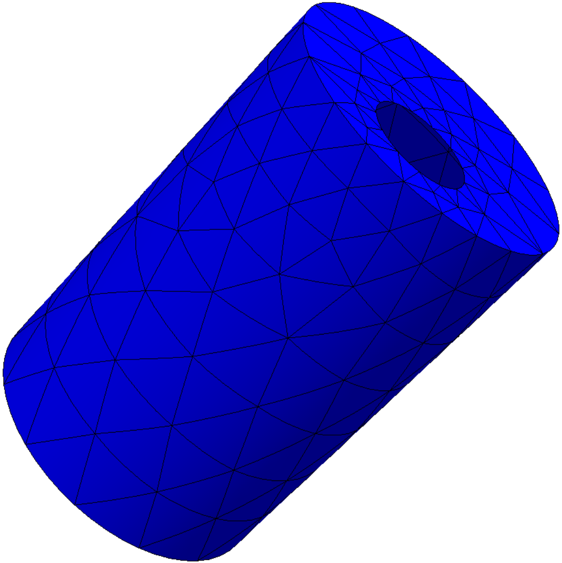
The Hellan–Herrmann–Johnson and TDNNS method for nonlinear Koiter and Naghdi shells

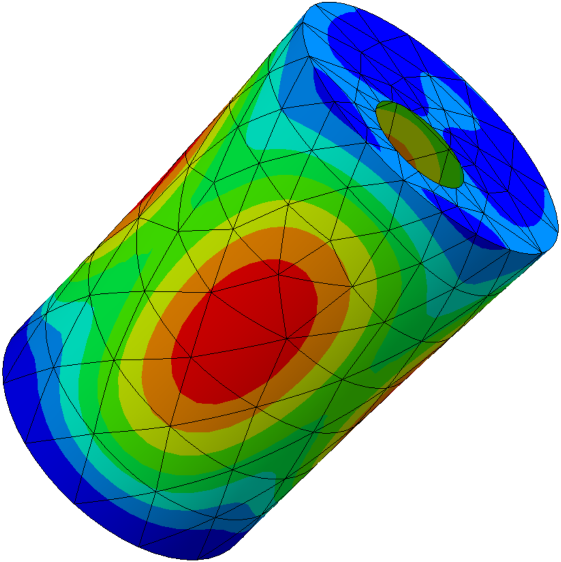
Michael Neunteufel (TU Wien)
Joachim Schöberl (TU Wien)

Luxembourg, Mai 2nd, 2023

Motivation



Motivation



Motivation

Motivation

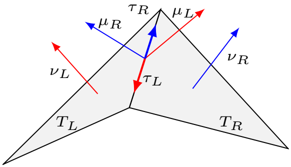
Contents

Distributional curvature

- Normal vector ν

  Tangent vector τ

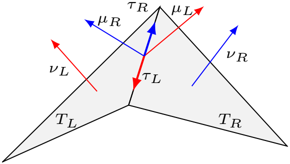
  Element normal vector $\mu = \nu \times \tau$

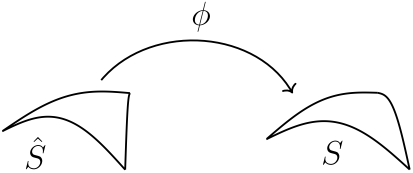


- Normal vector $\hat{\nu}$

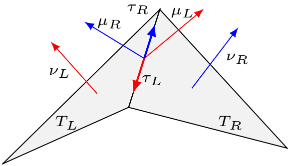
  Tangent vector $\hat{\tau}$

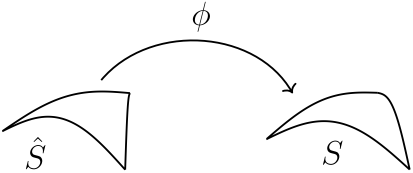
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



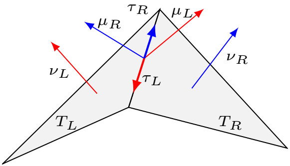


- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$
- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$

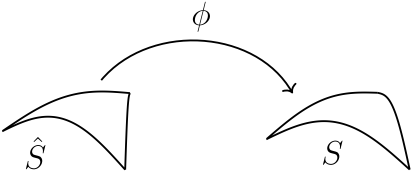




- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



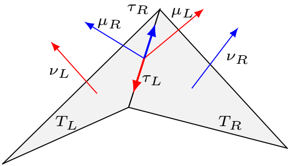
- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



- Normal vector $\hat{\nu}$

  Tangent vector $\hat{\tau}$

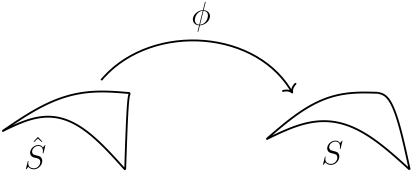
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$

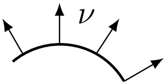
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$

  $$\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi = \frac{(\boldsymbol{F}^{\dagger})^{\top}\hat{\mu}}{\|(\boldsymbol{F}^{\dagger})^{\top}\hat{\mu}\|}$$



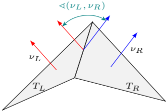
Distributional curvature

- Change of normal vector measures curvature $\nabla\nu$



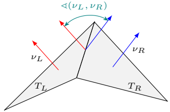
Distributional curvature

- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



GRINSPUN, GINGOLD, REISMAN AND ZORIN: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006), pp. 547–556.

- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



  - Distributional Weingarten tensor

$$\langle \nabla\nu, \boldsymbol{\sigma}\rangle_{\mathscr{T}} = \sum_{T\in\mathscr{T}_h}\int_T \nabla\nu|_T : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathcal{E}_h}\int_E \sphericalangle(\nu_L,\nu_R)\boldsymbol{\sigma}_{\mu\mu}\, ds$$

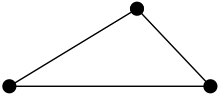
  - Measure jump of normal vector
  - Test function $\boldsymbol{\sigma}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *arXiv:2107.13794*.

Hellan–Herrmann–Johnson finite elements

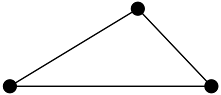
$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

$$H^1(\Omega) := \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$$
$$V_h^k := \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega)$$



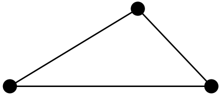
Hellan–Herrmann–Johnson finite elements

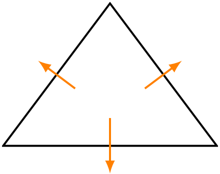
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathbf{div}\sigma \in L^2(\Omega)\}$$



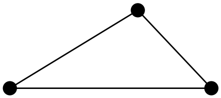
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \,|\, \mathbf{div}\sigma \in L^2(\Omega)\}$$

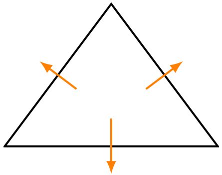
$BDM^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, \sigma_n$ is continuous over elements$\}$





$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \,|\, \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$

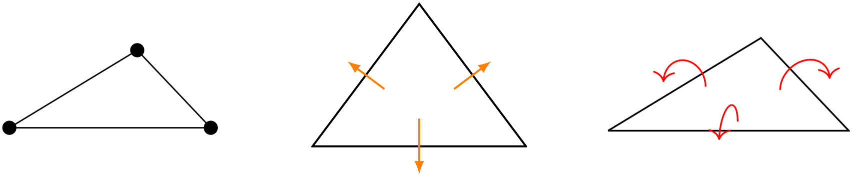




Hellan–Herrmann–Johnson finite elements

$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$

$$M_h^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{sym} \mid n^T\sigma n \text{ is continuous over elements}\}$$



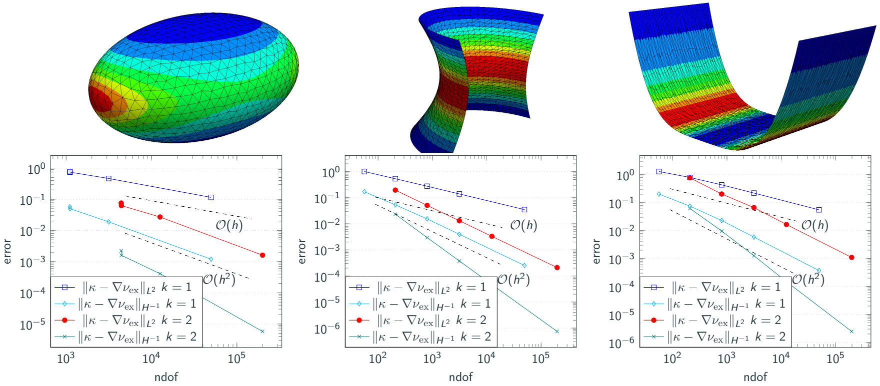
A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Lifted distributional curvature

Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathscr{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

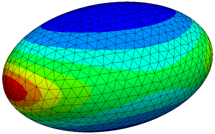
$$\int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}}$$

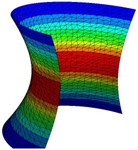


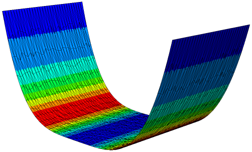
Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathscr{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

$$\int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}}$$



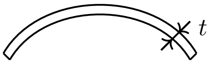




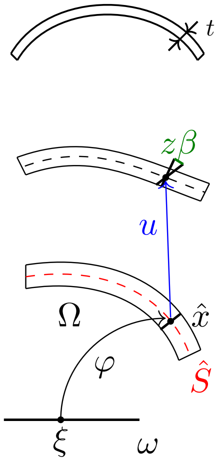
Analysis technical but possible

GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Nonlinear shells



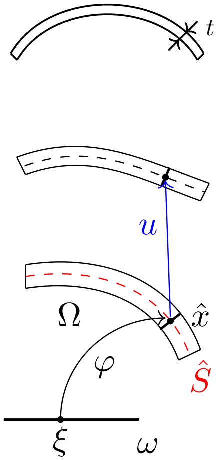
- Model of reduced dimensions



- Model of reduced dimensions

- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$

- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z \underbrace{(\nu + \beta) \circ \phi}_{=\tilde{\nu}\circ\phi}(\hat{x})$



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\,\nu \circ \phi(\hat{x})$

Shell model

$$\mathcal{W}(u,\quad) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

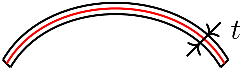
u . . . displacement of mid-surface

t . . . thickness

$\boldsymbol{M}$ . . . material tensor

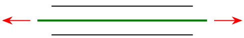
$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



Shell model

$$\mathcal{W}(u,\quad) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$



membrane energy

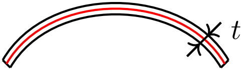
u ... displacement of mid-surface

t ... thickness

$\boldsymbol{M}$ ... material tensor

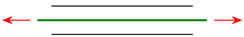
$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



Shell model

$$\mathcal{W}(u,\quad) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$



membrane energy

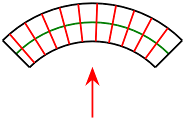
$u \ldots$ displacement of mid-surface

$t \ldots$ thickness

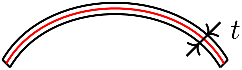
$\boldsymbol{M} \ldots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$

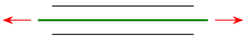


bending energy



Shell model

$$\mathcal{W}(u,\beta) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{tG\kappa}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2 + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla(\tilde{\nu}\circ\phi) - \nabla\hat{\nu})\|^2_{\boldsymbol{M}}$$



membrane energy

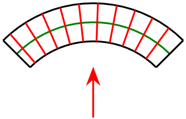
u ... displacement of mid-surface

t ... thickness

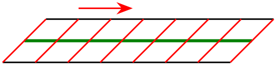
$\boldsymbol{M}$ ... material tensor

$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

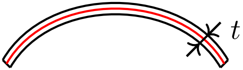
$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



bending energy



shearing energy



- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\boldsymbol{M}} - \langle f, u\rangle$$
$$+ \sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma} \in M_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u, \boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

TECHNISCHE UNIVERSITÄT WIEN

Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for ($\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$)

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} - \langle f, u\rangle \\ &+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ &+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds\end{aligned}$$

Mixed saddle-point problem $\to$ indefinite stiffness matrix

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

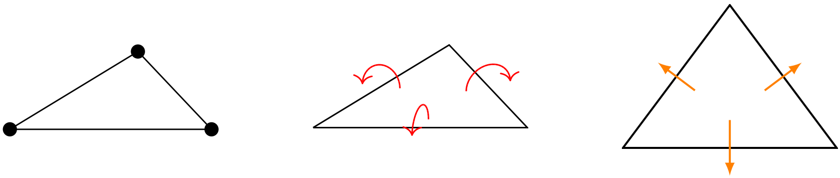
Shell problem (Hybridization)

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1,\mathrm{dc}}$ and $\alpha \in W_h^{k-1}$ for

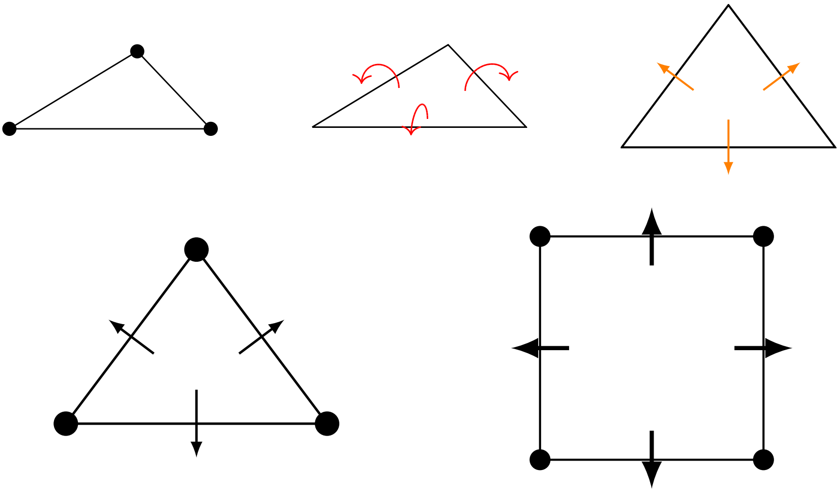
$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma}) = &\frac{t}{2}\|E(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} - \langle f, u\rangle\\ &+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx\\ &+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} + \alpha_{\hat{\mu}}[\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\,ds\end{aligned}$$

$$\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} = \tfrac{1}{2}(\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} + \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R}), \qquad [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = \boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} - \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R}$$

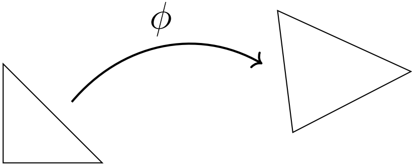
N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*



Shell element



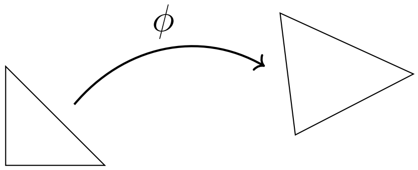
Mapping to the surface



Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \det(\boldsymbol{F})$$



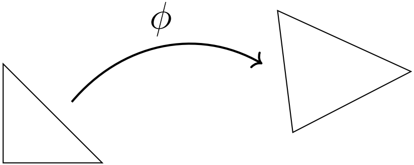
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi, \; J = \det(\boldsymbol{F})$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$



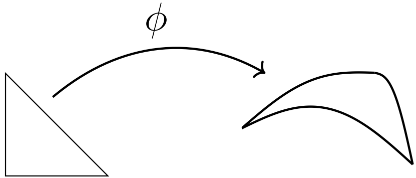
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \det(\boldsymbol{F})$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$

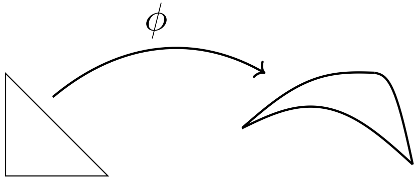


- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \sqrt{\det(\boldsymbol{F}^T\boldsymbol{F})}$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$



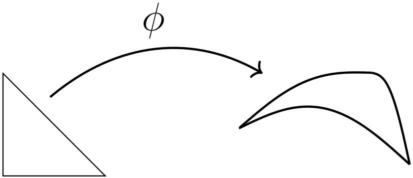
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \| \operatorname{cof}(\boldsymbol{F}) \|$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$



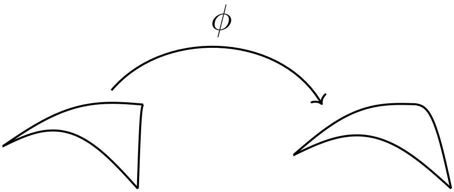
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \|\operatorname{cof}(\boldsymbol{F})\|$$

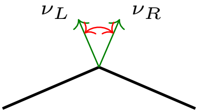
- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$

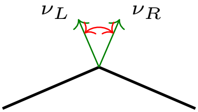


Computational aspect

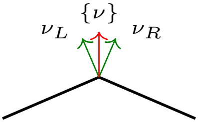
$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



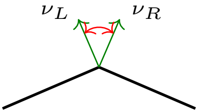
$$\int_{\partial T} (\sphericalangle(\{\nu\}, \nu) - \sphericalangle(\{\hat{\nu}\}, \hat{\nu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



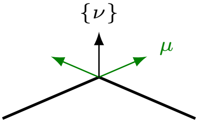
$$\{\nu\} := \frac{1}{\|\nu_L + \nu_R\|}(\nu_L + \nu_R)$$

Computational aspect

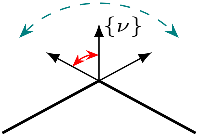
$$\int_E \left(\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R)\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\int_{\partial T} \left(\sphericalangle(\{\nu\}, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu})\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

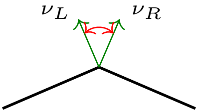


$$\{\nu\} := \frac{1}{\|\nu_L + \nu_R\|}(\nu_L + \nu_R)$$

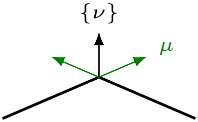


Computational aspect

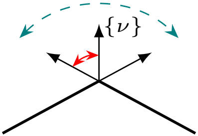
$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\int_{\partial T} (\sphericalangle(\{\nu\} \;, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

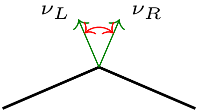


$$\{\nu\} \;:= \frac{\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R}{\|\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R\|}$$

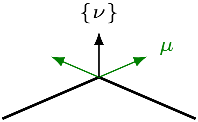


Computational aspect

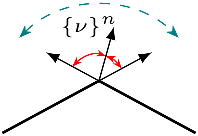
$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



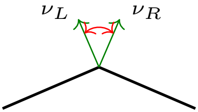
$$\int_{\partial T} (\sphericalangle(\{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



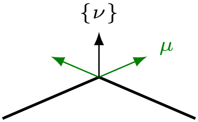
$$\{\nu\}^n := \frac{1}{\|{\nu_L}^n + {\nu_R}^n\|}({\nu_L}^n + {\nu_R}^n)$$



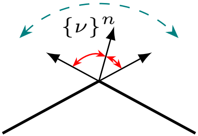
$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



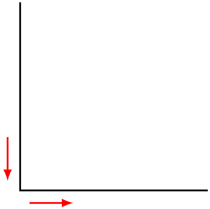
$$\int_{\partial T} (\sphericalangle(\boldsymbol{P}_\tau^\perp \{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\{\nu\}^n := \frac{1}{\|\nu_L{}^n + \nu_R{}^n\|}(\nu_L{}^n + \nu_R{}^n)$$

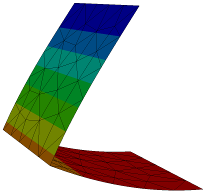


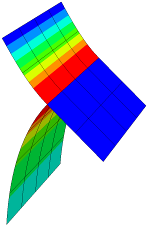
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$



$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \stackrel{!}{=} 0$$

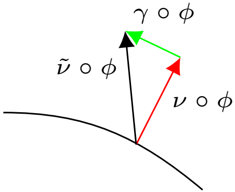
$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$





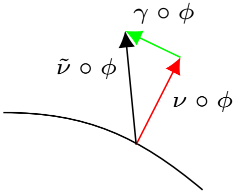
Naghdi shell model

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi+\gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu}+(\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg* (2013) 254, pp. 170–180.

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi+\gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu}+(\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = {}& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}}\\
&+ \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right) : \boldsymbol{\sigma}\,dx\\
&+ \sum_{E\in\mathcal{E}_h}\int_E \left(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R) + [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

Linearization

$$\mathcal{L}^{\text{shell}}_{\text{lin}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\text{cov}} u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat\nu} : \boldsymbol{\sigma}\,dx - \int_{\partial T}(\nabla u^\top \hat\nu)_{\hat\mu}\boldsymbol{\sigma}_{\hat\mu\hat\mu}\,ds\right)$$

$$\mathcal{L}^{\text{plate}}_{\text{lin}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\,dx - \int_{\partial T}\frac{\partial w}{\partial\hat\mu}\boldsymbol{\sigma}_{\hat\mu\hat\mu}\,ds\right)$$

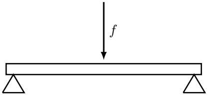
Kirchhoff–Love shell and plate

TECHNISCHE UNIVERSITÄT WIEN

$$\mathcal{L}_{\text{lin}}^{\text{shell}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\text{cov}} u)\|_{\boldsymbol{M}}^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\, dx - \int_{\partial T} (\nabla u^\top \hat{\nu})_{\hat{\mu}} \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

$$\mathcal{L}_{\text{lin}}^{\text{plate}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\, dx - \int_{\partial T} \frac{\partial w}{\partial \hat{\mu}} \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

$$\mathbf{divdiv}\nabla^2 w = f \Leftrightarrow \begin{cases} \boldsymbol{\sigma} = \nabla^2 w, \\ \mathbf{divdiv}\boldsymbol{\sigma} = f, \end{cases}$$



M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

$$
\begin{aligned}
\mathcal{L}_{\text{lin}}^{\text{shell}}(u,\boldsymbol{\sigma},\hat{\gamma}) &= \frac{t}{2}\|\text{sym}(\nabla^{\text{cov}} u)\|_{\boldsymbol{M}}^2 + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 \\
&\quad + \sum_{T\in\mathscr{T}_h}\Big(\int_T (\boldsymbol{H}_{\hat{\nu}} - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}((\nabla u^\top \hat{\nu})_{\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big) \\
\mathcal{L}_{\text{lin}}^{\text{plate}}(w,\boldsymbol{\sigma},\hat{\gamma}) &= \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\boldsymbol{M}^{-1}}^2 \\
&\quad + \sum_{T\in\mathscr{T}_h}\Big(\int_T (\nabla^2 w - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}(\frac{\partial w}{\partial\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\Big)
\end{aligned}
$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017) 137, pp. 713–740.

Membrane locking

Membrane locking

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

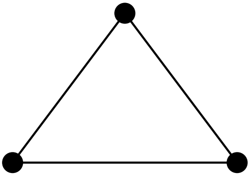
$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

Membrane locking

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$



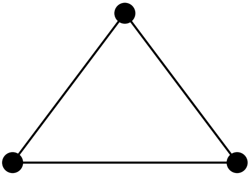
$V_h = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

TECHNISCHE UNIVERSITÄT WIEN

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



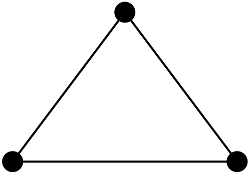
$V_h = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

Membrane locking

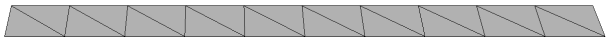
$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

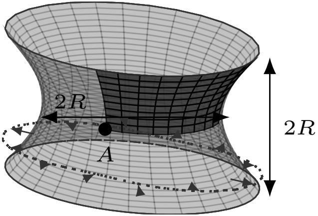
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$

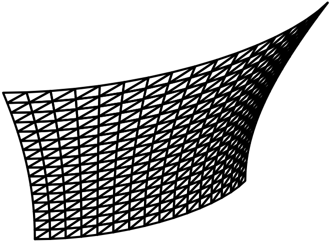


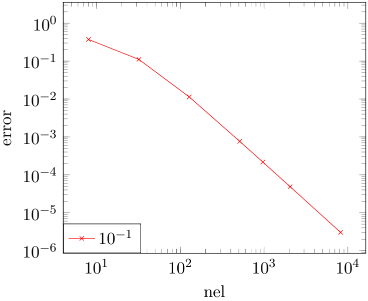
$$V_h = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$



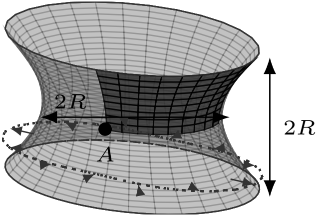
Hyperboloid with free ends

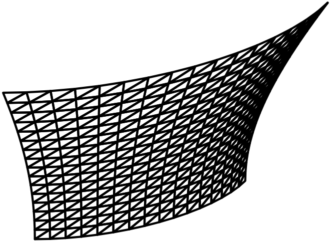


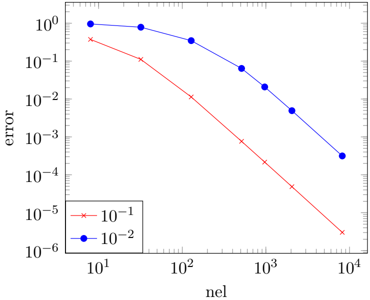




Hyperboloid with free ends

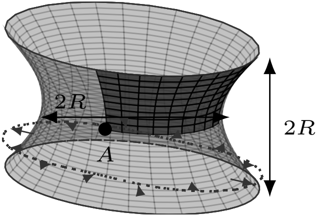


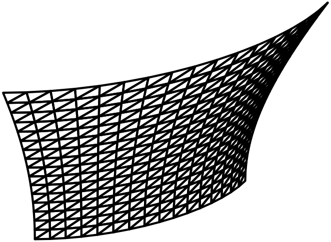


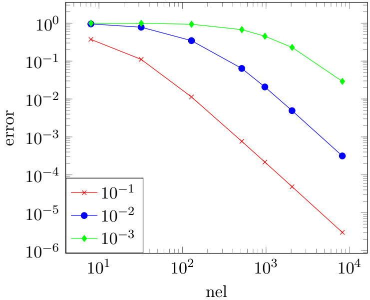


- Pre-asymptotic regime

Hyperboloid with free ends

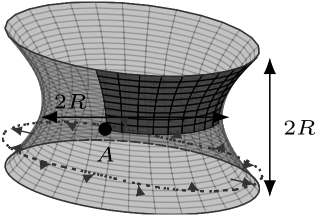


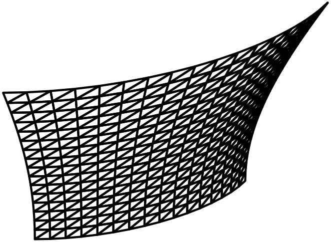


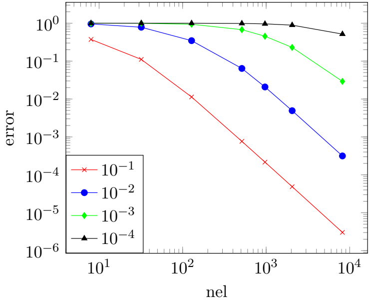


- Pre-asymptotic regime

Hyperboloid with free ends



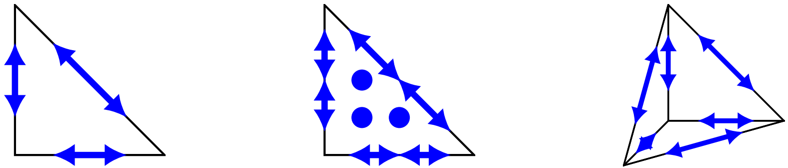




- Pre-asymptotic regime

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathrm{curl}\ (\mathrm{curl}\ \boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{2d-3\times 2d-3}\}$$



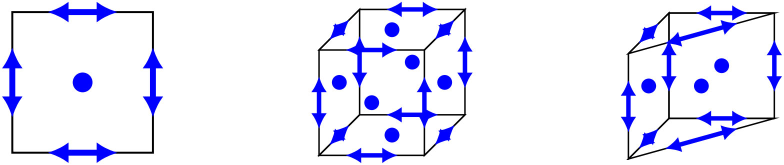
Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

Regge elements

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\mathcal{P}^k(\mathscr{T}_h)]_{sym}^{d\times d} \mid t^T \boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]_{sym}^{d\times d} \mid \mathrm{curl}\,(\mathrm{curl}\,\boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{2d-3\times 2d-3}\}$$



N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$

$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

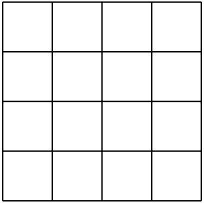
$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|^2_{\boldsymbol{M}}$$

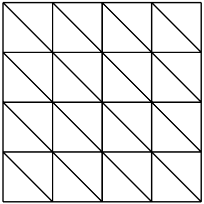
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\boldsymbol{M}}$$



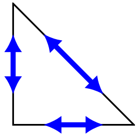
- Reduced integration for quadrilateral meshes

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}(u_h)\|_{\boldsymbol{M}}^2$$

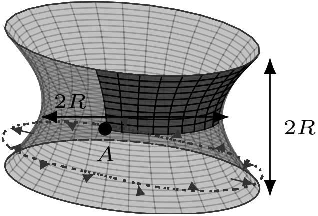


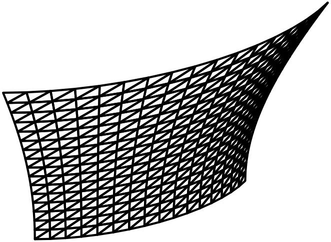
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

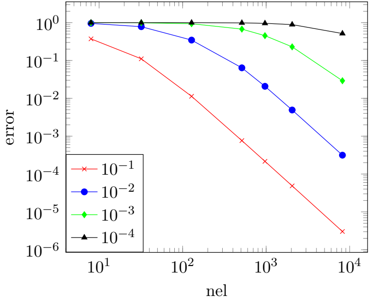


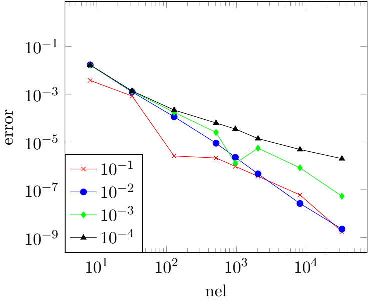
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends

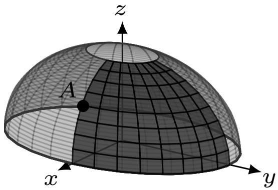


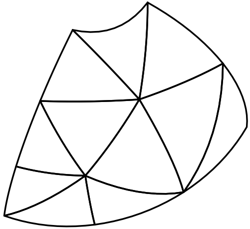


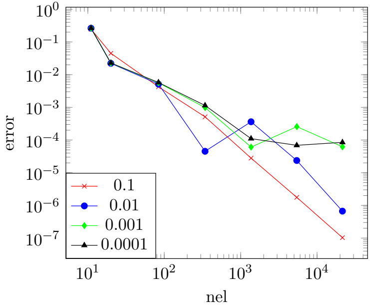


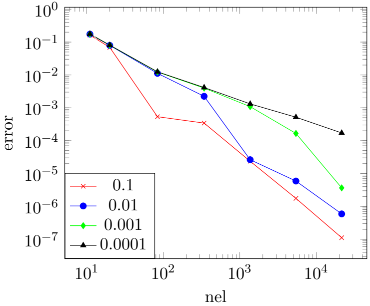


Open hemisphere with clamped ends



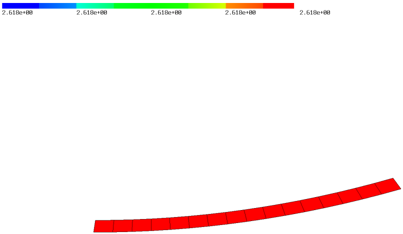


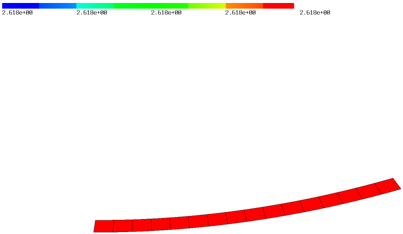




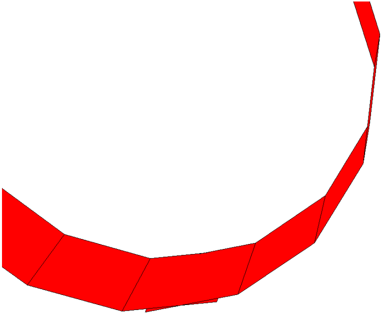
Numerical examples

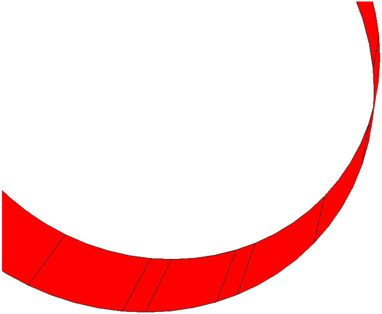
Cantilever subjected to end moment



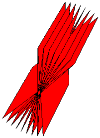


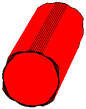
Cantilever subjected to end moment



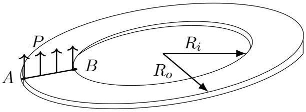


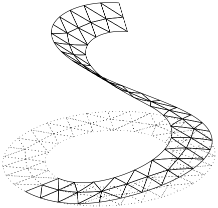
Cantilever subjected to end moment

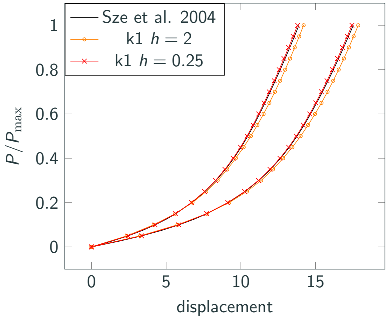




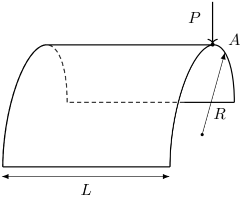
Slit annular plate

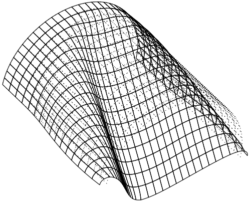


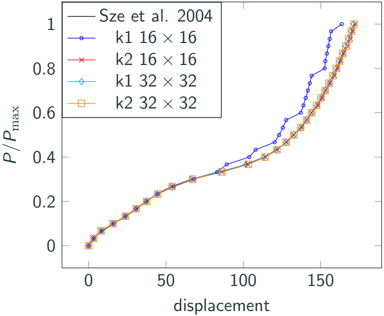




Pinched cylinder







Summary & outlook

- Distributional curvature for bending energy
- Method for (non-)linear Koiter and Naghdi shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

Summary & outlook

TECHNISCHE UNIVERSITÄT WIEN

- Distributional curvature for bending energy
- Method for (non-)linear Koiter and Naghdi shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

- Coupling for 3D elasticity
- Computing high-precision reference values
- NGSolve Add-On

Literature

- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.
- N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *arXiv:2107.13794*.

July 9-11, Portland, Oregon, USA

Portland State UNIVERSITY

www.ngsolve.org

www.youtube.com/@NGSolve

Registration open!

July 9-11, Portland, Oregon, USA

Portland State UNIVERSITY

www.ngsolve.org

www.youtube.com/@NGSolve

Registration open!

Thank You for Your attention!