Regge finite elements in differential geometry: metric, geodesic, and curvature approximation

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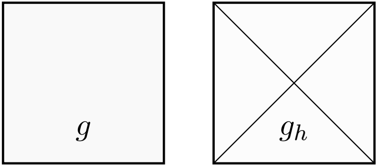
Der Wissenschaftsfonds.

Feb 15th, 2023

Motivation

- How and with which FE to approximate the metric of a Riemannian manifold?
- How to compute geodesics on such discrete, non-smooth metrics?
- How to compute curvature quantities like Gauss curvature on non-smooth metrics?



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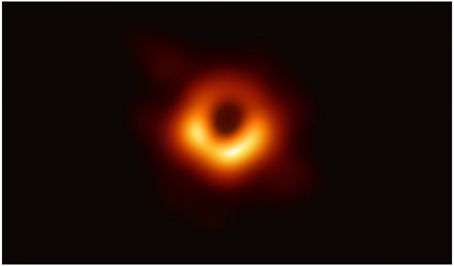

By Event Horizon Telescope (EHT)

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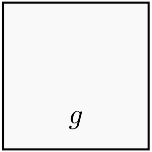
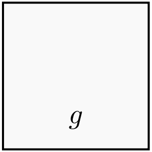
Riemannian manifold (M, g)

Riemannian manifolds



Riemannian manifold $(M \subset \mathbb{R}^2, g)$

Riemannian manifolds



Riemannian manifold (M, g)

Levi-Civita connection ∇

$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

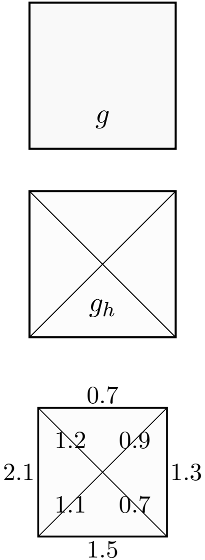
Riemannian manifolds

Riemannian manifold (M, g)

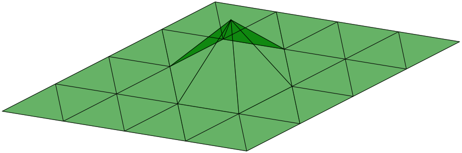
Levi-Civita connection ∇

$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

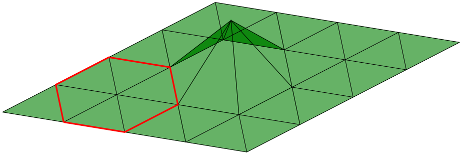
- Approximation of g on a triangulation
- Compute lengths and angles
- Regge calculus and finite elements

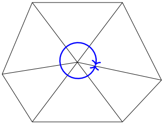


Regge calculus and elements



Regge: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).





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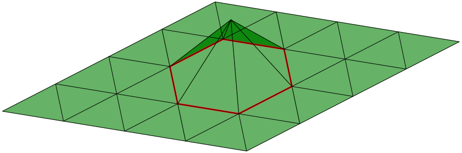

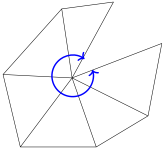

- angle defect

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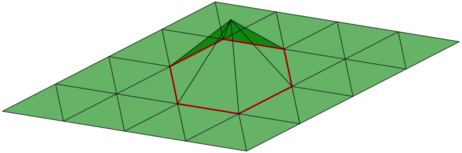

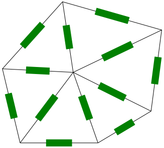

- metric tensor

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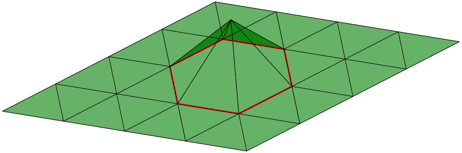
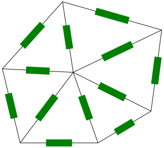
SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D* *12* (1975).





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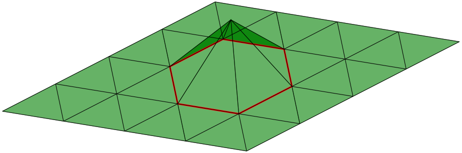
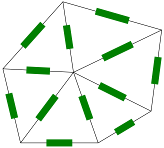
CHEEGER, MÜLLER, SCHRADER: On the curvature of piecewise flat spaces, *Communications in Mathematical Physics, 92(3) (1984).*

Regge calculus and elements





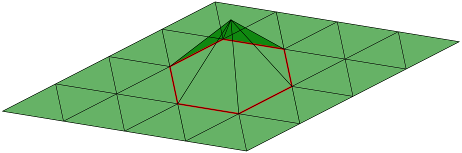
- metric tensor (tangential-tangential continuous)

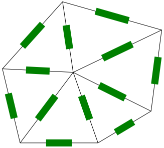
$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) = \{\varepsilon \in L^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, \mathrm{curl}^\top \mathrm{curl}(\varepsilon) \in H^{-1}(\Omega, \mathbb{R}^{(2d-3)\times(2d-3)})\}$$

Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Regge calculus and elements





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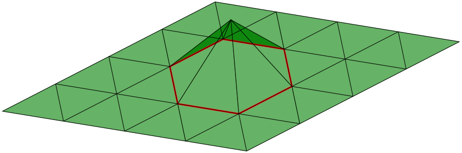

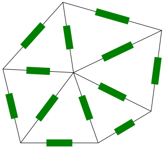

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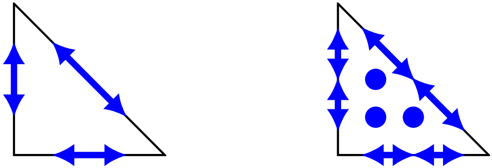
N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*

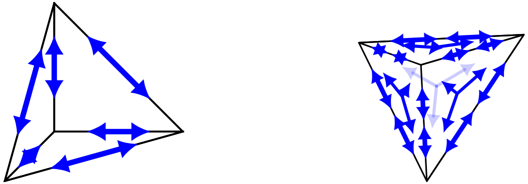


$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k, \qquad i \neq j \neq k$$

$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2})$$



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$$\int_F (g - \mathcal{R}_h^k g) : Q_F\, da = 0 \text{ for all } Q_F \in \mathcal{P}^{k-1}(F, \mathbb{R}^{3\times 3} \cap TF)$$

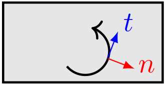
$$\int_T (g - \mathcal{R}_h^k g) : Q_T\, dx = 0 \text{ for all } Q_T \in \mathcal{P}^{k-2}(T, \mathbb{R}^{3\times 3})$$

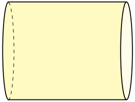
Geodesics

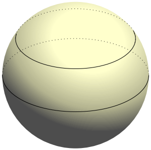
Geodesic curvature

$\kappa(g) = g(\nabla_t t, n)$

$n = t \times \nu$ (for embedded surface)







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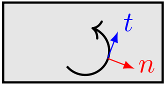
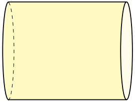
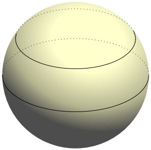




Christoffel symbols: $\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l$

$$\Gamma^k_{ij}(g) = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) = g^{kl}\Gamma_{ijl}(g)$$

$$\nabla_X Y = (X^i\partial_i Y^k + X^i Y^j \Gamma^k_{ij})\partial_k$$



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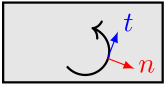

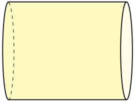

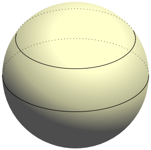

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$$\kappa(g) = \frac{\sqrt{\det g}}{g_{\hat{t}\hat{t}}^{3/2}}\left(\partial_{\hat{t}}\hat{t}\cdot\hat{n} + \Gamma^{\hat{n}}_{\hat{t}\hat{t}}\right)$$

$$\Gamma^{\hat{n}}_{\hat{t}\hat{t}} = \Gamma^k_{ij}\hat{t}^i\hat{t}^j\hat{n}_k$$

$\gamma : [a, b] \to M$ is geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$, i.e.

$$\ddot\gamma^k + \dot\gamma^i\dot\gamma^j\Gamma^k_{ij} = 0, \qquad k = 1, \ldots, n$$

Geodesics are critical points of energy

$$E(\gamma) = \frac{1}{2}\int_0^T g_{ij}(\gamma(s))\,\dot\gamma^i(s)\dot\gamma^j(s)\,ds$$

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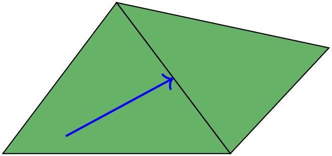

Geodesic curve

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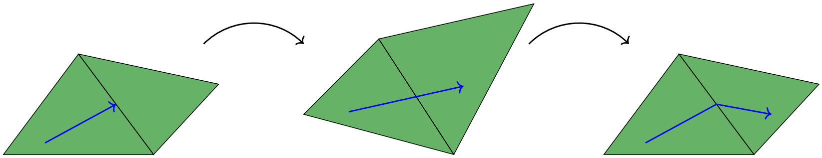

Discrete local geodesic

Let $g \in \mathrm{Reg}_h^k$. A pw smooth $\gamma : [a, b] \to \mathscr{T}$ not intersecting faces of dimension $\leq (N-2)$ is a local geodesic iff it satisfies the geodesic equation in T and where γ intersects a facet F

$$g_{ij}^+ \dot{\gamma}_+^i t^j = g_{ij}^- \dot{\gamma}_-^i t^j, \qquad g_{ij}^+ \dot{\gamma}_+^i n_+^j = -g_{ij}^- \dot{\gamma}_-^i n_-^j.$$

Its kinematic energy $g_{ij}\dot{\gamma}^i\dot{\gamma}^j$ is constant.

Update formula: $\dot{\gamma}_-^i = \dot{\gamma}_+^i - (g_{ij}^+ \dot{\gamma}_+^j n_+^k)(n_+^i + n_-^i)$

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

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Convergence geodesic

Assume that $g_h \in \mathrm{Reg}_h^k$ is a family of Regge metrics fulfilling $\|g - g_h\|_{L^\infty} \leq \frac{1}{2}\|g^{-1}\|_{L^\infty}^{-1}$ uniformly in h. Let $\gamma : [0, T] \to M$ be a smooth geodesic and γ_h a family of geodesics with respect to g_h with the same initial conditions as γ. Then

$$|\dot\gamma(t) - \dot\gamma_h(t)| \leq C\left(\|g - g_h\|_{W^{1,\infty}(\mathscr{T}_h)} + h^{-1}\|g - g_h\|_{L^\infty}\right),$$
$$|\gamma(t) - \gamma_h(t)| \leq C\left(h\|g - g_h\|_{W^{1,\infty}(\mathscr{T}_h)} + \|g - g_h\|_{L^\infty}\right).$$

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$$|\dot{\gamma}(t) - \dot{\gamma}_h(t)| \leq C\, h^k |g|_{W^{k+1,\infty}},$$
$$|\gamma(t) - \gamma_h(t)| \leq C h^{k+1} |g|_{W^{k+1,\infty}}.$$

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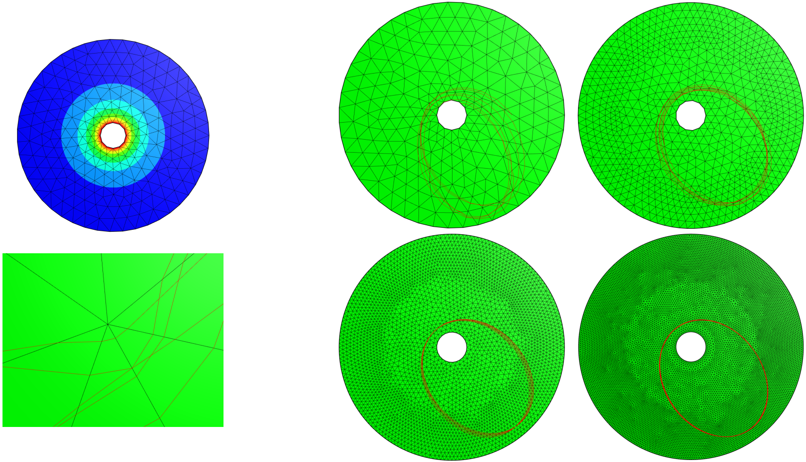
$E = E_{\mathrm{kin}} + V,\ V(q) = -|q|^{-1}$

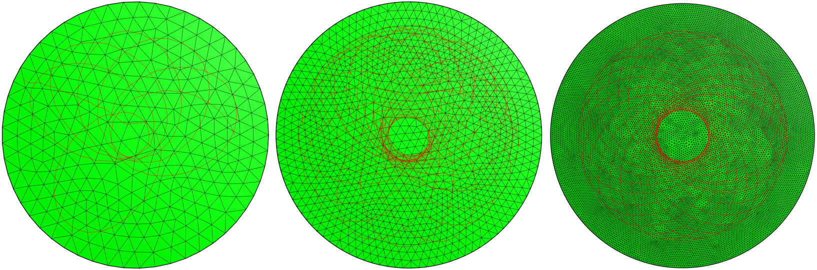
Jacobi metric: $g = 2(E - V)\delta = 2(E + |q|^{-1})\delta$

NGSolve



Taking into account of relativistic effects (Mercury orbit)
Star mass M, particle mass m, total energy $E \leq m$

$$ds = \left(E^2 - m^2 + \frac{2Mm^2}{r}\right)\left(\frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + \frac{r^2 d\Phi^2}{1 - \frac{2M}{r}}\right)$$



Distributional Gauss and scalar curvature

Riemann curvature tensor:

$$\mathfrak{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y,Z,W) = g(\mathfrak{R}(X,Y)Z, W)$$

$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma^p_{ik} \Gamma_{jpl} - \Gamma^p_{jk} \Gamma_{ipl}$$

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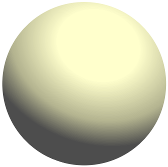
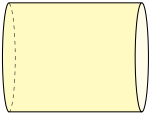


- **Gauss curvature**

$$K = \frac{g(R(u,v)v,u)}{\|u\|_g \|v\|_g - g(u,v)^2} = \frac{R_{1221}}{\det g}$$



- **Scalar curvature**

$$S = g^{ik} g^{jl} R_{ijkl}$$

Riemann curvature tensor:

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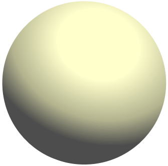
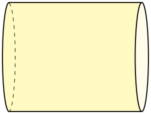


- **Ricci tensor**

$$\mathrm{Ric}_{ij} = g^{ab} R_{aibj}$$



- **Einstein tensor**

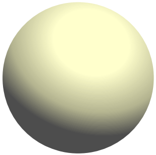
$$G = \mathrm{Ric} - \frac{1}{2} S\, g$$

Gauss–Bonnet

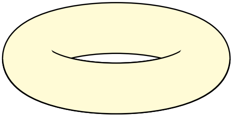
On manifold M:

$$\int_M K(g) + \int_{\partial M} \kappa(g) + \sum_V (\pi - \sphericalangle_V^M(g)) = 2\pi\chi_M$$

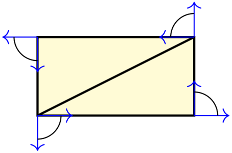
$$\chi_M(\mathscr{T}) = n_V - n_E + n_T$$



$\chi_M = 2$



$\chi_M = 0$



$\chi_M = 1$

Gauss–Bonnet

On triangle T:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle_{V_i}^T(g)) = 2\pi$$

$$\chi_T = 3 - 3 + 1 = 1$$

Gauss–Bonnet

On triangle T:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle_{V_i}^T(g)) = 2\pi$$

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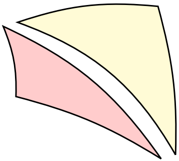

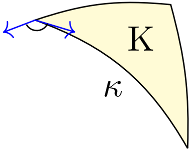

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Gauss–Bonnet

On triangle T:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle_{V_i}^T(g)) = 2\pi$$

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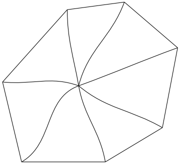

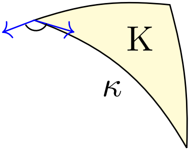

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Distributional densitized Gauss curvature

Let $g \in \mathrm{Reg}_h^k(\mathscr{T})$ and $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\langle (K\omega)(g), \varphi \rangle = \sum_{T\in\mathscr{T}} K_T(\varphi, g) + \sum_{E\in\mathscr{E}} K_E(\varphi, g) + \sum_{V\in\mathscr{V}} K_V(\varphi, g)$$

BERCHENKO-KOGAN, GAWLIK: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Distributional densitized Gauss curvature

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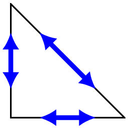
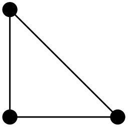


$$K_T(\varphi, g) = \int_T K(g)\, \varphi\, \omega_T$$

$$K_E(\varphi, g) = \int_E [\![\kappa(g)]\!] \varphi\, \omega_E$$

$$K_V(\varphi, g) = \Big(2\pi - \sum_{T: V \subset T} \sphericalangle_V^T(g)\Big)\varphi(V)$$

$$=: \Theta_V(g)\, \varphi(V)$$



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$$K_T(\varphi, g) = \int_T K(g)\,\varphi\,\omega_T$$

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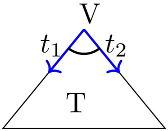

$$\sphericalangle_V^T(g) = \arccos\left(\frac{t_1^\top g t_2}{\|t_1\|_g \|t_2\|_g}\right)$$

Distributional scalar curvature

- Second fundamental form $\mathrm{I\!I}(X,Y) = g(\nabla_X n, Y)$
  2D $\mathrm{I\!I}(t,t) = \kappa(g)$
- mean curvature $H = \mathrm{tr}(\mathrm{I\!I}\,|_F)$

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- Second fundamental form $\mathrm{II}(X,Y)=g(\nabla_X n, Y)$
  2D $\mathrm{II}(t,t)=\kappa(g)$
- mean curvature $H=\mathrm{tr}(\mathrm{II}\,|_F)$

Distributional densitized scalar curvature

Let $g\in\mathrm{Reg}_h^k$ and $\varphi\in\mathring{\mathcal{V}}_h^q$

$$\langle (S\omega)(g),\varphi\rangle = \sum_{T\in\mathscr{T}}\int_T S\varphi\,\omega_T + 2\sum_{F\in\mathscr{F}}\int_F [\![H]\!]\varphi\,\omega_F + 2\sum_{E\in\mathscr{E}}\int_E \Theta_E(g)\,\varphi\,\omega_E$$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

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Convergence scalar curvature 2D

Let $\{g_h\}_{h>0}$ Regge metrics on shape-regular $\{\mathcal{T}_h\}_{h>0}$ triangulations of Ω. Assume $\lim_{h\to 0}\|g_h - g\|_{L^\infty(\Omega)} = 0$ and $C_1 = \sup_{h>0}\max_{T\in\mathcal{T}_h}\|g_h\|_{W^{1,\infty}(T)} < \infty$. Then

$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}} \leq C\Big(1 + \max_T h_T^{-1}\|g_h - g\|_{L^\infty(T)} + \max_T |g_h - g|_{W^{1,\infty}(T)}\Big)\|\!|g_h - g|\!\|$$

$$\|\!|\sigma|\!\|^2 = \|\sigma\|^2_{L^2(\Omega)} + \sum_T h_T^2 |\sigma|^2_{H^1(T)}$$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature 2D

Let $\{g_h\}_{h>0} \in \mathrm{Reg}_h^k$ optimal-order interpolants on shape-regular $\{\mathcal{T}_h\}_{h>0}$ triangulations of Ω. Assume $\lim_{h\to 0} \|g_h - g\|_{L^\infty(\Omega)} = 0$ and $C_1 = \sup_{h>0} \max_{T\in\mathcal{T}_h} \|g_h\|_{W^{1,\infty}(T)} < \infty$. Then

$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}} \leq C\, h^{k+1} |g|_{H^{k+1}}$$

$|\!|\!|\sigma|\!|\!|^2 = \|\sigma\|^2_{L^2(\Omega)} + \sum_T h_T^2 |\sigma|^2_{H^1(T)}$

Optimal-order interpolant: $|\mathcal{I}_h g - g|_{H^m(T)} \leq C h_T^{k-m} |g|_{H^m(T)}$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

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Convergence scalar curvature $N > 2$

Assume additionally $C_2 = \sup_{h>0} \max_{T\in\mathcal{T}_h} |g_h|_{W^{2,\infty}(T)} < \infty$.

$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}(\Omega)} \leq C\Big(1 + \max_T h_T^{-2}\|g_h - g\|_{L^\infty(T)} + \max_T h_T^{-1}|g_h - g|_{W^{1,\infty}(T)}\Big)\|\!|g_h - g|\!\|.$$

$$\|\!|\sigma|\!\|^2 = \|\sigma\|^2_{L^2(\Omega)} + \sum_T h_T^2|\sigma|^2_{H^1(T)} + \sum_T h_T^4|\sigma|^2_{H^2(T)}$$

Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence scalar curvature $N>2$

Assume additionally $C_2 = \sup_{h>0} \max_{T\in\mathcal{T}_h} |g_h|_{W^{2,\infty}(T)} < \infty$.

$$\|(S\omega)(g_h) - (S\omega)(g)\|_{H^{-2}(\Omega)} \leq C\, h^{k+1} |g|_{H^{k+1}}$$

for $k>0$.

$$|\!|\!|\sigma|\!|\!|^2 = \|\sigma\|^2_{L^2(\Omega)} + \sum_T h_T^2 |\sigma|^2_{H^1(T)} + \sum_T h_T^4 |\sigma|^2_{H^2(T)}$$

GAWLIK, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

Convergence result (Gauss curvature)

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Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathring{\mathcal{V}}_h^q$ such that for all $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Convergence Gauss curvature

$\lim_{h\to 0} \|g_h - g\|_{L^\infty(\Omega)} = 0$, $\lim_{h\to 0} h^{-1} \log h^{-1} \|g_h - g\|_{L^2(\Omega)} = 0$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l-1} \Big(\|g_h - g\|_{H_h^1} + \inf_{u_h \in \mathring{\mathcal{V}}_h^k} \|K - u_h\|_{L^2} \Big)$$

GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Convergence result (Gauss curvature)

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathring{\mathcal{V}}_h^q$ such that for all $\varphi \in \mathring{\mathcal{V}}_h^q$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Convergence Gauss curvature

$g_h \in \mathrm{Reg}_h^k$ opt-order interp, $-1 \leq l \leq k-2$, $q \geq \max\{1, k-2\}$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k-1}\left(|g|_{H^{k+1}} + |K|_{H^k}\right)$$

Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Convergence result (Gauss curvature)

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in \mathring{\mathcal{V}}_h^{k+1}$ such that for all $\varphi \in \mathring{\mathcal{V}}_h^{k+1}$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Convergence Gauss curvature

$g_h \in \mathrm{Reg}_h^k$ canonical interpolant, $-1 \leq l \leq k-1$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k}(|g|_{W^{k+1,\infty}} + |K|_{H^k})$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

Numerical example (scalar curvature 2D and 3D)

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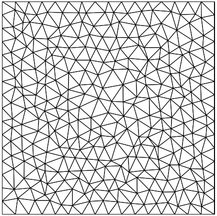
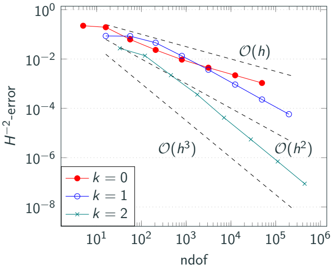
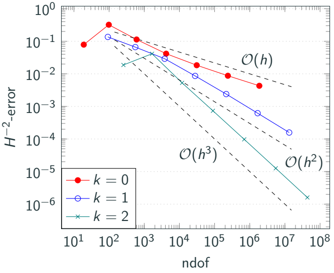
NGSolve



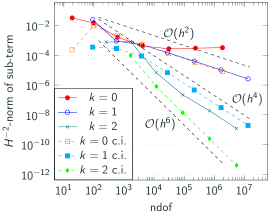


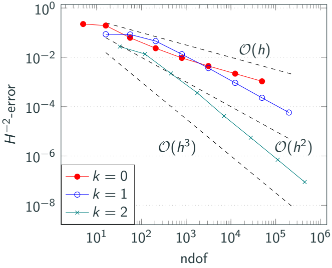


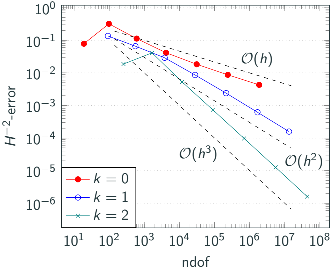
2D

3D

Numerical example (scalar curvature 2D and 3D)



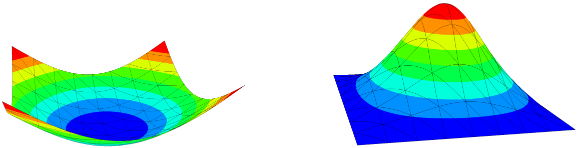




2D

3D

Numerical example (Gauss curvature)

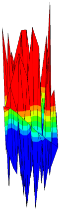


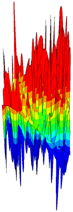
$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \qquad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

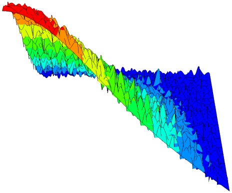
$$K(g) = \frac{81(1 - x^2)(1 - y^2)}{(9 + x^2(x^2 - 3)^2 + y^2(y^2 - 3)^2)^2}$$

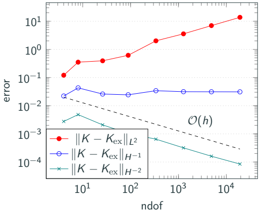
Numerical example (Gauss curvature)

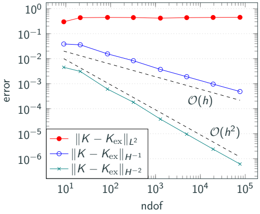
Optimal-order interpolant ($q = k + 1$)

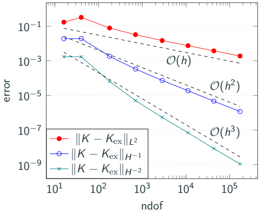












$k = 0$ $k = 1$ $k = 2$

Numerical example (Gauss curvature)

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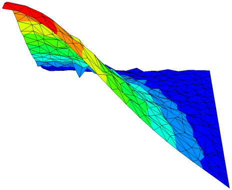
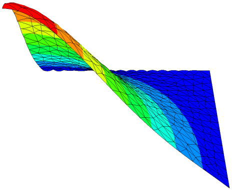
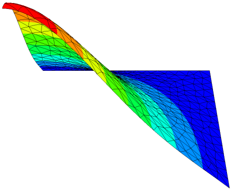
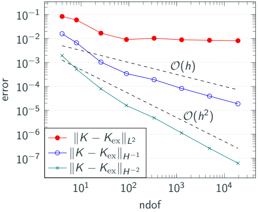
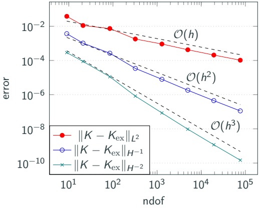
Canonical interpolant

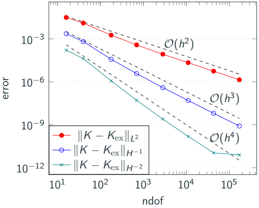












$k = 0$ $k = 1$ $k = 2$

Idea of proofs: Integral representation

$$\langle (K\omega)(g), \varphi \rangle = \sum_{T\in\mathscr{T}} \int_T K(g)\,\varphi\,\omega_T + \sum_{E\in\mathscr{E}} \int_E [\![\kappa(g)]\!]\varphi\,\omega_E + \sum_{V\in\mathscr{V}} \Theta_V(g)\,\varphi(V)$$

- Consistency: For $g \in C^2(M, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \varphi \in \mathring{\mathcal{V}}_h^q$ there holds $\langle (K\omega)(g), \varphi \rangle = \int_{\mathscr{T}} K(g)\varphi\,\omega$

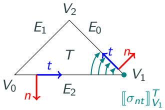
Idea of proofs: Integral representation

$$\langle (K\omega)(g), \varphi\rangle = \sum_{T\in\mathscr{T}} \int_T K(g)\,\varphi\,\omega_T + \sum_{E\in\mathscr{E}} \int_E [\![\kappa(g)]\!]\varphi\,\omega_E + \sum_{V\in\mathscr{V}} \Theta_V(g)\,\varphi(V)$$

- Consistency: For $g \in C^2(M, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \varphi \in \mathring{\mathcal{V}}_h^q$ there holds $\langle (K\omega)(g), \varphi\rangle = \int_{\mathscr{T}} K(g)\varphi\,\omega$
- Observation: $(\mathbb{S}_g\sigma = \sigma - \mathrm{tr}_g(\sigma)\,g)$

$$\frac{d}{dt}\big((K\omega)(g + t\sigma)\big)|_{t=0} = \frac{1}{2}\,\mathrm{div}_g\mathrm{div}_g(\mathbb{S}_g\sigma)\,\omega_g$$

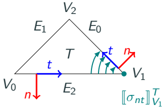
Hellan–Herrmann–Johnson method



$$\langle \mathrm{divdiv}\mathbb{S}\sigma, u_h\rangle = \sum_{T\in\mathscr{T}} \int_T \mathrm{divdiv}\mathbb{S}\sigma\, u_h\, dx - \sum_{E\in\mathscr{E}} \int_E [\![(\mathrm{div}\mathbb{S}\sigma)_n + \nabla_t(\sigma_{nt})]\!] u_h\, ds + \sum_{V\in\mathscr{V}} \sum_{T\supset V} [\![\sigma_{nt}]\!]_V^T u_h(V)$$

Idea of proofs: Integral representation

Hellan–Herrmann–Johnson method



$$\langle \mathrm{div}\mathrm{div}\mathbb{S}\sigma, u_h\rangle = \sum_{T\in\mathscr{T}}\int_T \mathrm{div}\mathrm{div}\mathbb{S}\sigma\, u_h\,dx - \sum_{E\in\mathscr{E}}\int_E [\![(\mathrm{div}\mathbb{S}\sigma)_n + \nabla_t(\sigma_{nt})]\!] u_h\,ds$$
$$+ \sum_{V\in\mathscr{V}}\sum_{T\supset V}[\![\sigma_{nt}]\!]_V^T u_h(V)$$

$$G(t) = g_h + t(g-g_h),\ \sigma_h = G'(t) = g - g_h$$

$$\langle (K\omega)(g_{)}, u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = \frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)}\mathrm{div}_{G_h(t)}\mathbb{S}_{G_h(t)}(\sigma_h), u_h\rangle\,dt$$

GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Idea of proofs: Integral representation

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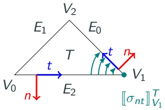
Hellan–Herrmann–Johnson method



$$\langle \mathrm{div}\mathrm{div}\mathbb{S}\sigma, u_h\rangle = \sum_{T\in\mathscr{T}}\int_T \mathrm{div}\mathrm{div}\mathbb{S}\sigma\, u_h\,dx - \sum_{E\in\mathscr{E}}\int_E [\![(\mathrm{div}\mathbb{S}\sigma)_n + \nabla_t(\sigma_{nt})]\!] u_h\,ds + \sum_{V\in\mathscr{V}}\sum_{T\supset V}[\![\sigma_{nt}]\!]_V^T u_h(V)$$

$$G(t) = g_h + t(g - g_h),\ \sigma_h = G'(t) = g - g_h$$

$$\langle (K\omega)(g), u_h\rangle - \langle (K\omega)(g_h), u_h\rangle = \frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)}\mathrm{div}_{G_h(t)}\mathbb{S}_{G_h(t)}(\sigma_h), u_h\rangle\,dt$$

Show $|\langle \mathrm{div}_{G_h(t)}\mathrm{div}_{G_h(t)}\mathbb{S}_{G_h(t)}(\sigma_h), u_h\rangle| \leq Ch^{-1}\|g_h - g\|_{H^1_h}\|u_h\|_{H^1}$

GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).

Integral representation scalar curvature

$$\frac{d}{dt}\langle (S\omega)(g(t)), u\rangle = \langle \mathrm{div}_g \mathrm{div}_g \mathbb{S}_g \sigma, u\rangle - a_h(g;\sigma,u)$$

$$\overline{\mathrm{I\!I}} = \mathrm{I\!I} - H\, g$$

$$a_h(g;\sigma,u) = \sum_{T\in\mathscr{T}} \int_T \langle G,\sigma\rangle u\omega_T + \sum_{F\in\mathscr{F}} \int_F \langle [\![\overline{\mathrm{I\!I}}]\!]_F, \sigma|_F\rangle u\omega_F - \sum_{E\in\mathscr{E}} \int_E \langle \Theta_E g|_E, \sigma|_E\rangle u\omega_E$$

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leads to distributional Einstein tensor for $u = 1$

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Orthogonality properties of canonical interpolant

Alternative representation: $\langle \mathrm{div}_g \mathrm{div}_g \mathbb{S}_g \sigma, u_h \rangle = -\langle \mathrm{inc}_g\, \sigma, u_h \rangle$

$$\int_E (\sigma - \mathcal{R}_h^k \sigma)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (\sigma - \mathcal{R}_h^k \sigma) : q\, da = 0 \text{ for all } q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2})$$

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$$|\langle \mathrm{inc}_g(\sigma - \mathcal{R}_h^k \sigma), u_h\rangle| \leq C \|\sigma - \mathcal{R}_h^k \sigma\| \|u_h\|_{H^1} \text{ for all } u_h \in \mathring{\mathcal{V}}_h^{k+1}$$

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Distributional Riemann curvature

Extension of Gauss curvature to 3D and nD

- Riemann curvature tensor R_{ijkl} has 6 independent entries
- Curvature operator $Q : M \to \bigwedge^2(M) \odot \bigwedge^2(M)$

$$\langle Q(u \wedge v), w \wedge z \rangle = \langle R(u,v)z, w \rangle \text{ for all } u, v, w, z \in \mathfrak{X}(M)$$

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- No Gauss–Bonnet theorem in 3D

Distributional Riemann curvature

B fourth order tensor with same (skew-)symmetries as R

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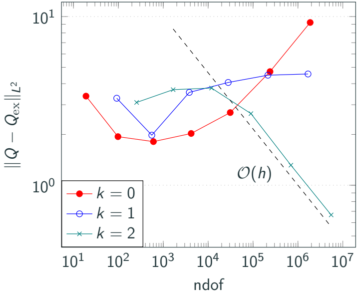
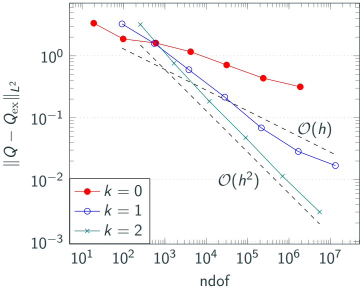
Analysis: WIP

Numerical example (Curvature operator 3D)



optimal-order interpolant



canonical interpolant

Summary

TECHNISCHE UNIVERSITÄT WIEN

- Regge finite elements approximating metrics
- Approximation of geodesics and curvature quantities

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- Regge finite elements approximating metrics
- Approximation of geodesics and curvature quantities

- Analysis of distributional Riemann curvature tensor
- Analysis of Einstein tensor
- Application to general relativity (Einstein field equation)

Literature

- Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota* (2018).
- Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM J. Numer. Anal.* (2020).
- Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.
- Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *arXiv:2301.02159*.

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Thank You for Your attention!