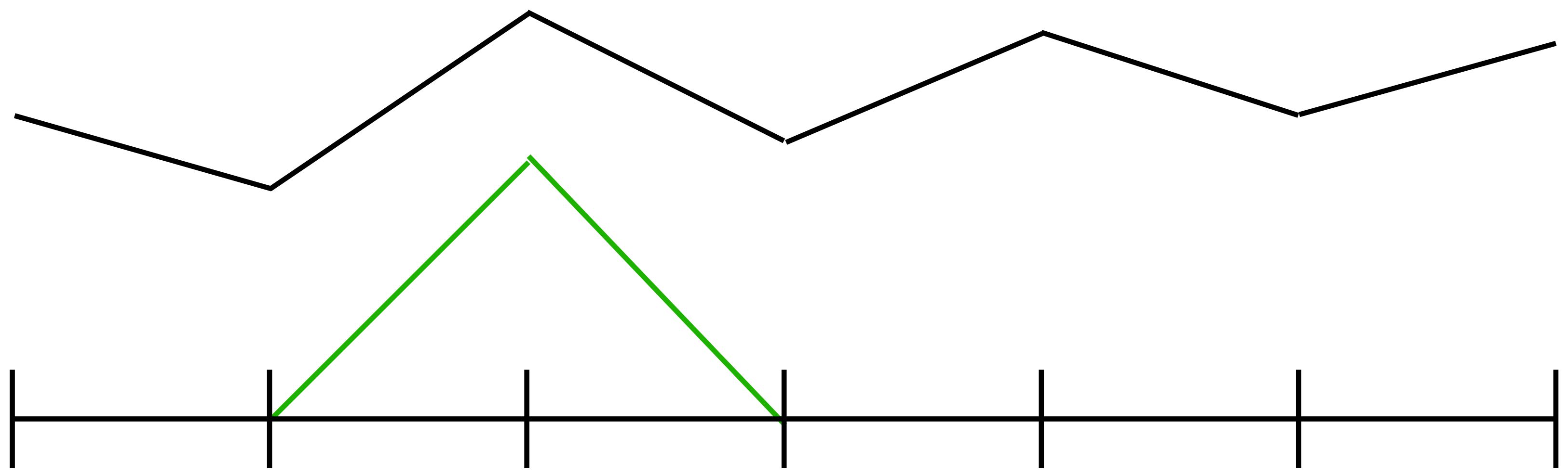
Distributional Gauss curvature



- Lagrange finite element space $V_h^k = \{u_h \in \mathscr{P}^k(\mathscr{T}) \ : \ u_h \text{ is continuous}\}$

Distributional Gauss curvature



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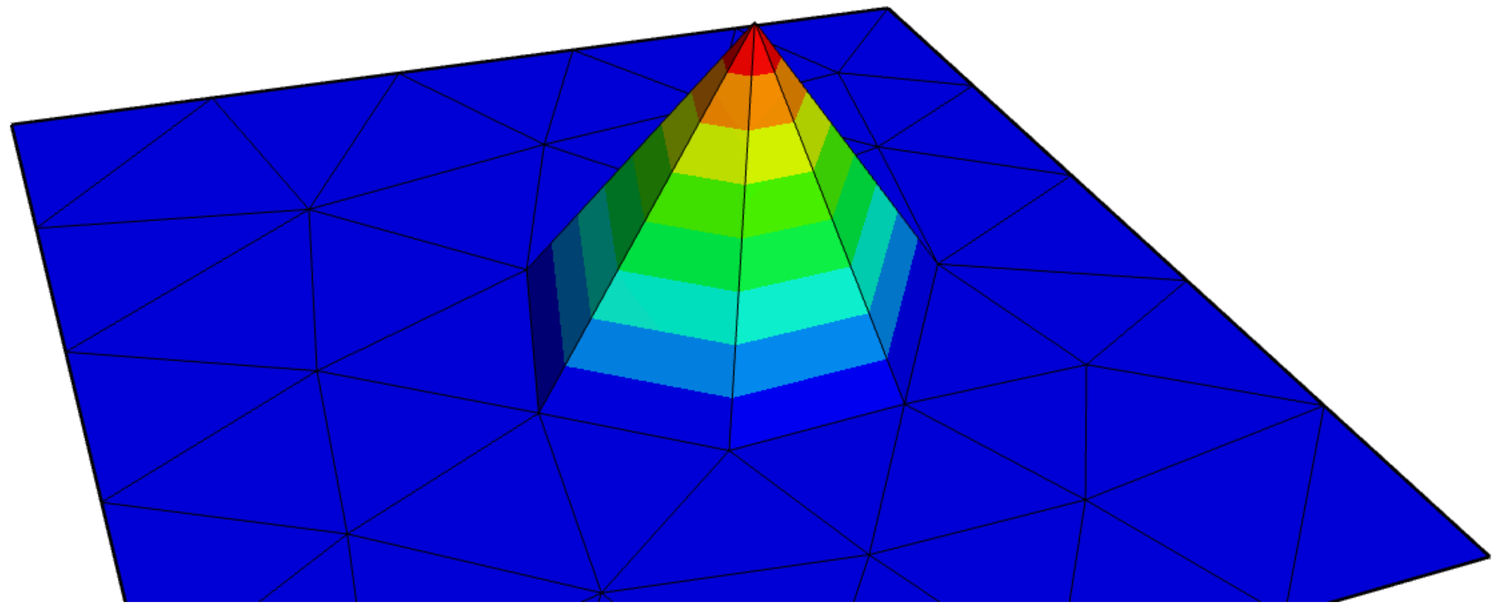
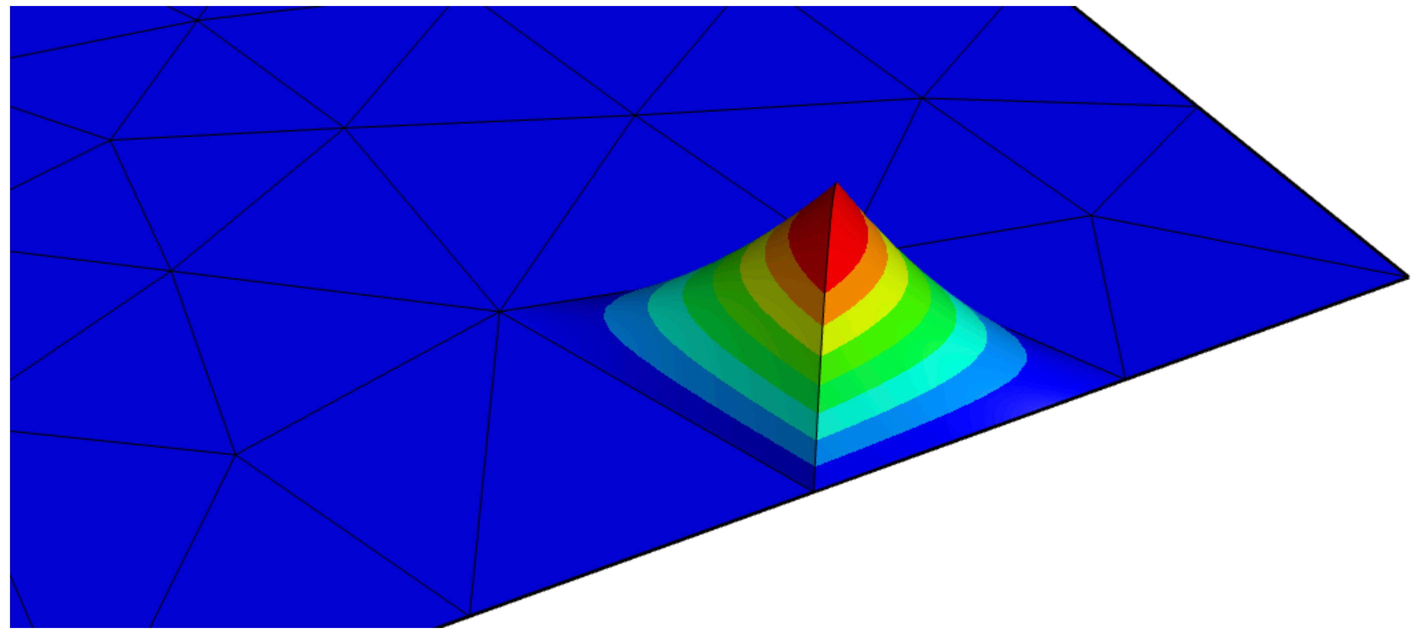
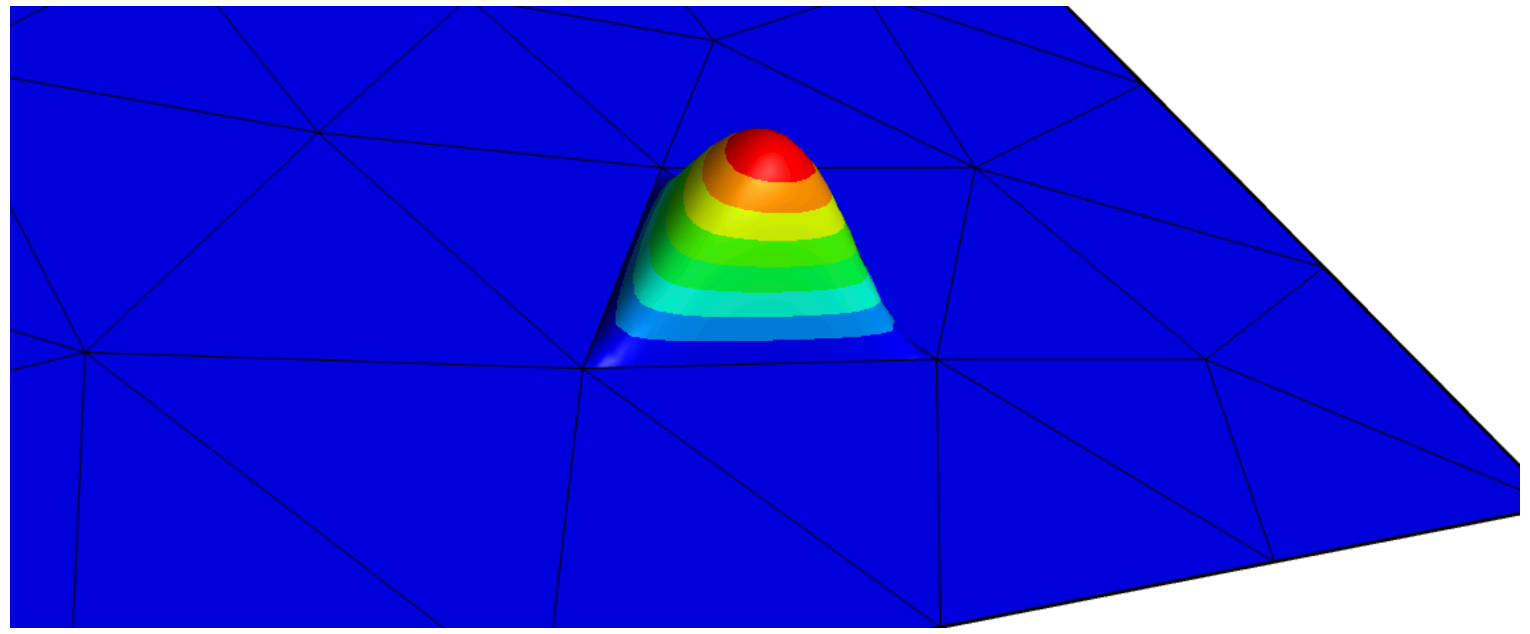

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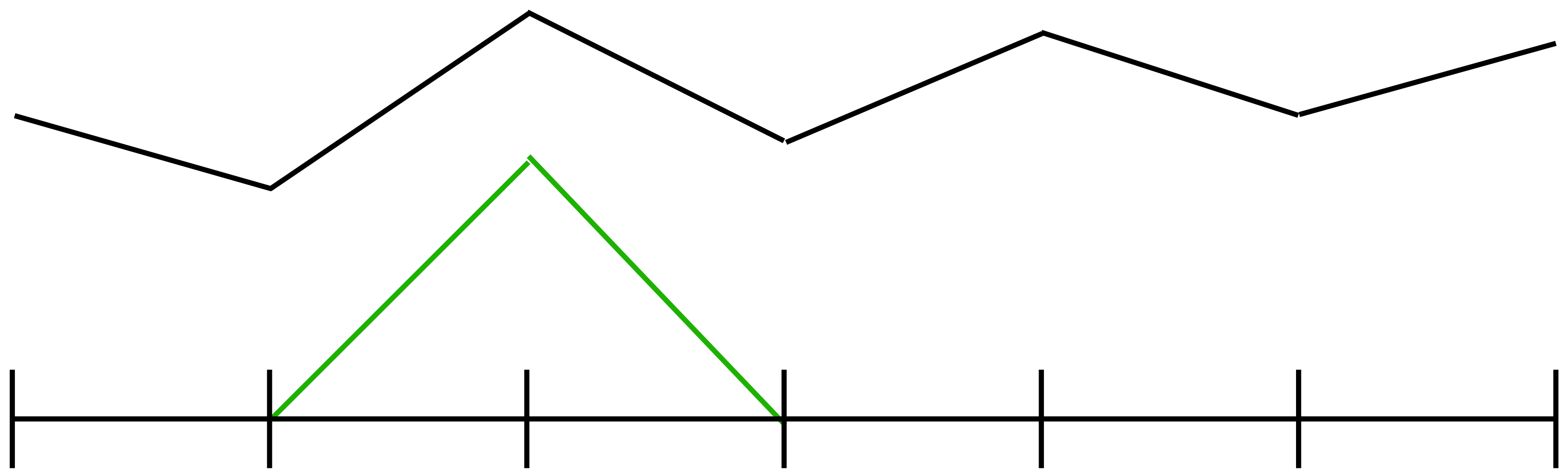

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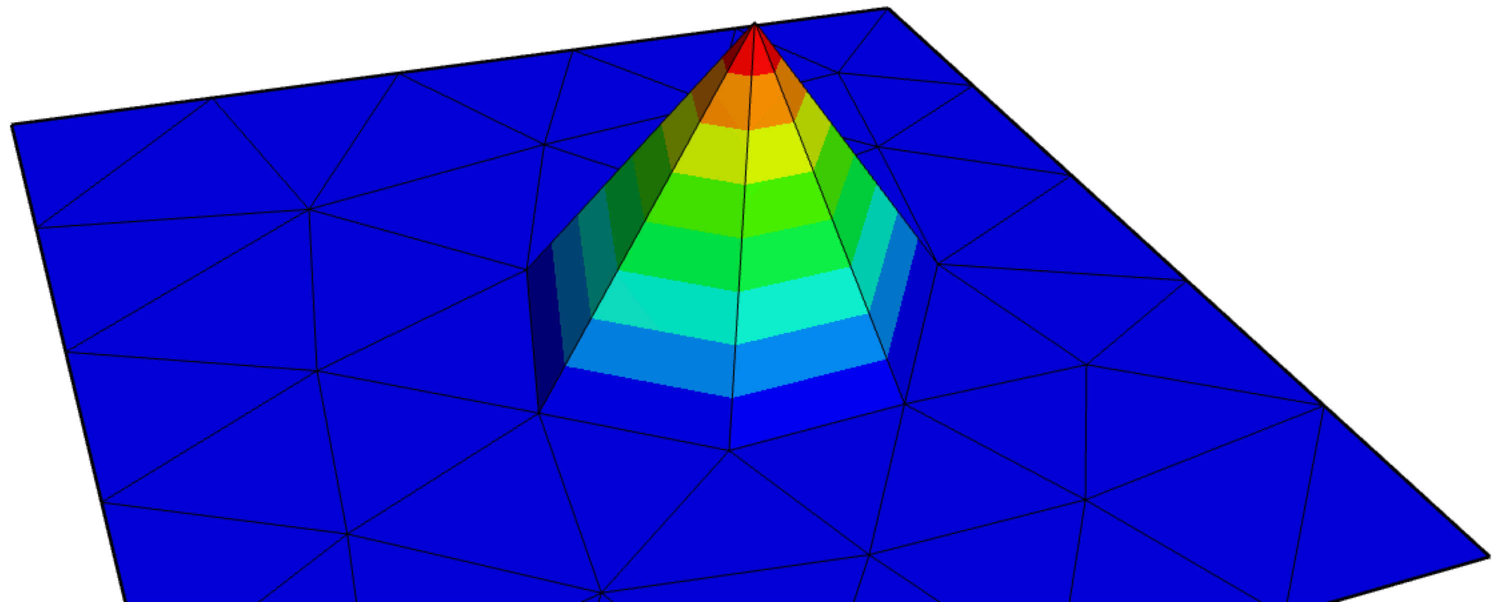

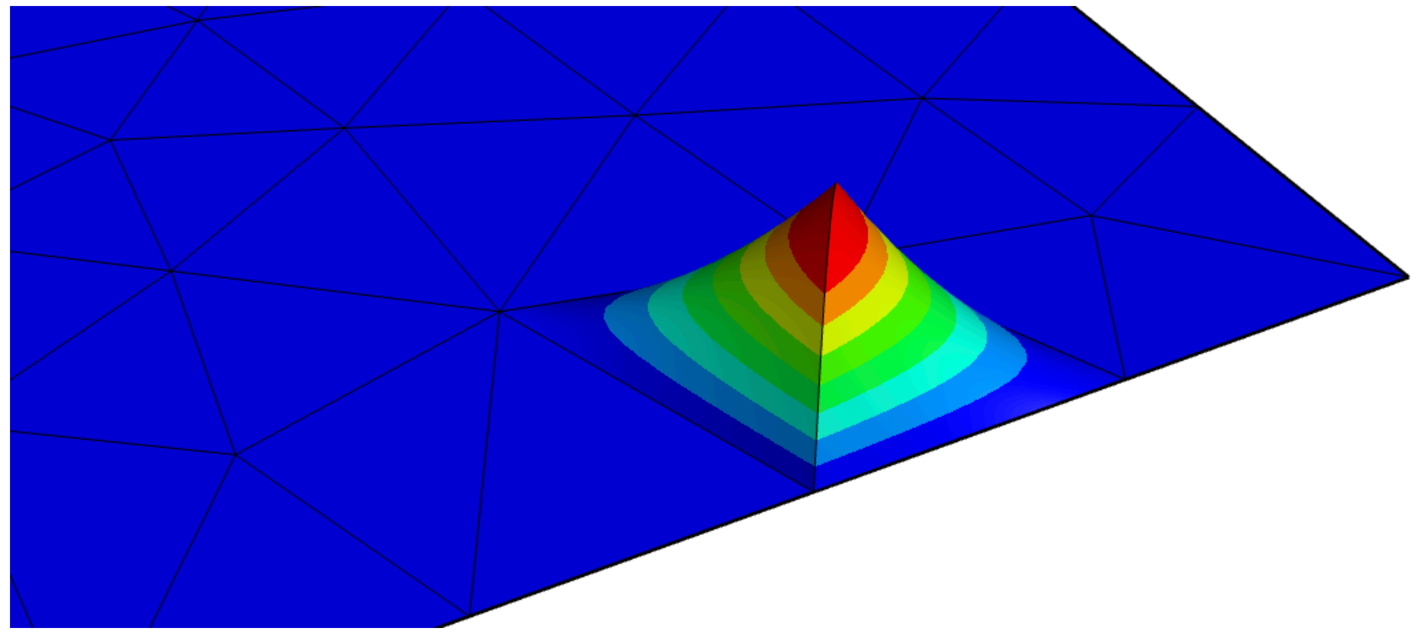

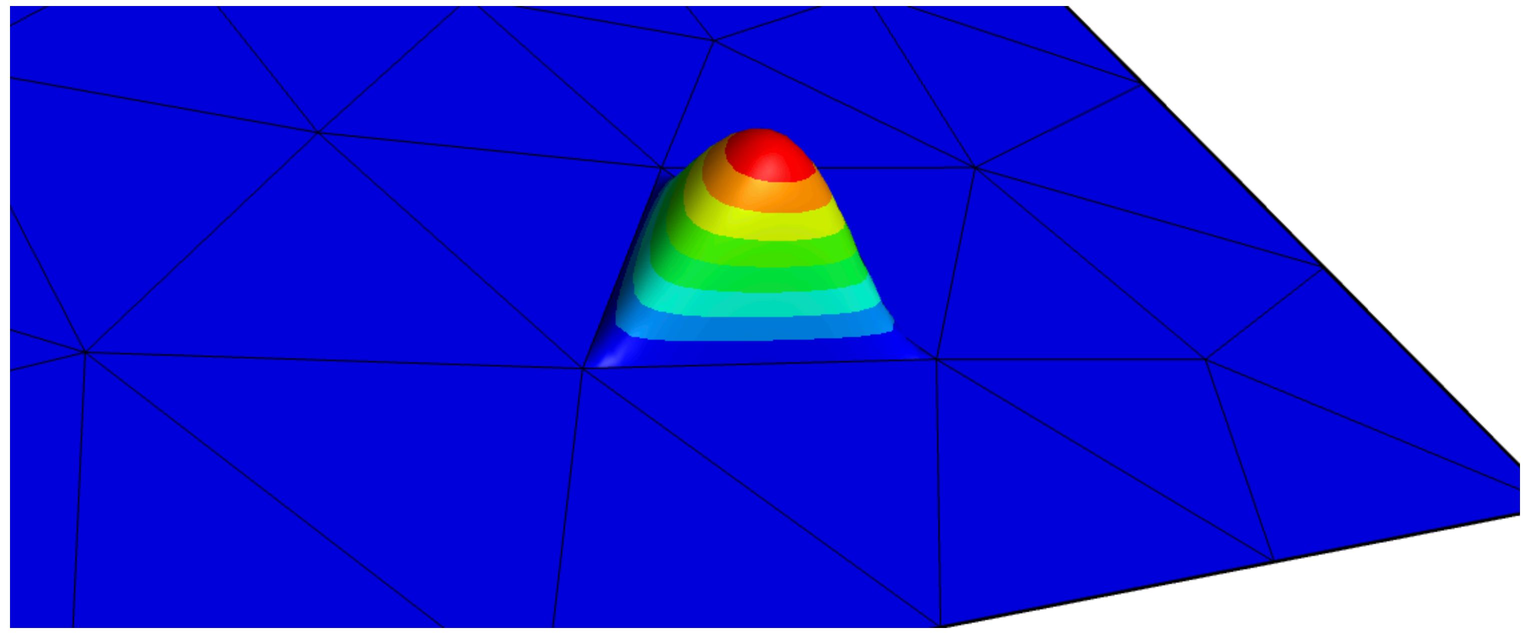

- Lagrange finite elements as test functions

$$(K\omega)_{\mathrm{dist}}(u_h) = \sum_T \int_T K|_T u_h \omega_T + \sum_E \int_E [\![\kappa_g]\!] u_h \omega_E + \sum_V \sphericalangle_V(g)\, u_h(V)$$

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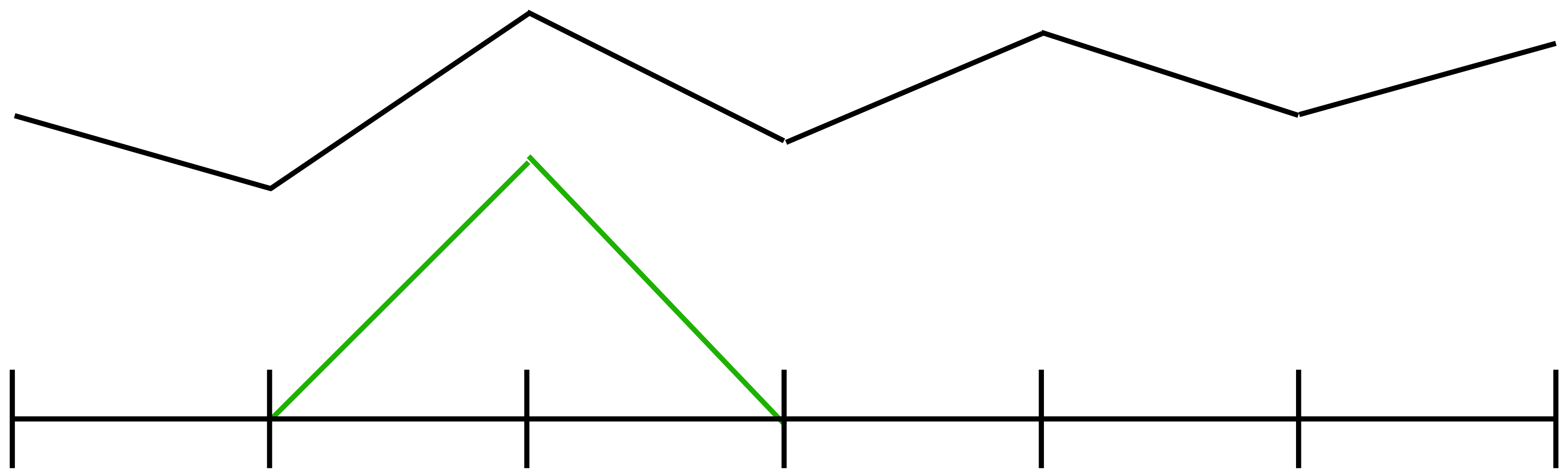

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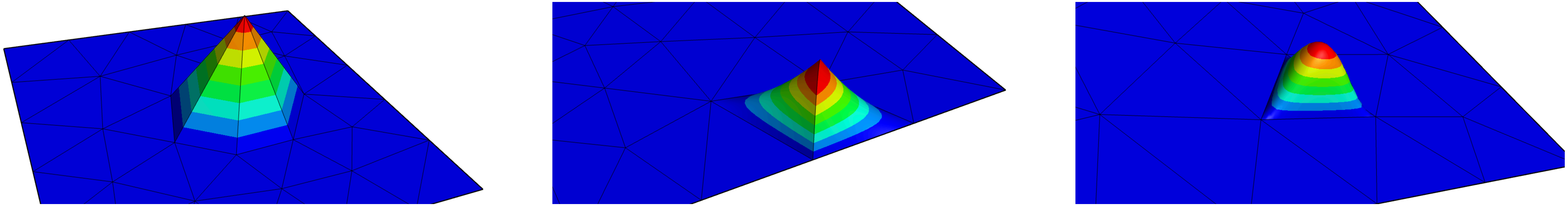

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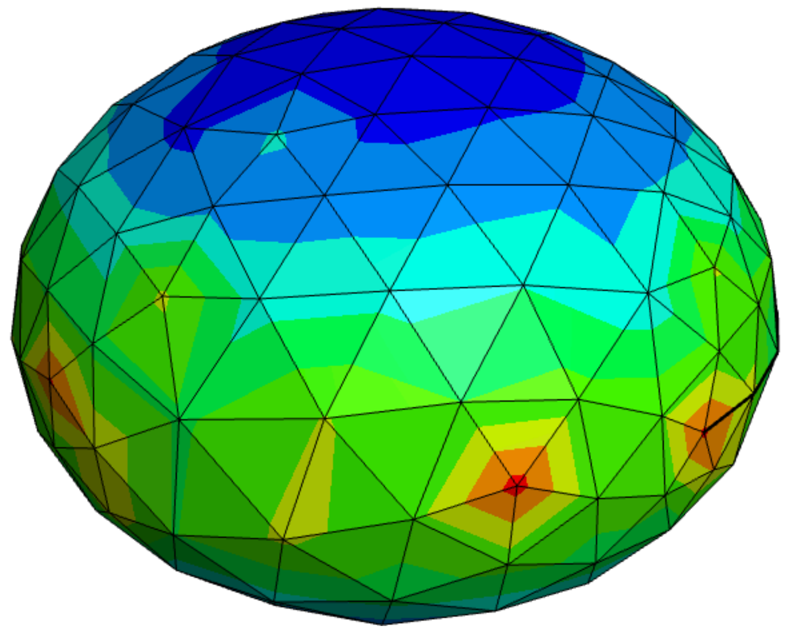
- Let $g_h \in \mathrm{Reg}_h^k$. Find the discrete L^2-Riesz representative $K_h \in V_h^{k+1}$ such that for all $u_h \in V_h^{k+1}$

$$\int_\Omega K_h u_h \omega = (K\omega)_{\mathrm{dist}}(u_h)\,.$$

Example: Gauss curvature of surface



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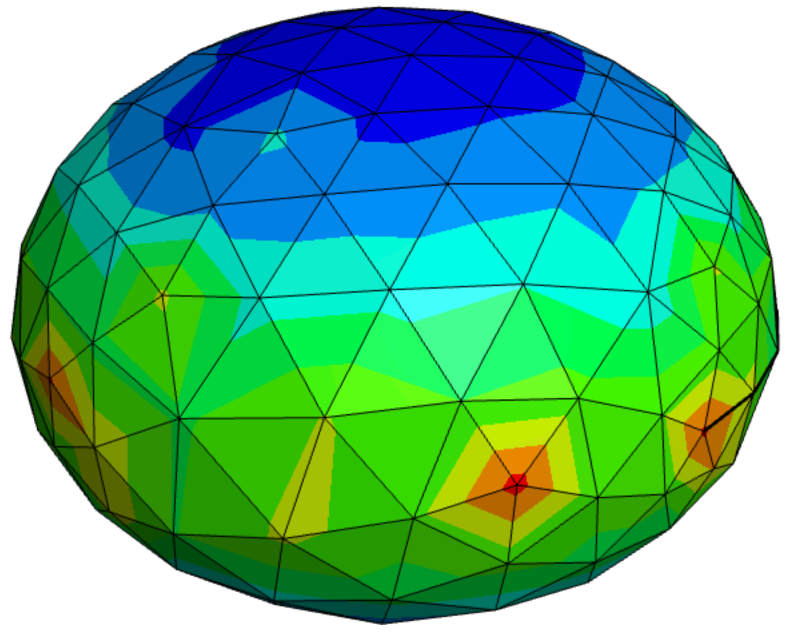

NGSolve

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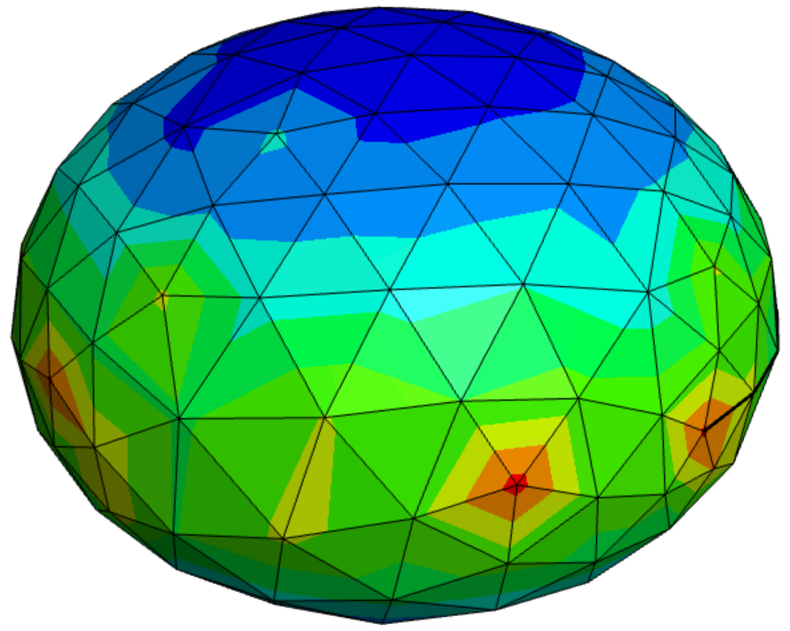

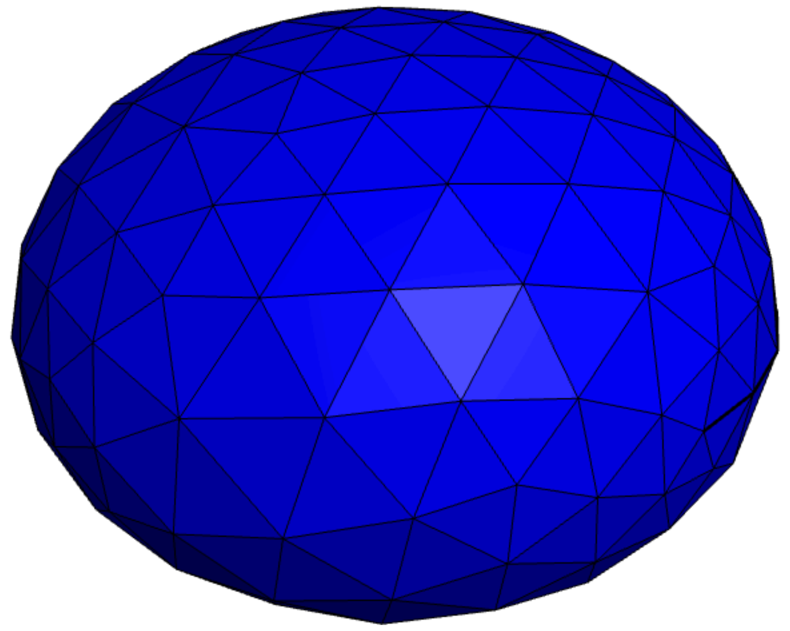

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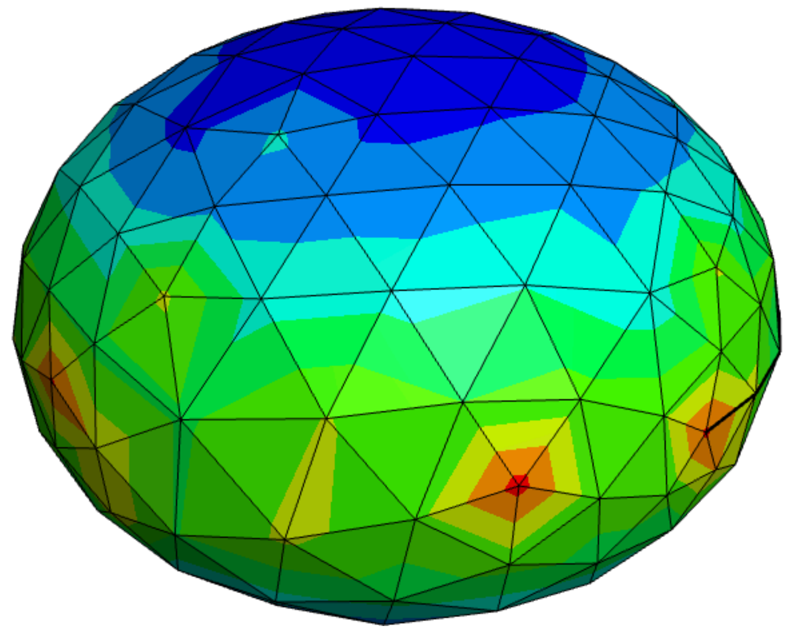

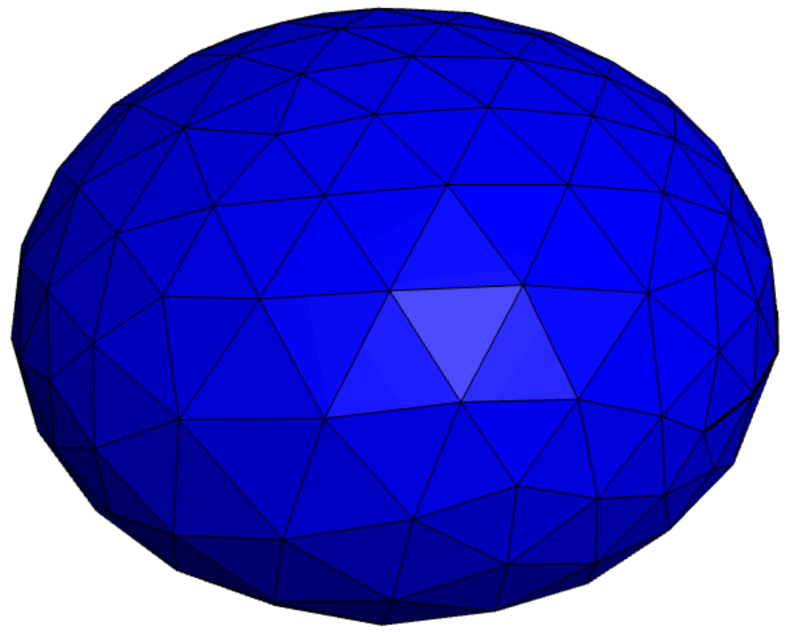

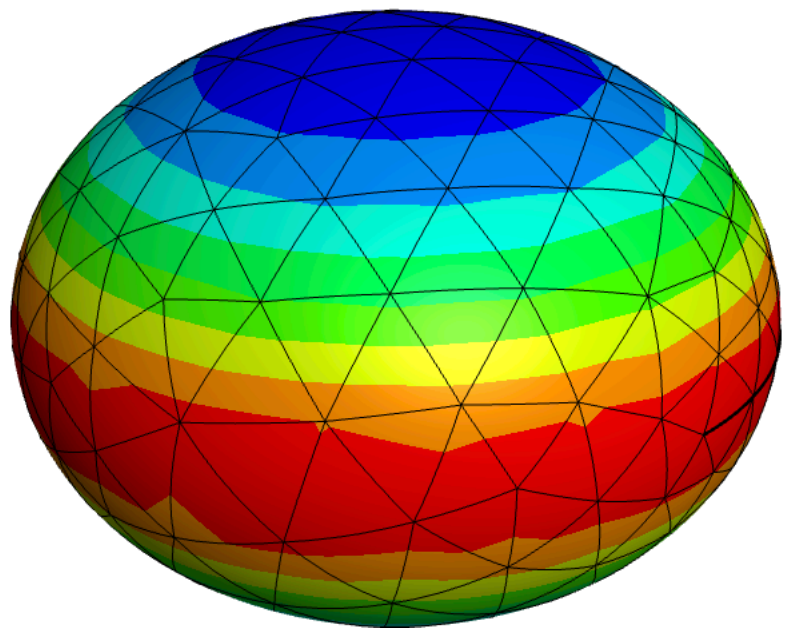

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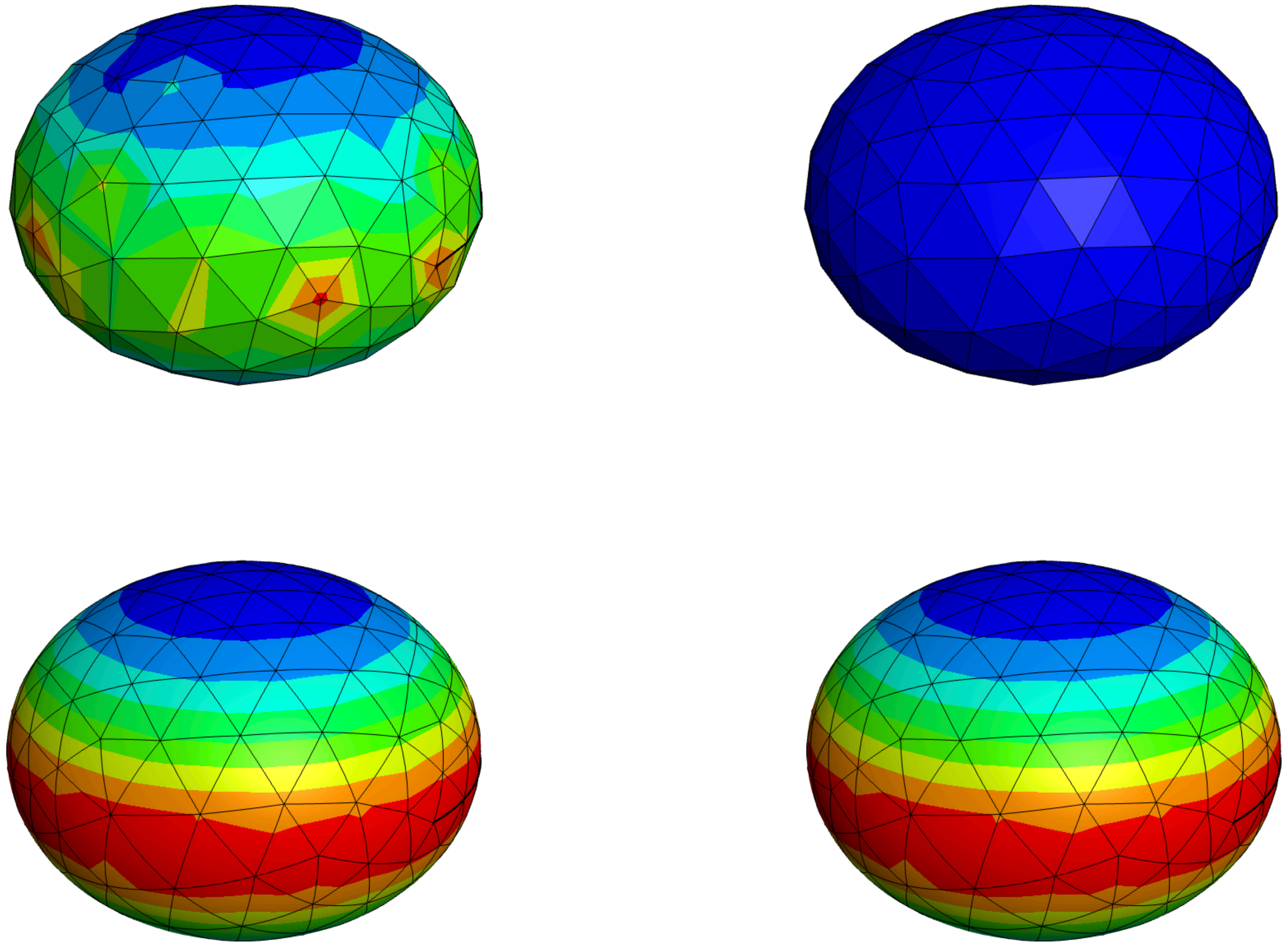

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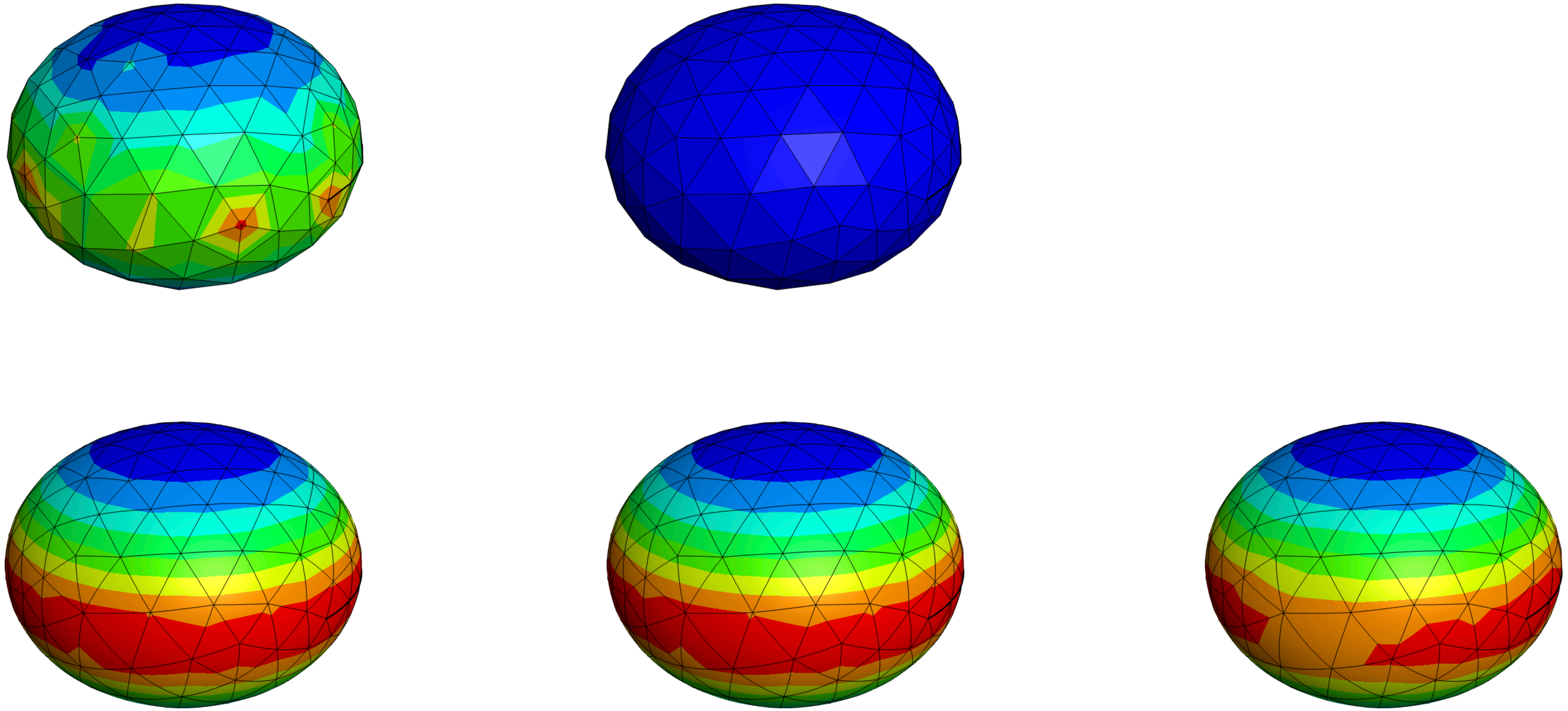

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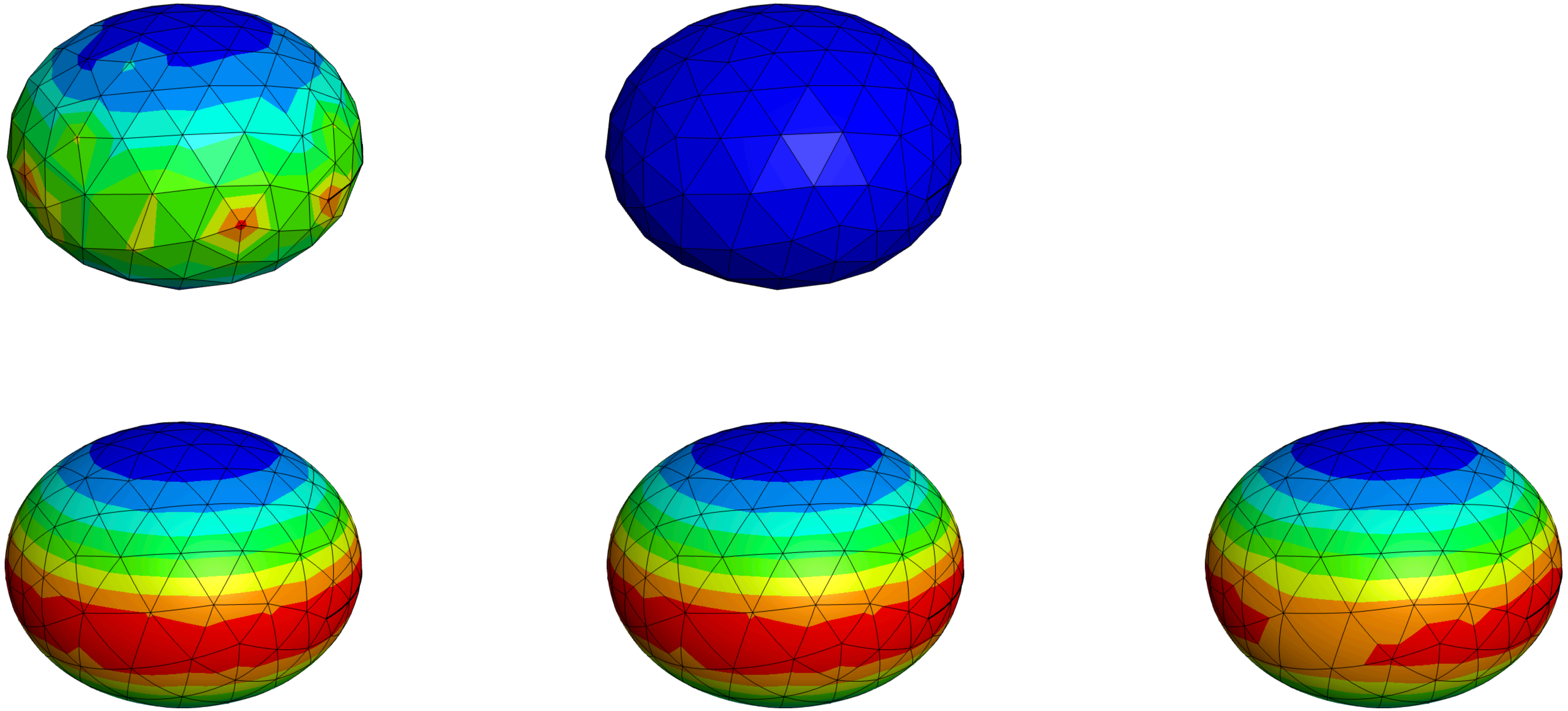

NGSolve

- Removing distributional terms yields reduced accuracy

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Numerical analysis: integral representation

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Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, SIAM J. Numer. Anal., 2020.

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- Estimate integrand

$$|\langle \mathrm{div}_{g(t)}\mathrm{div}_{g(t)}(\mathbb{S}_{g(t)}\sigma)\omega_{g(t)}, u_h\rangle| \leq C\|g - g_h\|_{L^2}\|u_h\|_{H^2}$$

Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, SIAM J. Numer. Anal., 2020.

Convergence results

Theorem: Let $k \in \mathbb{N}_0$, $g \in H^{k+1}(\Omega)$, and $g_h \in \mathrm{Reg}_h^k$ be a sequence of Regge metrics approximating g optimally $\|g_h - g\|_{L^2} \leq Ch^{k+1}\|g\|_{H^{k+1}}$. Then there holds for the lifted Gauss curvature $K_h(g_h) \in V_h^{k+1}$ for sufficiently small h

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Corollary: There holds for $1 \leq l \leq k$

$$\|K_h(g_h) - K(g)\|_{L^2} \leq Ch^{k-1}(\|g\|_{H^{k+1}} + \|K(g)\|_{H^{k-1}}),$$

$$\|K_h(g_h) - K(g)\|_{H_h^l} \leq Ch^{k-l-1}(\|g\|_{H^{k+1}} + \|K(g)\|_{H^{k-1}})\,.$$

$$\|\sigma\|_{H_h^l}^2 = \sum_T \|\sigma\|_{H^l(T)}^2$$

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Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, SMAI J. Comput. Math., 2023.

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One convergence rate more. It can further be increased depending on the choice of polynomial spaces.

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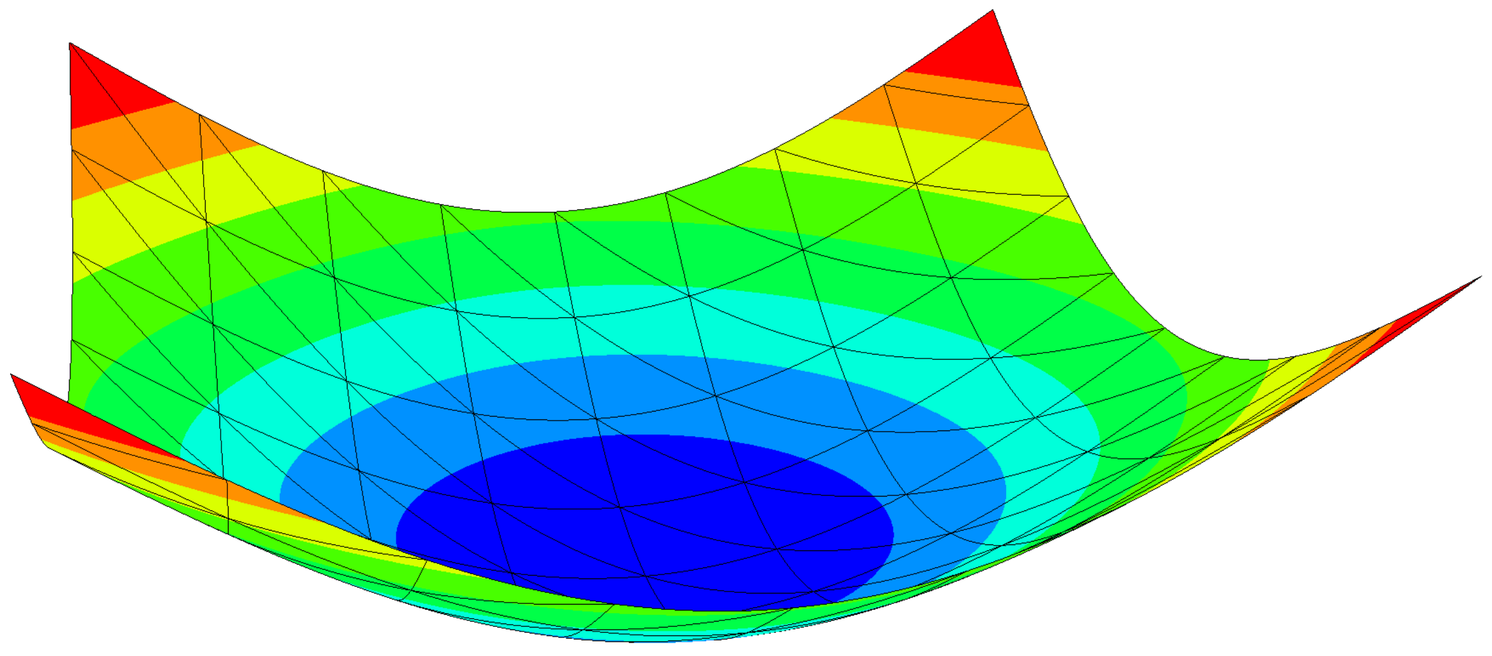
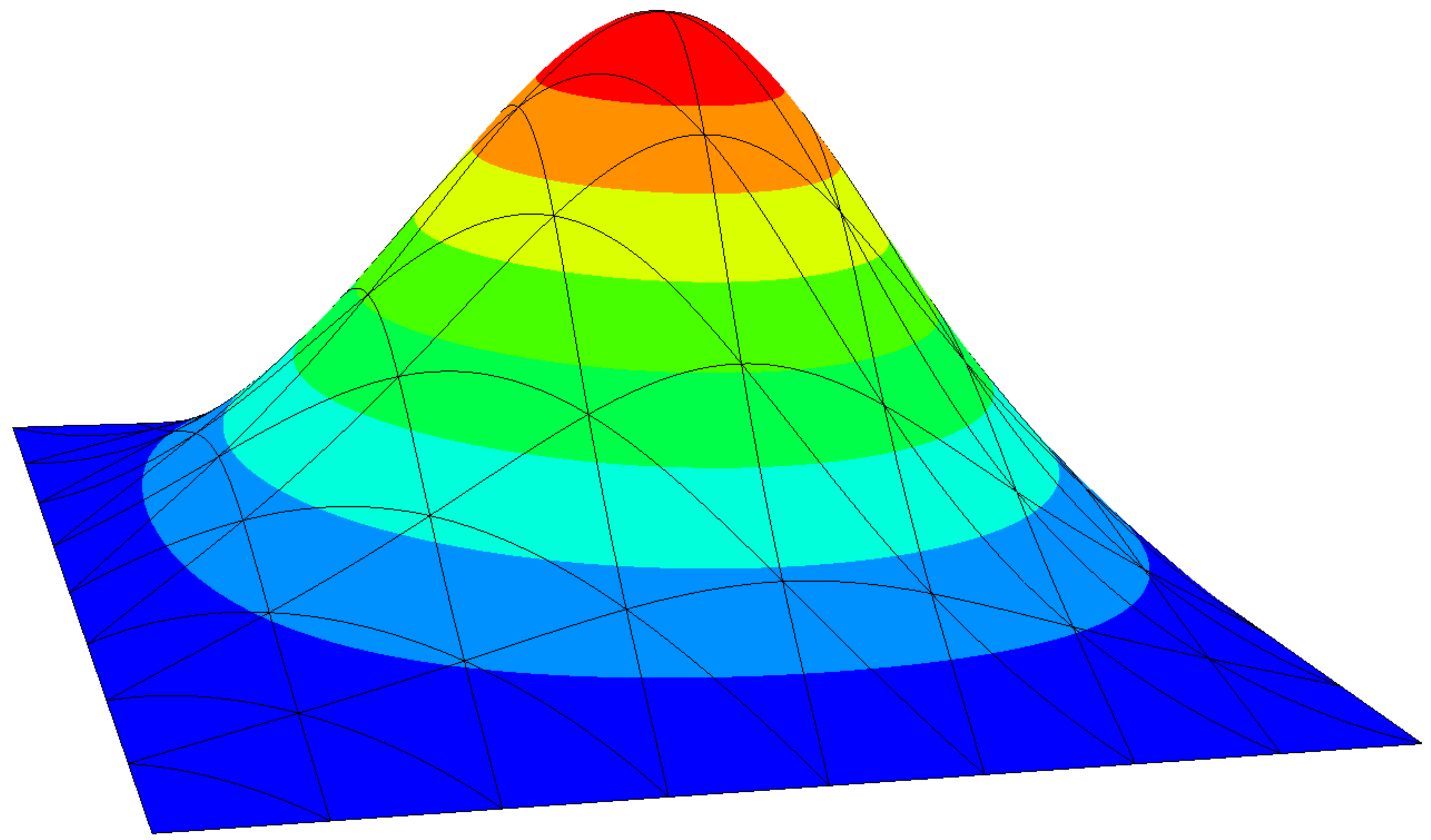
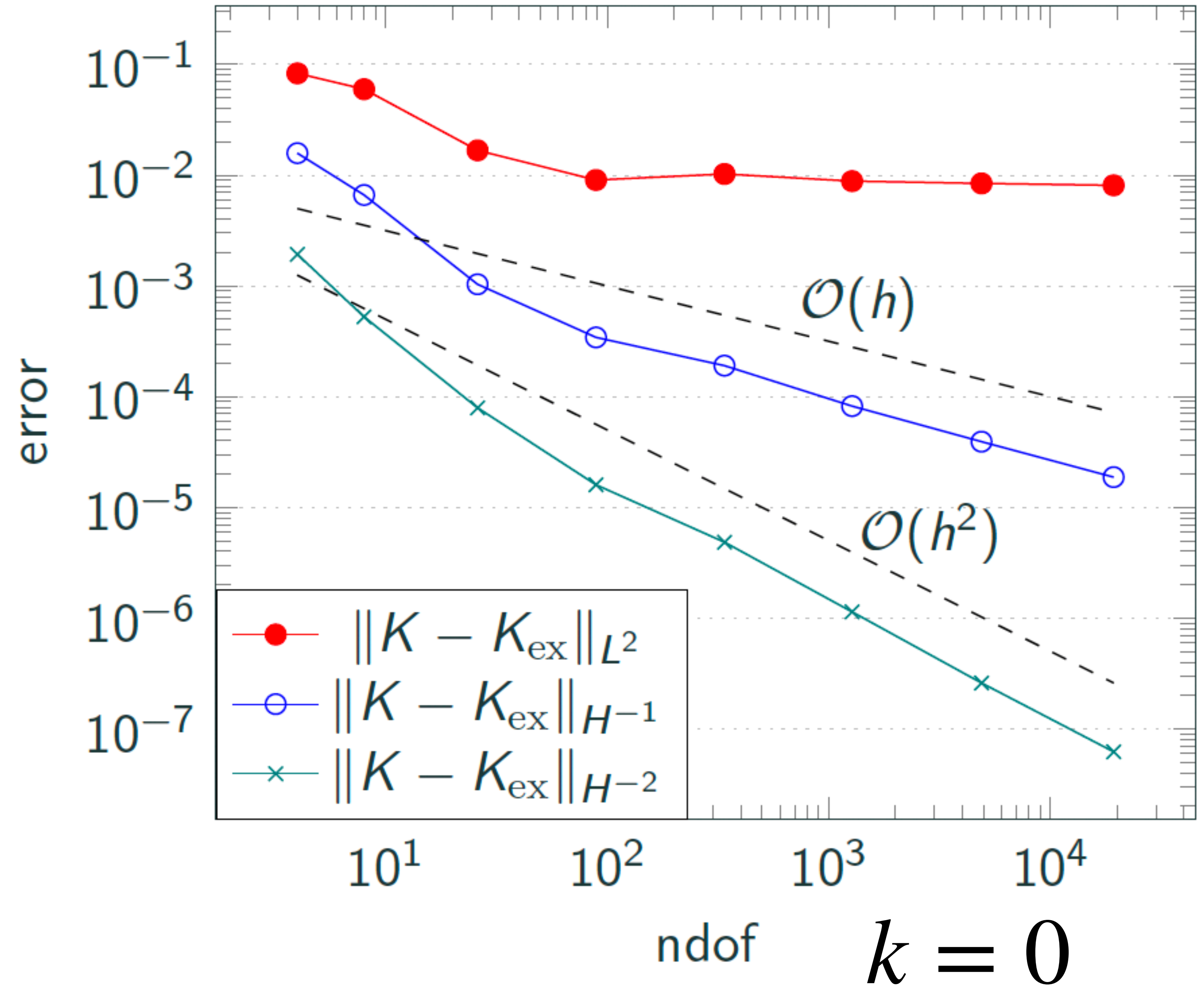
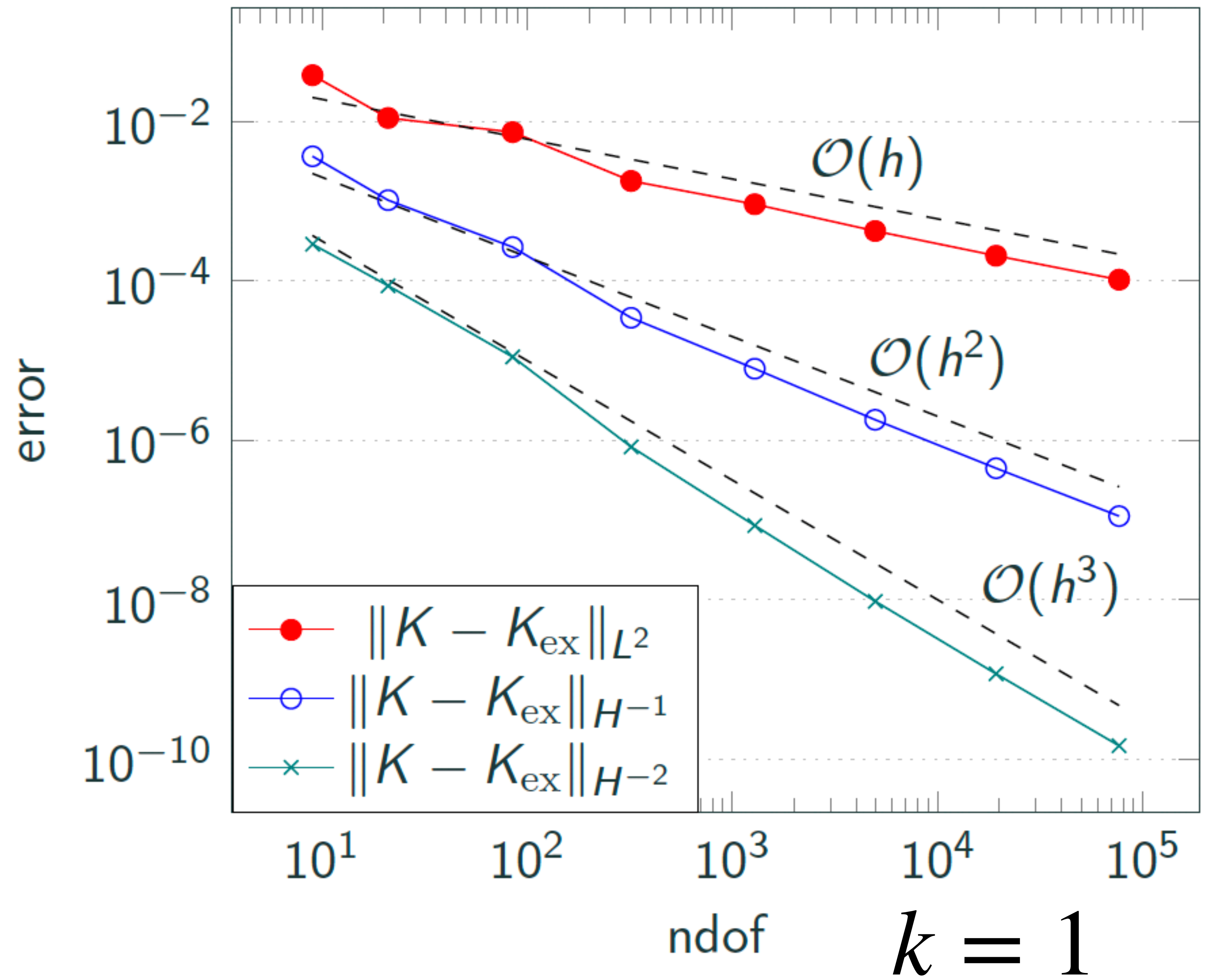
Numerical example: Gauss curvature



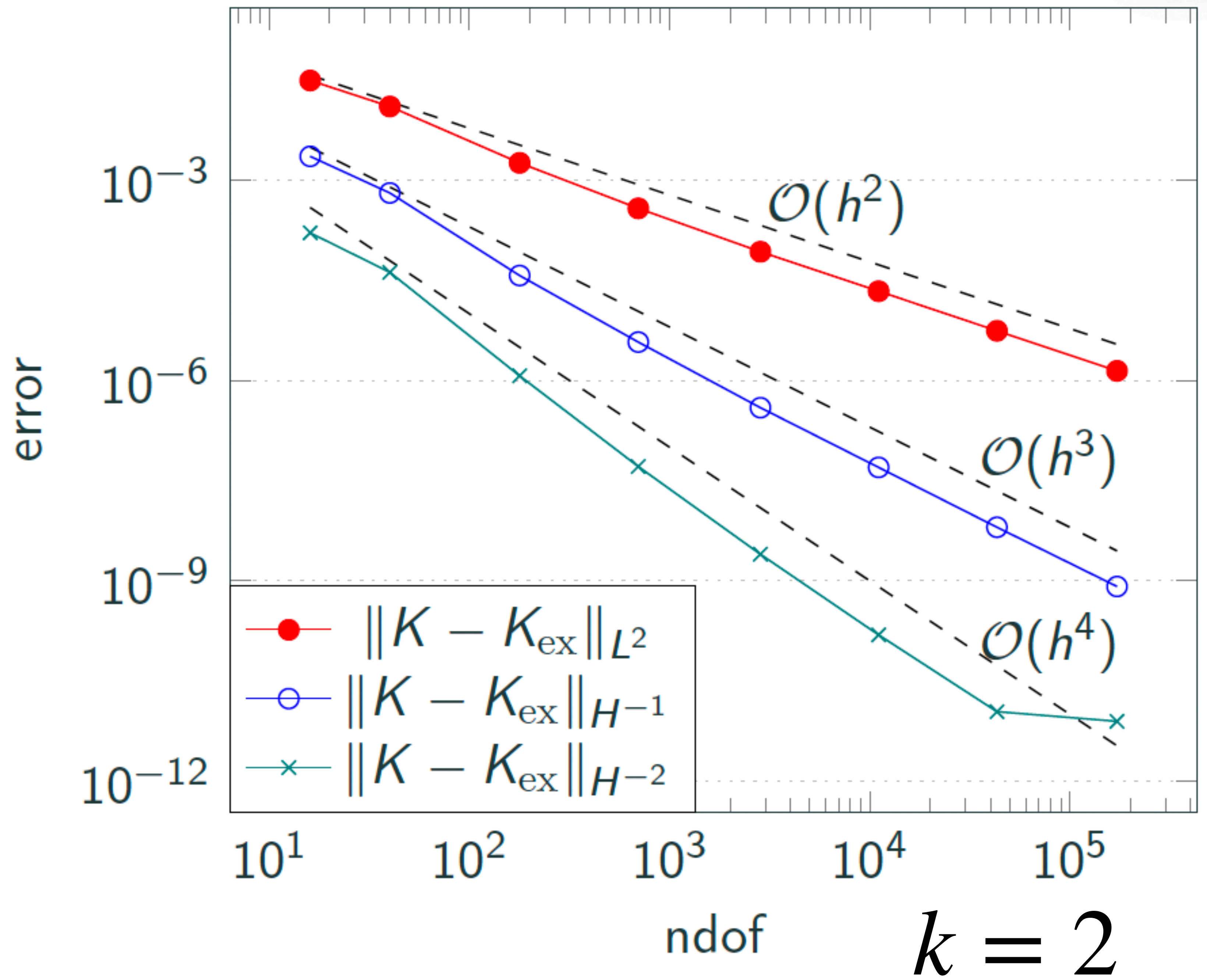




$k = 0$

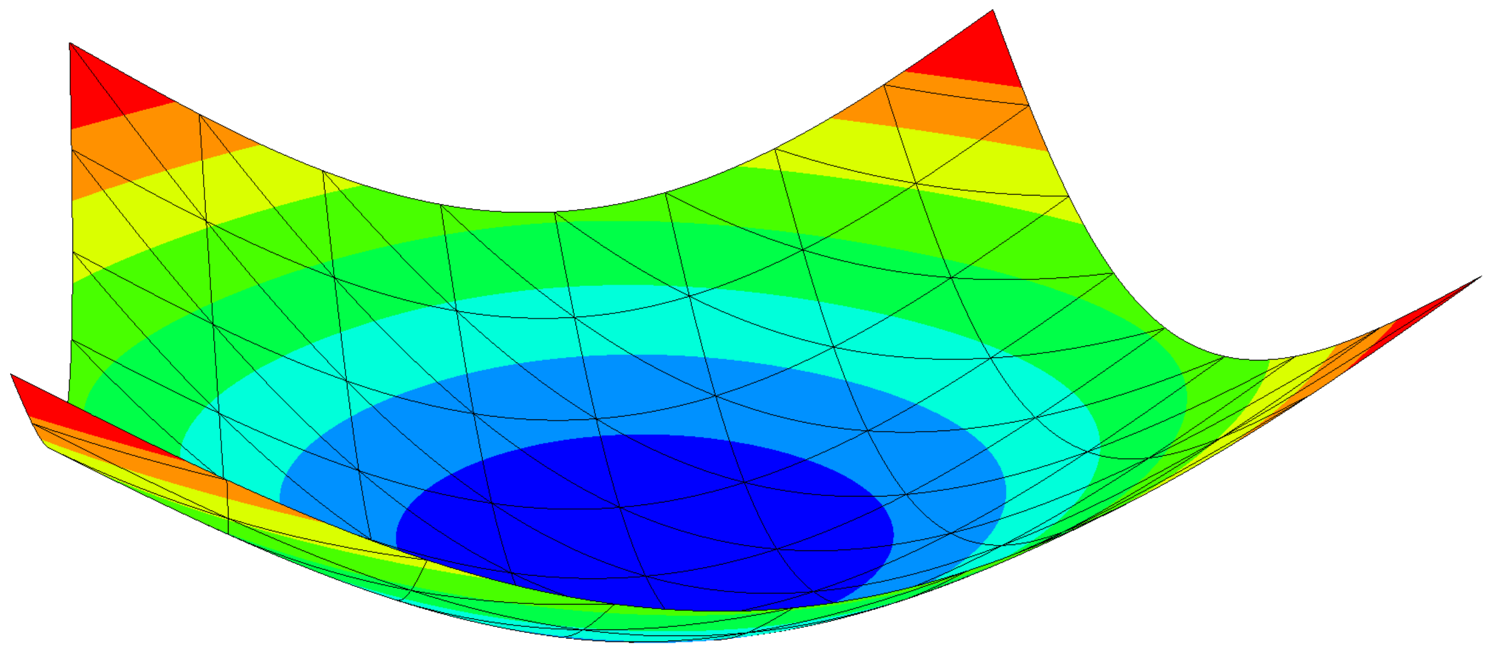


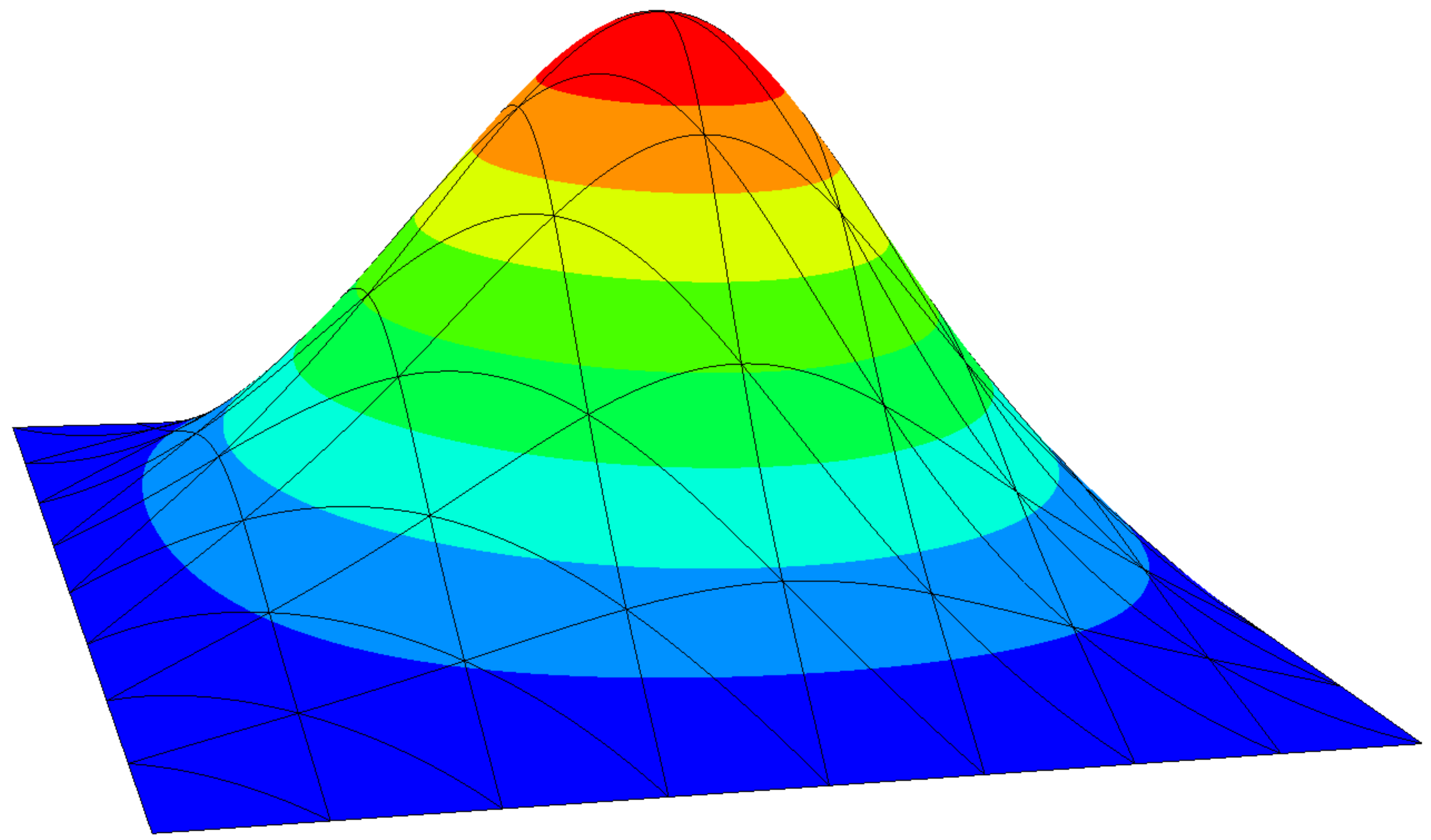
$k = 1$

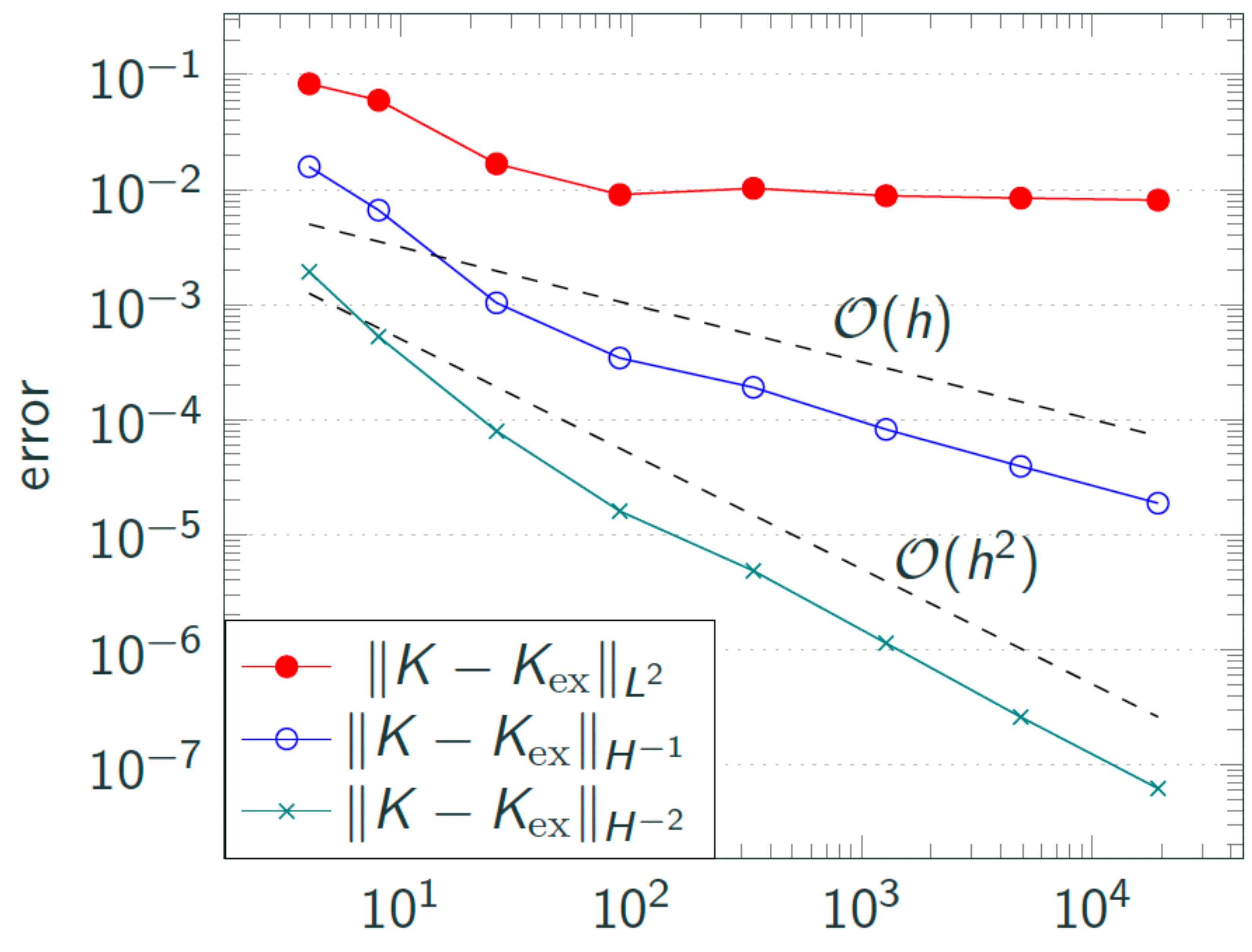


$k = 2$

Numerical example: Gauss curvature







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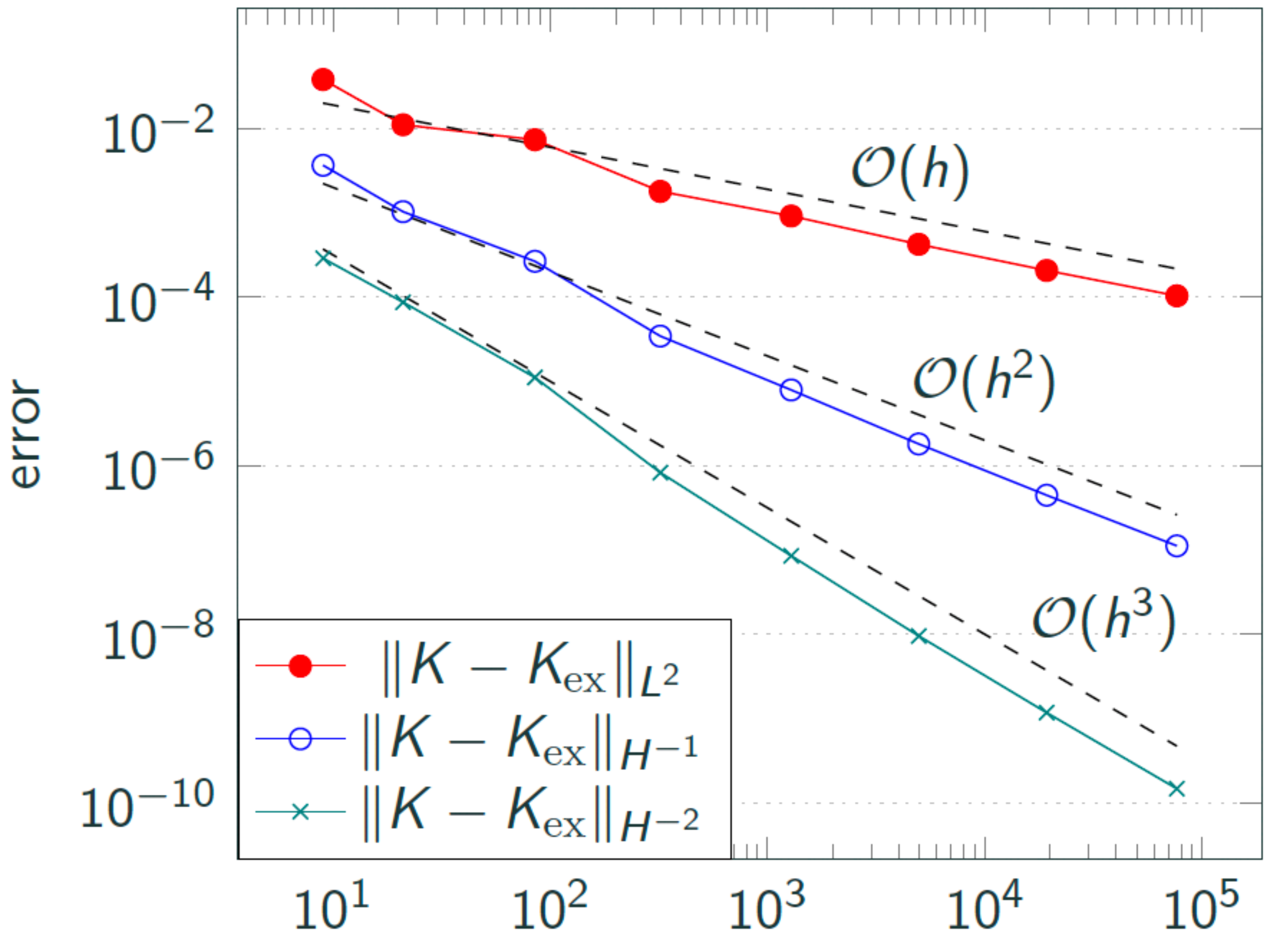

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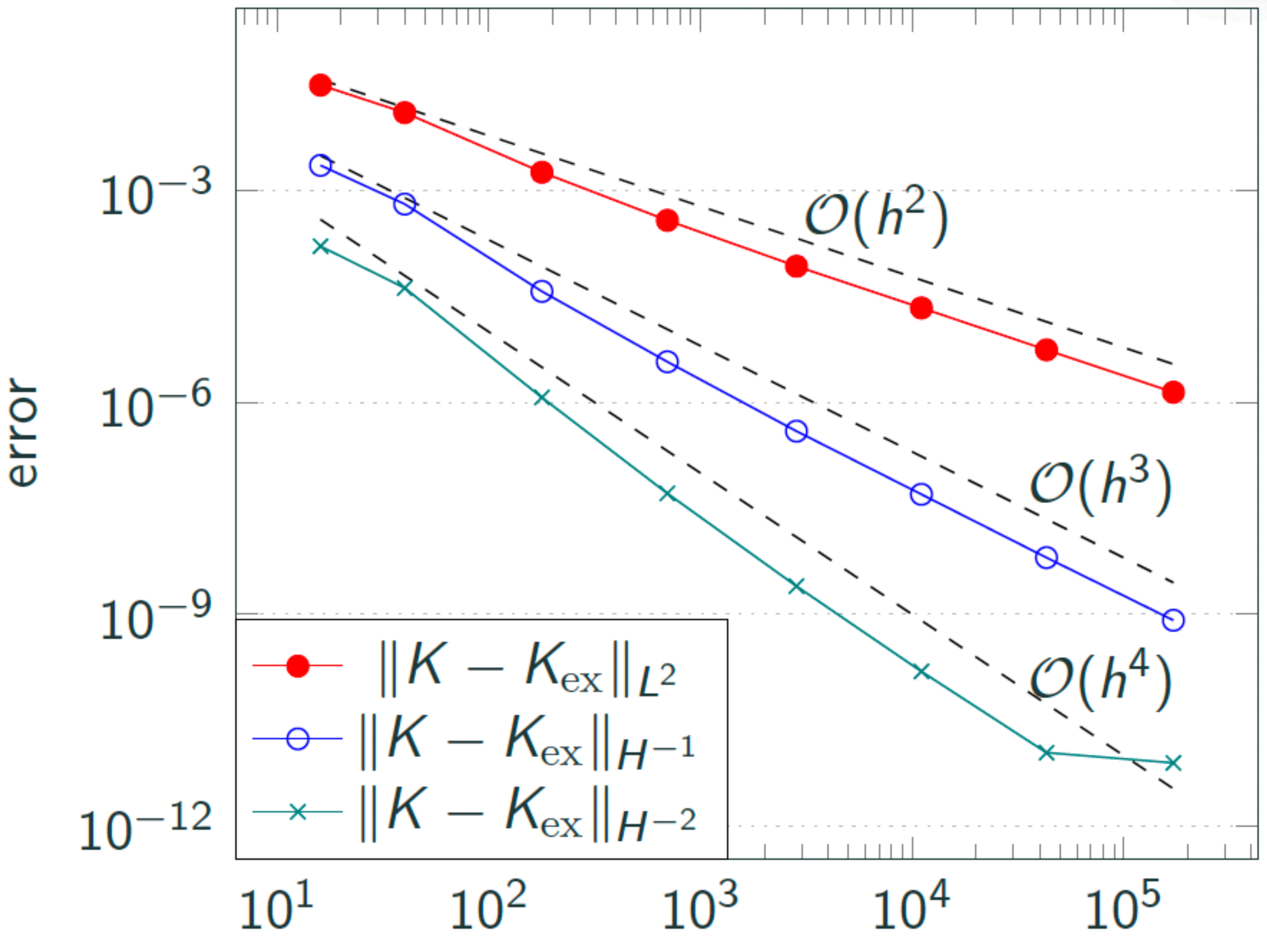

$k = 2$

NGSolve

Extensions

- Scalar curvature $S = g^{ij} g^{kl} \mathfrak{R}_{ikjl}$ (scalar-valued, any dimension)
- Einstein tensor $G_{ij} = g^{kl} \mathfrak{R}_{ikjl} - 0.5\, S\, g_{ij}$ (matrix-valued, any dimension)
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- How can we define the Riemann curvature tensor from a Regge metric?

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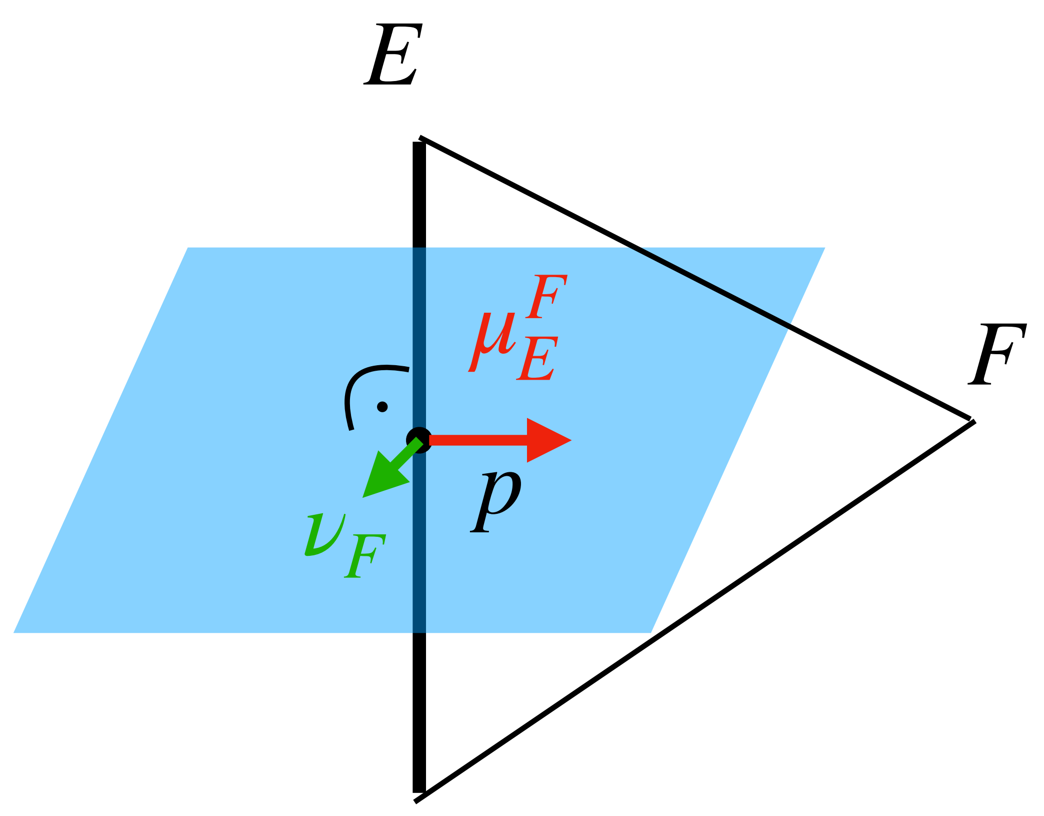
Distributional Riemann curvature tensor Part 1

- Angle defect in higher dimensions? Let $d = 3$ and consider edge E and face F
- 2D plane perpendicular to E
- Co-normal μ_E^F pointing inward F and ν_F normal of F span plane



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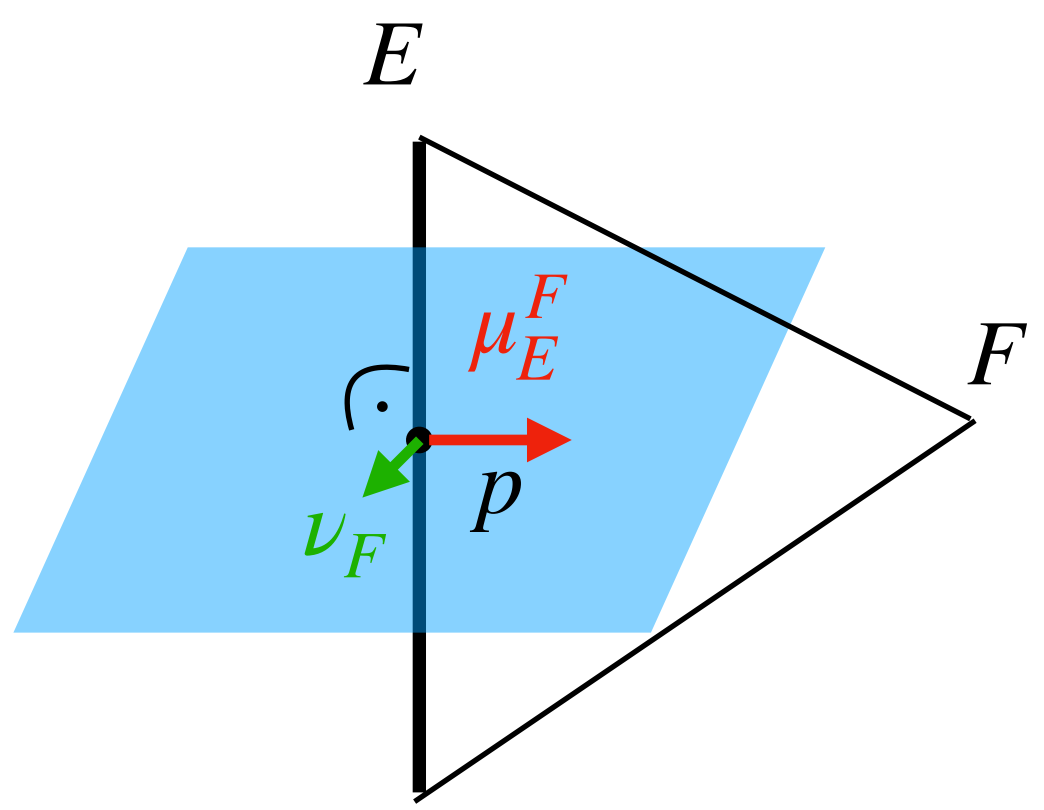

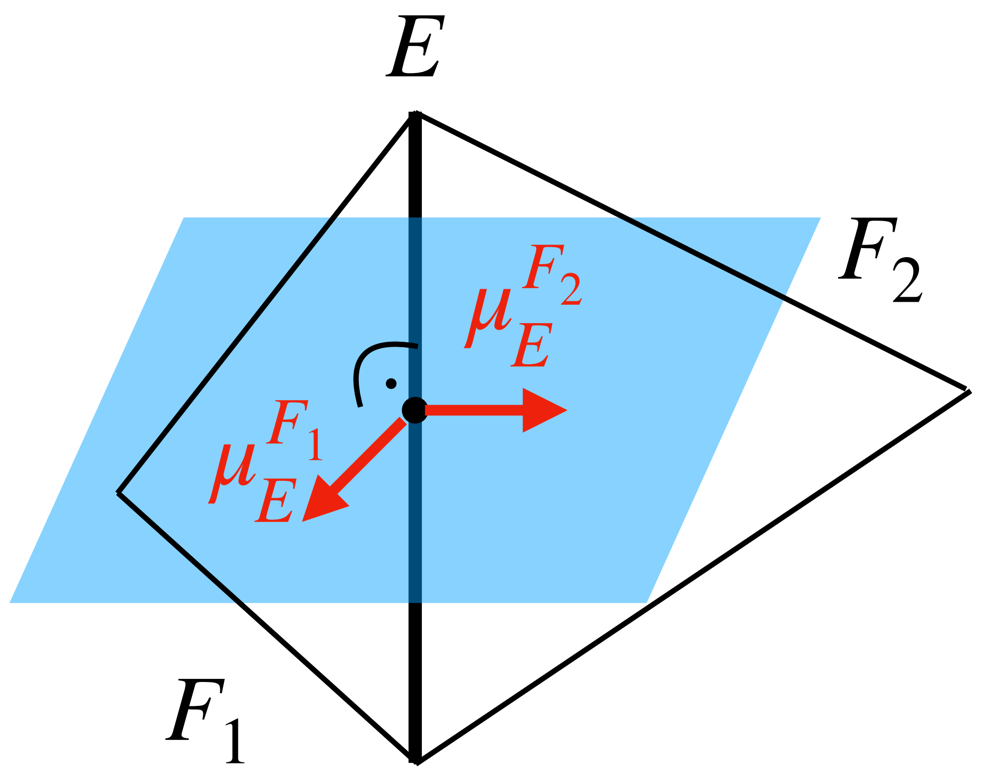

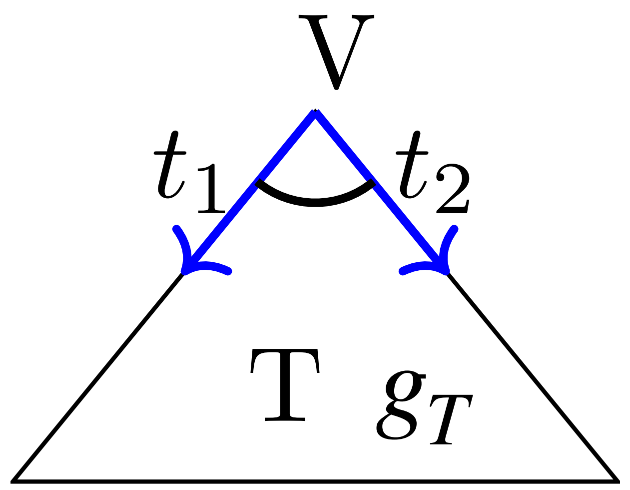

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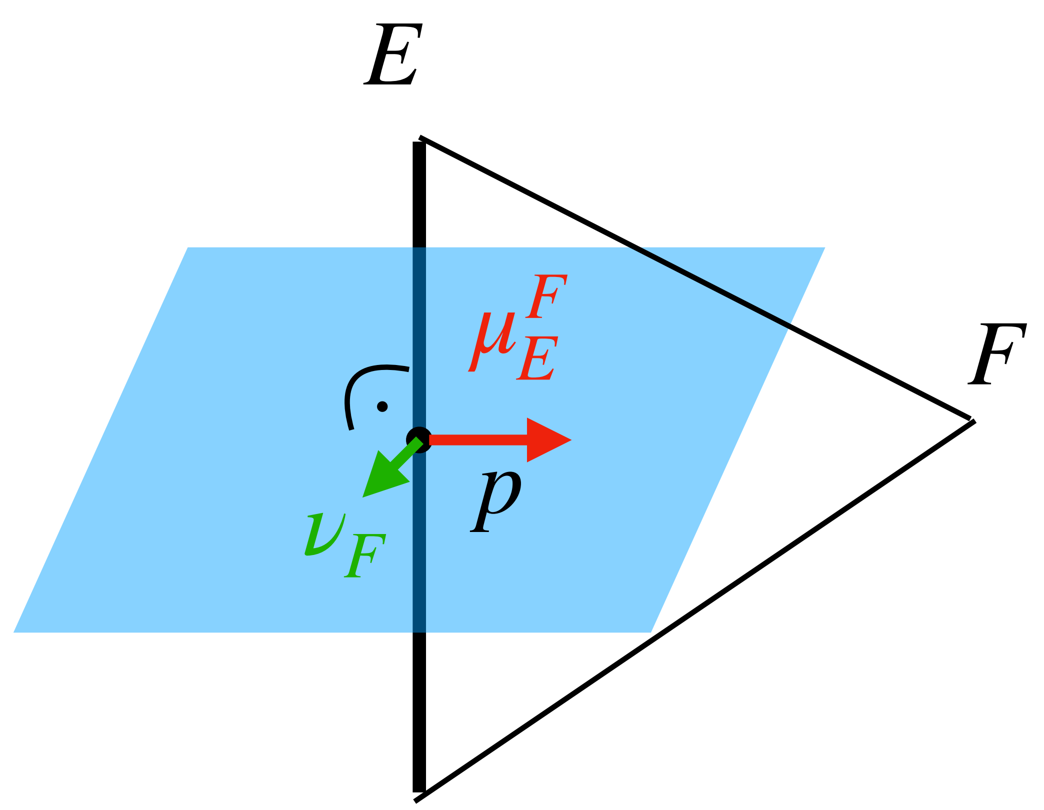

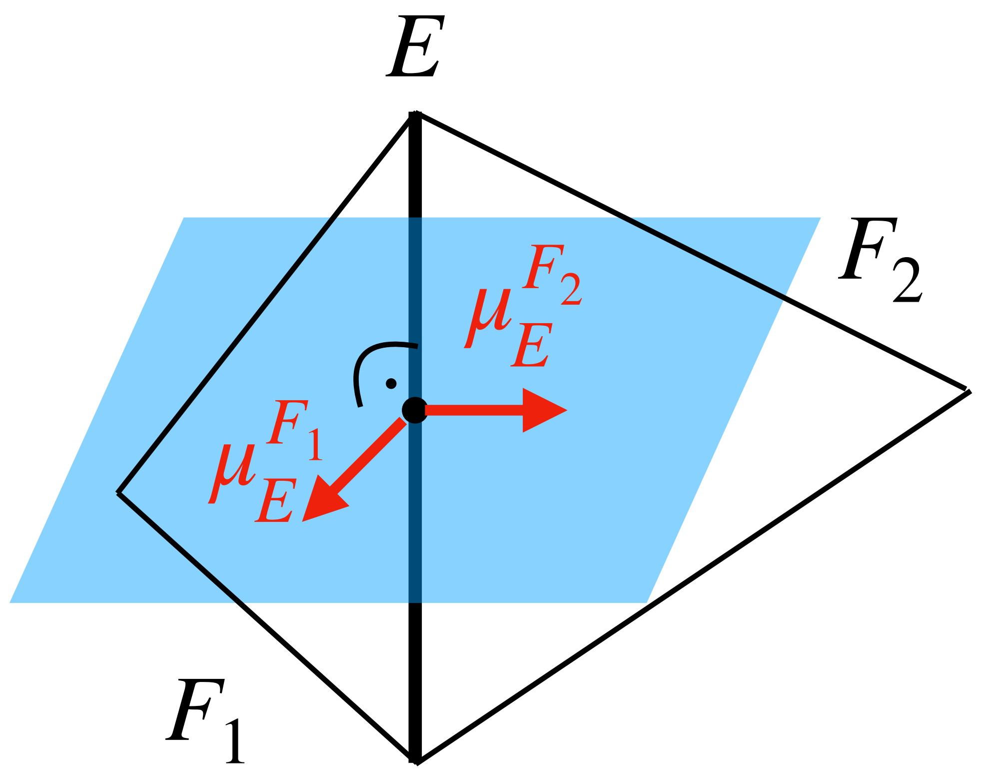

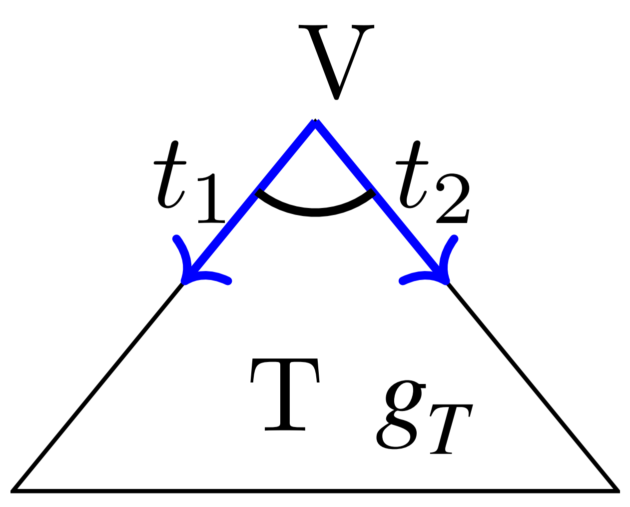

$$(\Re\omega)_{\mathrm{dist}}(A) = \sum_T \int_T \langle \Re_T, A\rangle\, \omega_T + \sum_E \int_E \sphericalangle^E(g)\; A(\mu,\nu,\nu,\mu)\, \omega_E$$

Distributional Riemann curvature tensor Part 2

- $\mathfrak{R}(X, \nu_\pm, \nu_\pm, x)$ is Gauss curvature in 2D plane spanned by $X, \nu_\pm$

- Jump of geodesic curvature

$$[\![\mathfrak{R}(X, \nu_\pm, \nu_\pm, X)]\!] = [\![\kappa_g]\!] = [\![I\!I(X, X)]\!]$$

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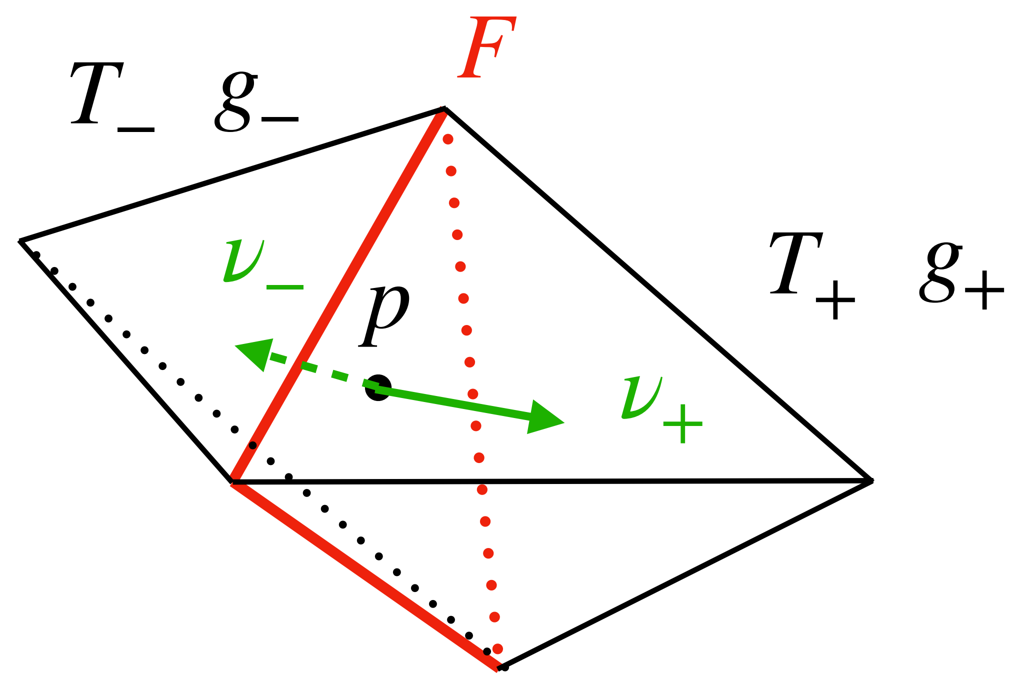

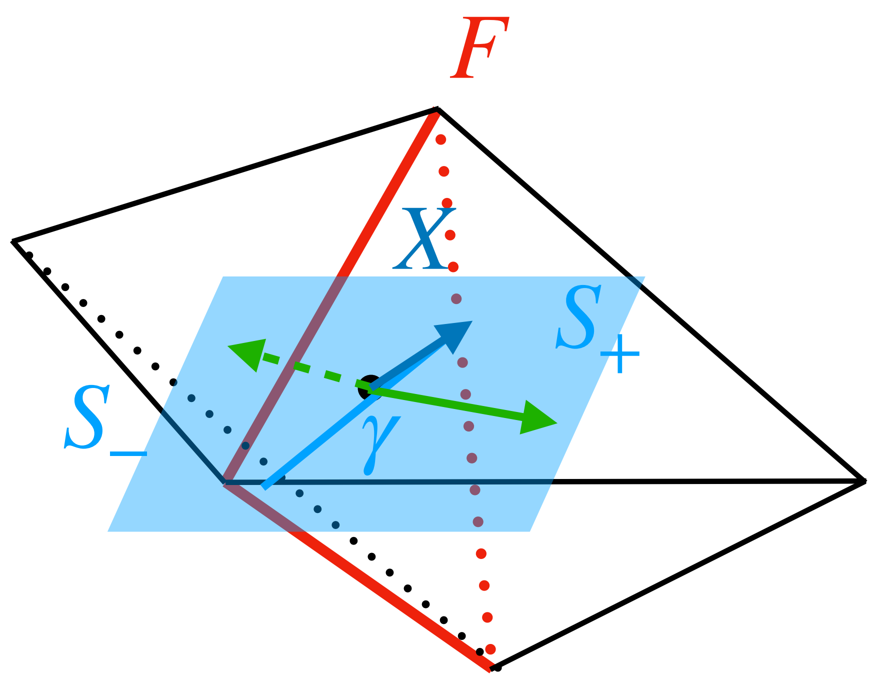

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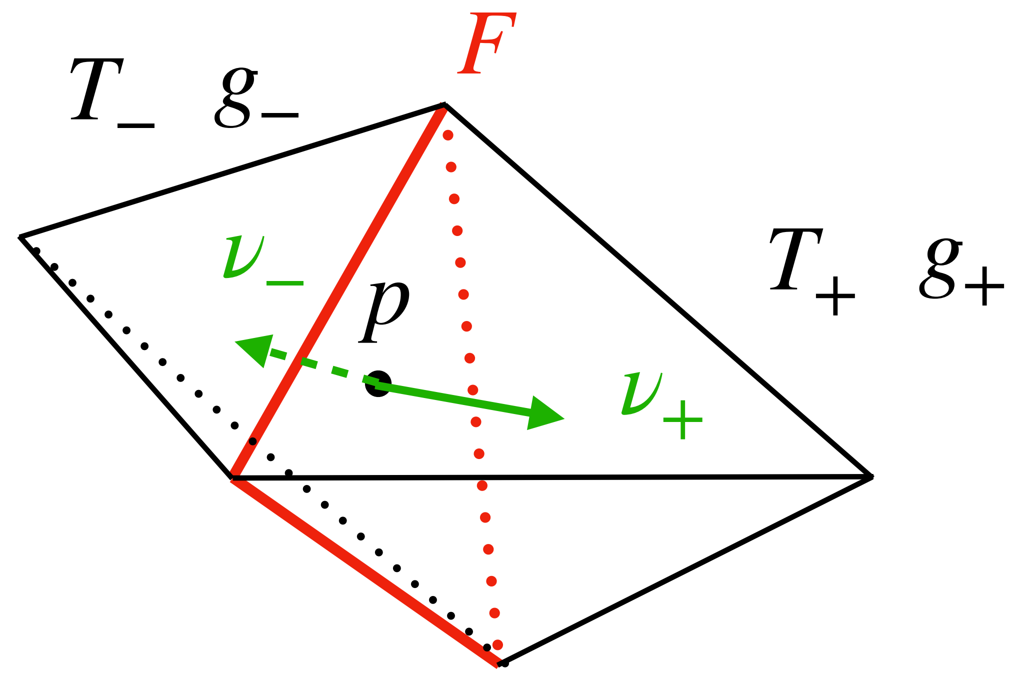

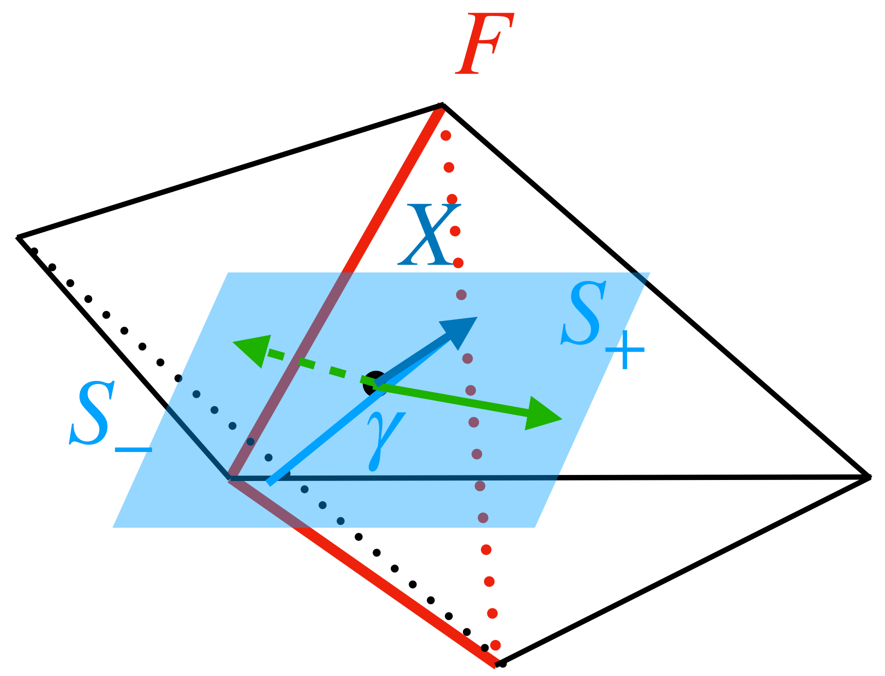

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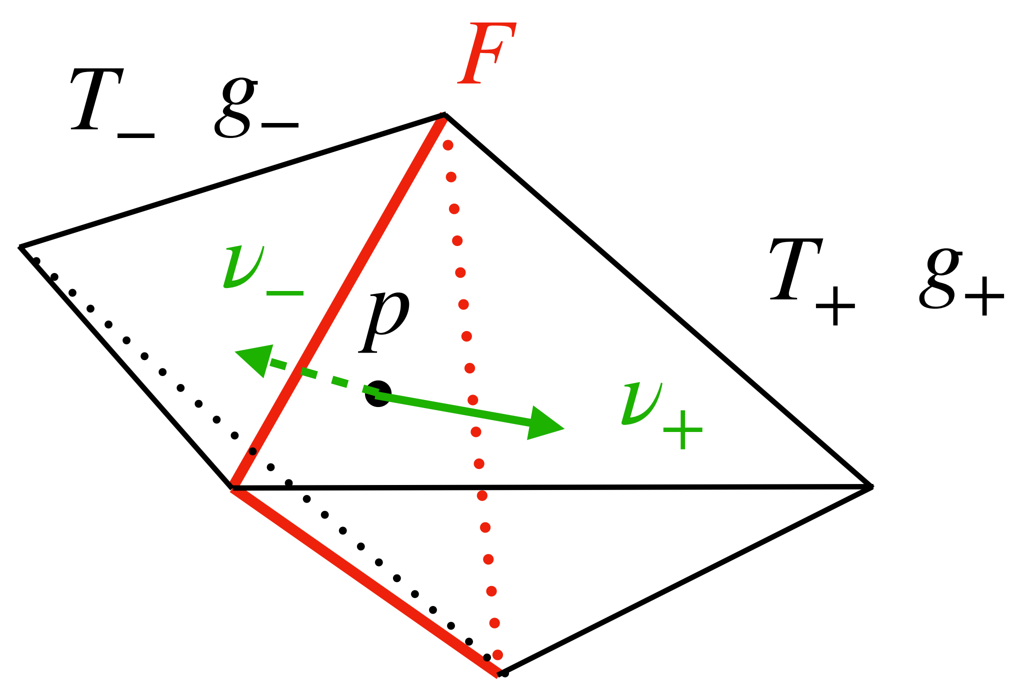

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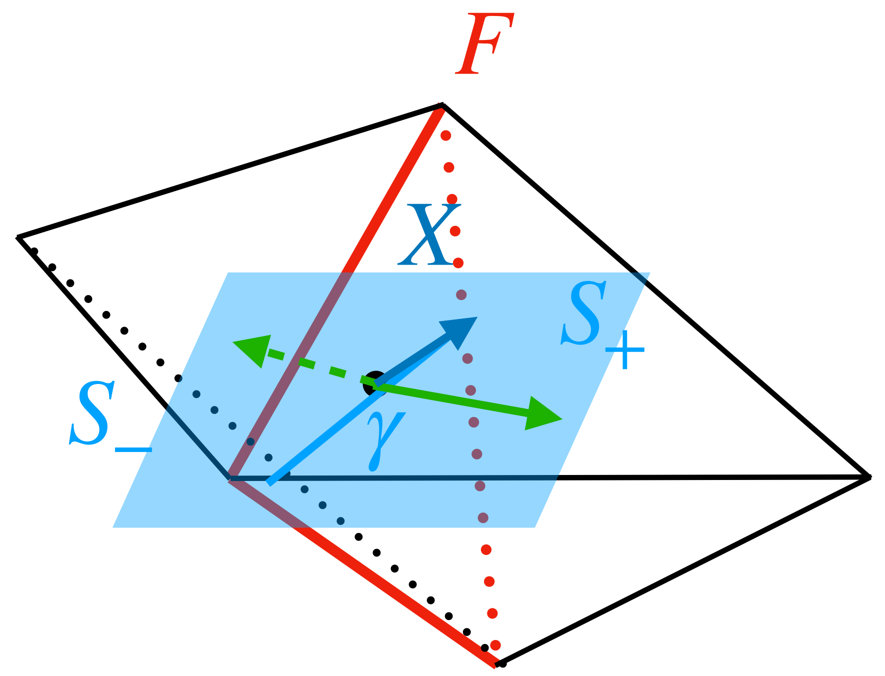

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- Distributional Riemann curvature in any dimension

$$(\mathfrak{R}\omega)_{\text{dist}}(A) = \sum_T \int_T \langle \mathfrak{R}_T, A \rangle \, \omega_T + \sum_F \int_F \langle [\![I\!I]\!], A(\,\cdot\,, \nu, \nu, \,\cdot\,) \rangle \, \omega_F + \sum_E \int_E \sphericalangle^E(g) \; A(\mu, \nu, \nu, \mu) \, \omega_E$$

Convergence

Theorem: Let $k \in \mathbb{N}_0$, $g \in W^{2,\infty}(\Omega)$, and $g_h \in \mathrm{Reg}_h^k$ be a sequence of Regge metrics such that $\lim_{h\to 0} \|g_h - g\|_{L^\infty} = 0$ and $\sup_{h>0} \max_T \|g_h\|_{W^{2,\infty}(T)} < \infty$. Then there holds for sufficiently small h

$$\|(\mathfrak{R}\omega)_{\mathrm{dist}}(g_h) - (\mathfrak{R}\omega)(g)\|_{H^{-2}} \leq C C_g \, |||g_h - g|||_2,$$

where

$$C_g = \begin{cases} 1 + \max_T(h_T^{-1}\|g_h - g\|_{L^\infty(T)}) + \|g - g_h\|_{W_h^{1,\infty}}, & N = 2, \\ 1 + \max_T(h_T^{-2}\|g_h - g\|_{L^\infty(T)} + h_T^{-1}\|g_h - g\|_{L^\infty(T)}), & N \geq 3. \end{cases}$$

$$|||\sigma|||_2^2 = \|\sigma\|_{L^2}^2 + h^2\|\sigma\|_{H_h^1}^2 + h^4\|\sigma\|_{H_h^2}^2$$

Gopalakrishnan, N., Schöberl, Wardetzky: Generalizing Riemann curvature to Regge metrics, arXiv:2311.01603.

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Convergence for $k \geq 1$ for $N \geq 3$ and $k \geq 0$ for $N = 2$ (Gauss curvature)

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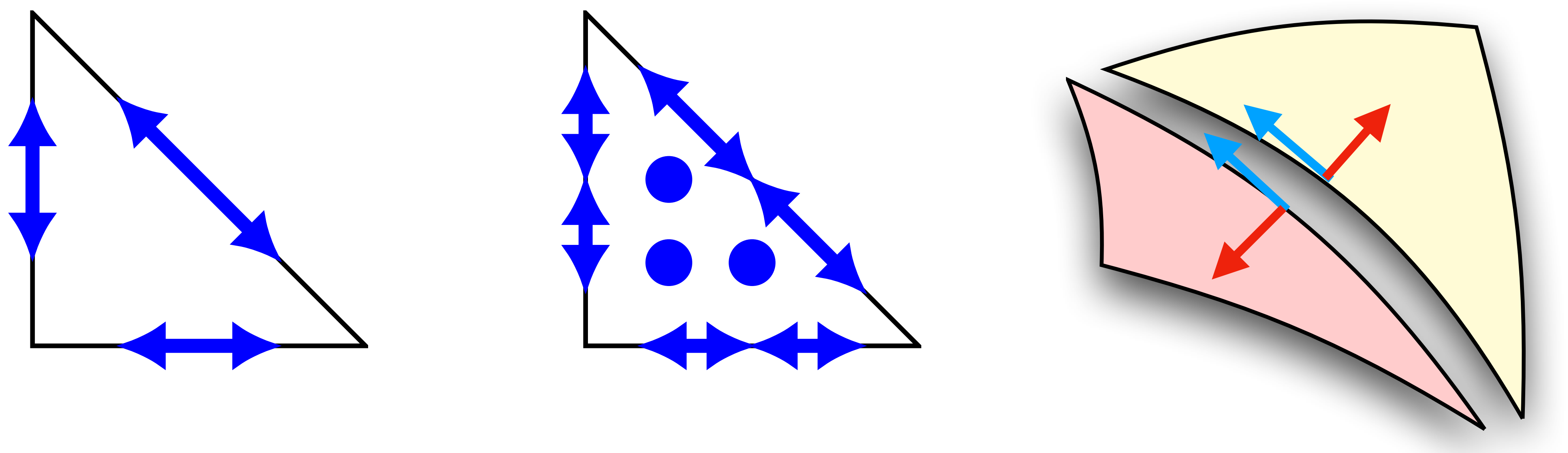
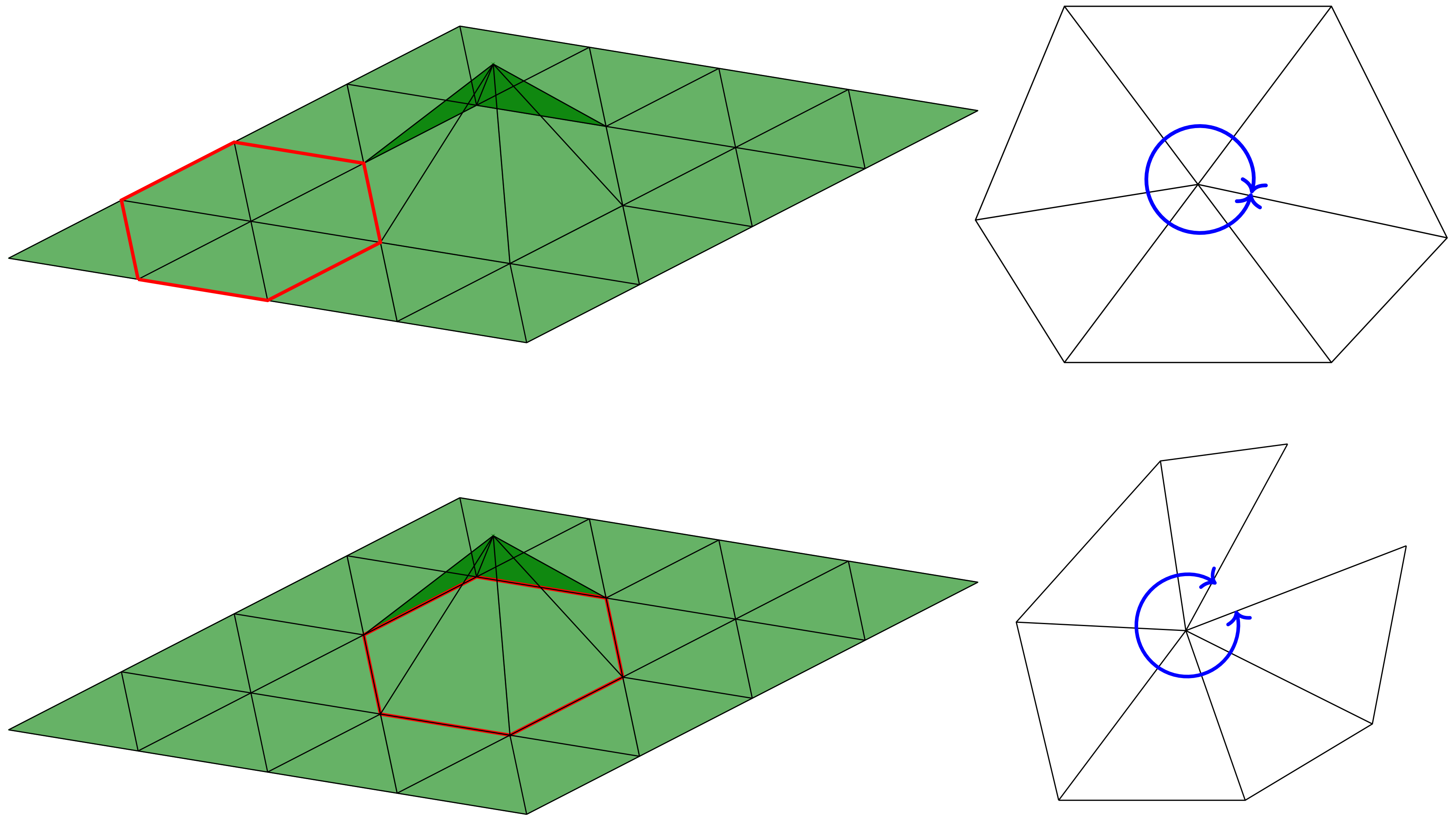
Summary & Outlook



- Regge finite elements for metric approximation
- Distributional Gauss curvature
- Extensions to scalar curvature, Einstein tensor, and Riemann curvature tensor



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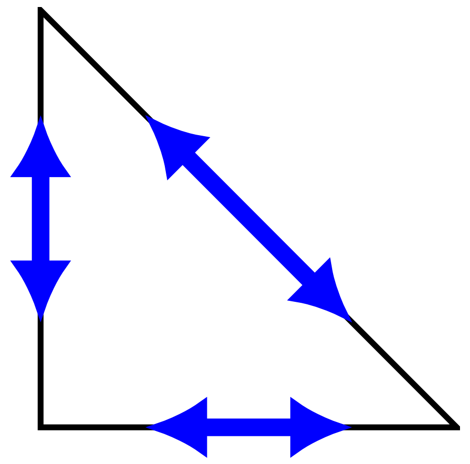
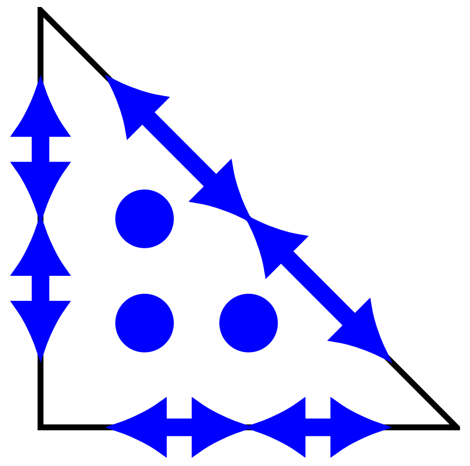
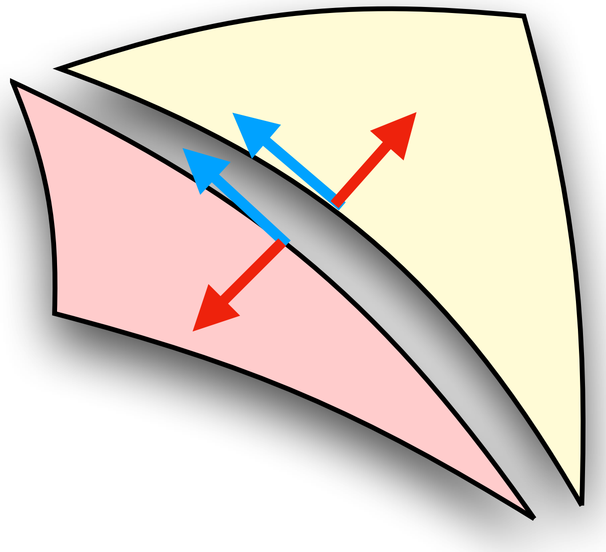

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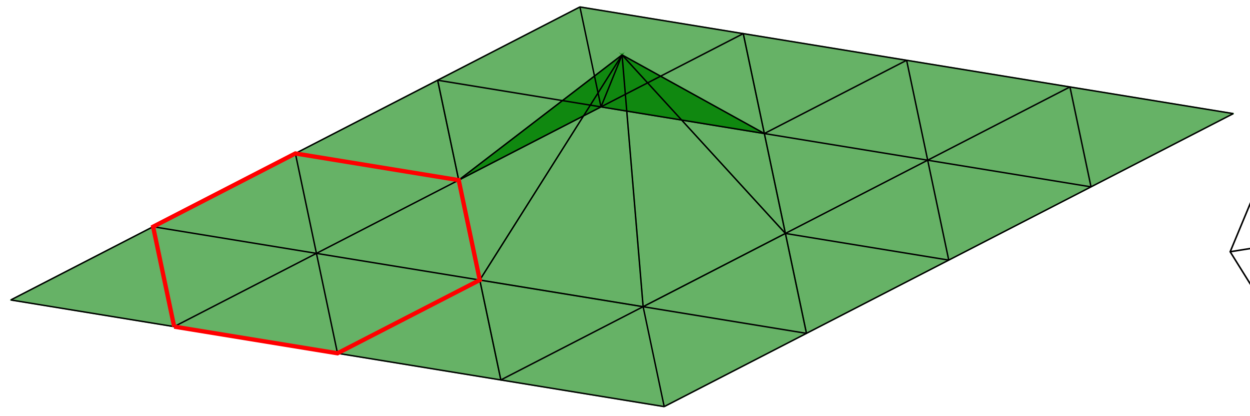
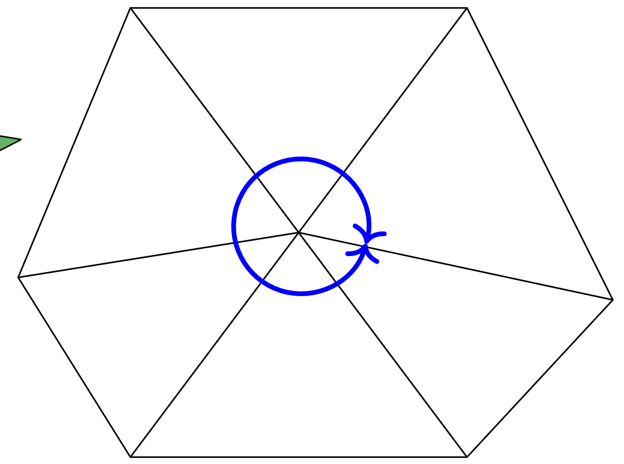

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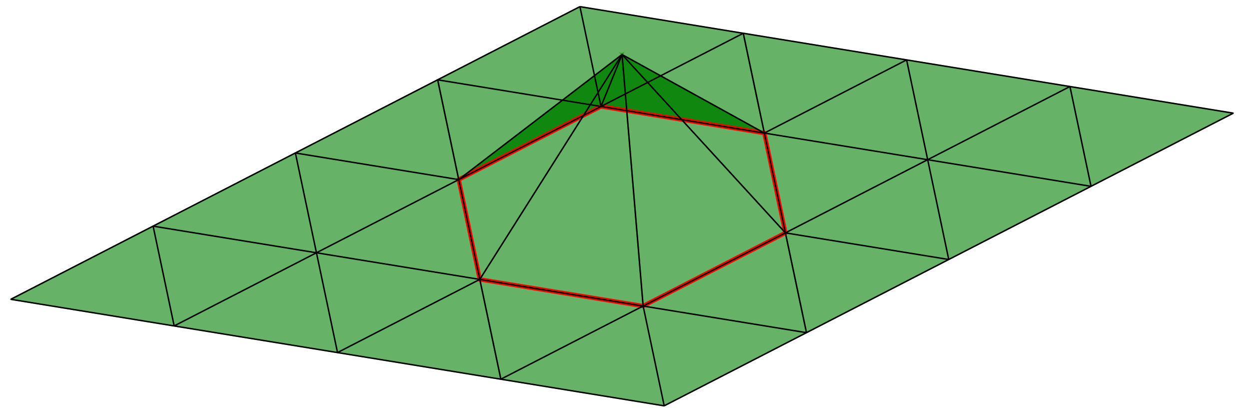
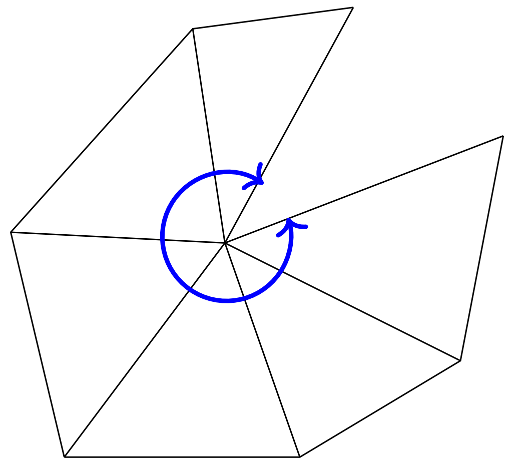
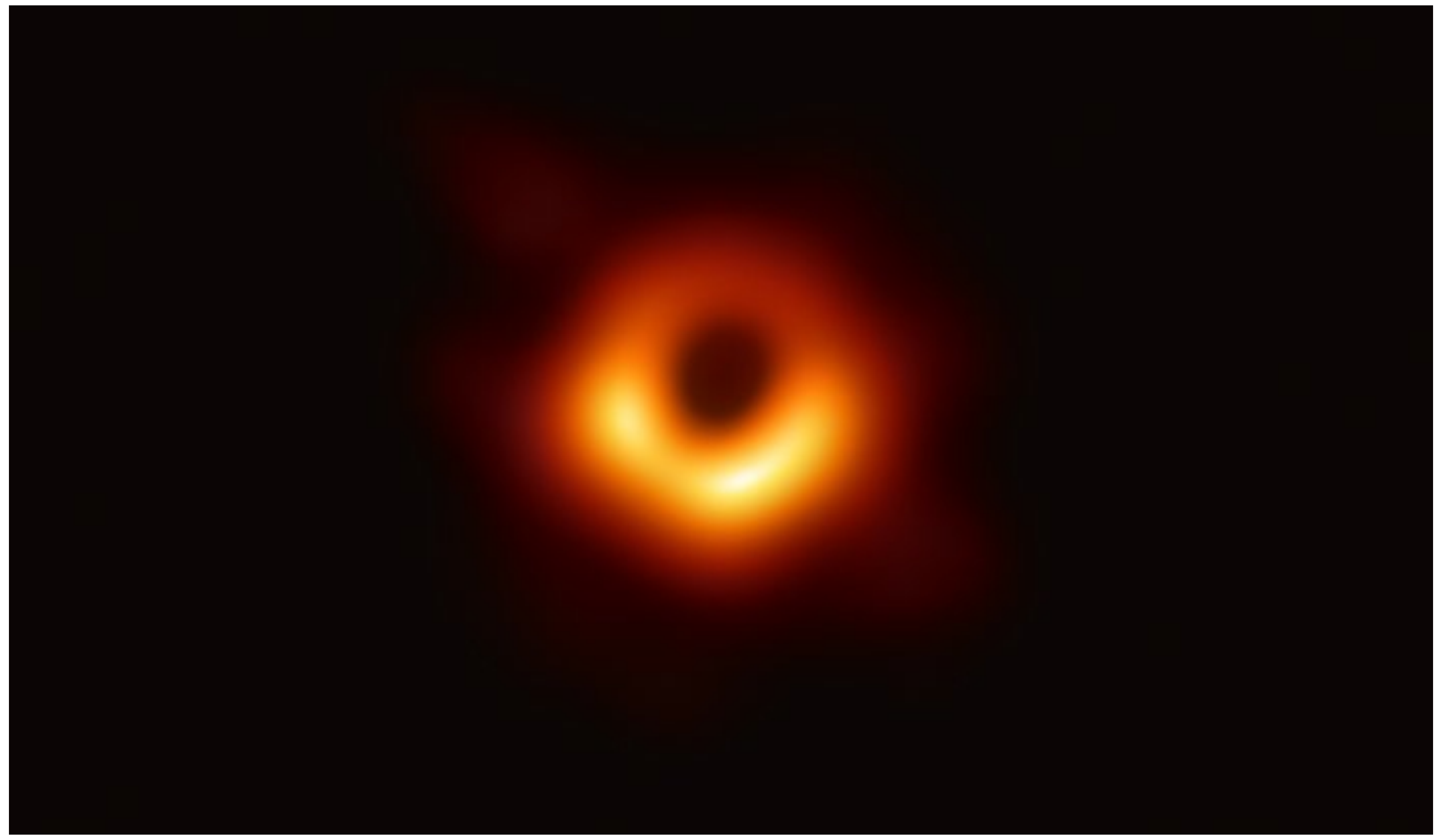



- Finite elements for Einstein and Riemann curvature tensor
- Combine finite element numerical analysis with discrete differential geometry
- Long-term goal: Application to geometric flows and numerical relativity



By Event Horizon Telescope (EHT)

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- Li: Finite Elements with Applications in Solid Mechanics and Relativity, PhD thesis, 2018.
- Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, SIAM J. Numer. Anal., 2020.
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, SMAI J. Comput. Math., 2023.
- Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, FoCM, 2022.
- Gopalakrishnan, N., Schöberl, Wardetzky: On the improved convergence of lifted distributional Gauss curvature from Regge elements, RINAM, 2024.
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Thank you for your attention!