Advanced Numerical Methods for Fluid-Structure Interaction

Michael Neunteufel (TU Wien)
Joachim Schöberl (TU Wien)

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Contents

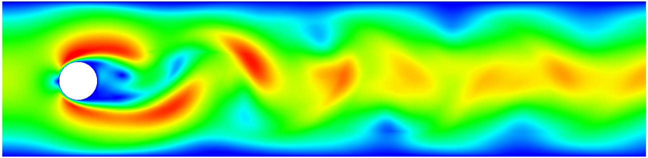
Navier-Stokes equations

Navier-Stokes equations

$$u(x,t)\ldots \text{ velocity}$$
$$p(x,t)\ldots \text{ pressure}$$

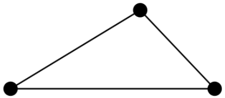
Navier-Stokes

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= f \\ \operatorname{div}(u) &= 0 \end{aligned}$$



Taylor-Hood elements

- H^1 elements for velocity and pressure
- $V := [\Pi^2(\mathcal{T}_h)]^2 \cap [C^0(\Omega)]^2$
- $P := \Pi^1(\mathcal{T}_h) \cap C^0(\Omega)$



$$
\begin{aligned}
\int_\Omega \frac{\partial u}{\partial t} \cdot v + (u \cdot \nabla) u \cdot v + \nu \nabla u : \nabla v - \mathrm{div}(v) p \, dx &= 0 \quad \forall v \in V \\
\int_\Omega \mathrm{div}(u) q \, dx &= 0 \quad \forall q \in P
\end{aligned}
$$

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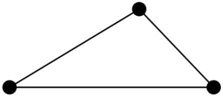

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- Only discrete divergence-freeness

$$\int_\Omega \operatorname{div}(u) q \, dx = 0 \quad \forall q \in P \quad \not\Rightarrow \quad \operatorname{div}(u) = 0$$

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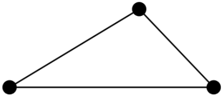

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v + (u \cdot \nabla) u \cdot v + \nu \nabla u : \nabla v - \mathrm{div}(v) p \, dx = 0 \quad \forall v \in V$$
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- Only discrete divergence-freeness

$$\int_\Omega \mathrm{div}(u) q \, dx = 0 \quad \forall q \in P \quad \not\Rightarrow \quad \mathrm{div}(u) = 0$$

$$\mathrm{div}(V) \quad \not\subseteq \quad P$$

H(div)-conforming HDG

- Velocity in H(div) $:= \{u \in [L^2(\Omega)]^n : \mathrm{div}(u) \in L^2(\Omega)\}$

H(div)-conforming HDG

- Velocity in $W := \{u \in [\Pi^k(\mathcal{T}_h)]^n : [\![u \cdot n]\!]_F = 0, \forall F \in \mathcal{F}_h\}$

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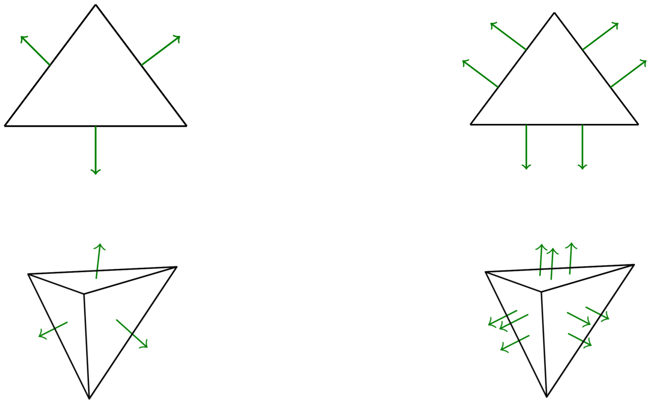
- Raviart-Thomas and BDM elements



H(div)-conforming HDG

- Pressure in $L^2(\Omega)$

- Pressure in $Q := \Pi^k[\mathcal{T}_h]$

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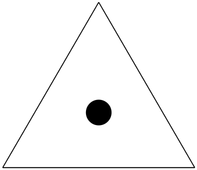
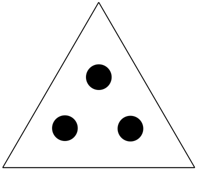




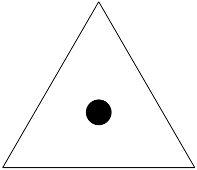
H(div)-conforming HDG

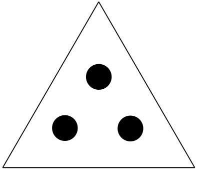
- Pressure in $Q := \Pi^k[\mathcal{T}_h]$

- There holds

$$\mathrm{div}(H(\mathrm{div})) \quad \subset \quad L^2(\Omega)$$

- Pressure in $Q := \Pi^k[\mathcal{T}_h]$





- There holds

$$\begin{aligned} \mathrm{div}(H(\mathrm{div})) &\subset L^2(\Omega) \\ \mathrm{div}(W) &\subset Q \end{aligned}$$

De Rham complex part 1

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

- (Exact) sequence in continuous setting
- Simply connected domains

De Rham complex part 1

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

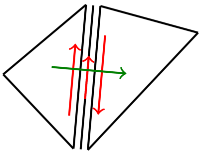
- (Exact) sequence in continuous setting
- Simply connected domains
- $\ker(\mathrm{div}) = \mathrm{range}(\mathrm{curl})$
- $\mathrm{div}(f) = 0 \quad \Rightarrow \quad \exists A : \mathrm{curl}(A) = f$

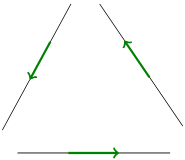
De Rham complex part 1

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
 & & & & \bigcup & & \bigcup \\
 & & & & W_h & \xrightarrow{\mathrm{div}_h} & Q_h
\end{array}
$$

- (Exact) sequence in continuous setting
- Simply connected domains
- $\ker(\mathrm{div}) = \mathrm{range}(\mathrm{curl})$
- $\mathrm{div}(f) = 0 \quad \Rightarrow \quad \exists A : \mathrm{curl}(A) = f$
- Mimic this sequence in discrete setting

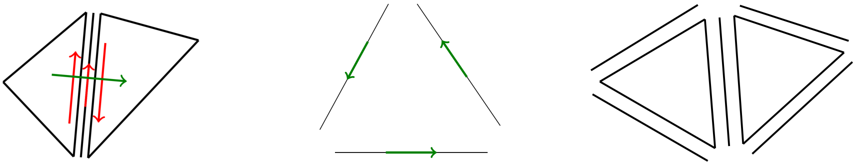
- Additional facet variables for tangential continuity





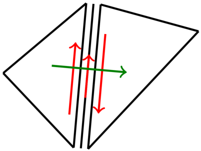
H(div)-conforming HDG

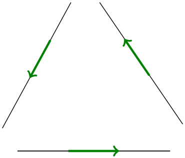
- Additional facet variables for tangential continuity
- $F := \{\hat{u} \in [\Pi^k(\mathcal{F}_h)]^n : \; \hat{u} \cdot n = 0\}$

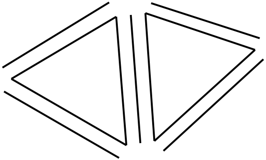


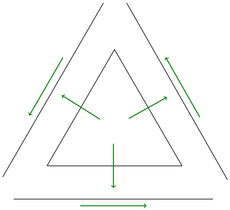
H(div)-conforming HDG

- Additional facet variables for tangential continuity
- $F := \{\hat{u} \in [\Pi^k(\mathcal{F}_h)]^n : \ \hat{u} \cdot n = 0\}$
- $V := W \times F$









- Polynomials on each triangle T

$$
\begin{aligned}
&-\int_T \Delta u v \\
&= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} v
\end{aligned}
$$

LEHRENFELD *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

H(div)-conforming Laplace

- Polynomials on each triangle T

$$
\begin{aligned}
&-\int_T \Delta u v \\
&= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v - \bar{v})
\end{aligned}
\qquad \bar{v} := v_n + \hat{v}_\tau
$$

Lehrenfeld *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

H(div)-conforming Laplace

- Polynomials on each triangle T

$$
\begin{aligned}
&-\int_T \Delta u v \\
&= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} \underbrace{(v - \bar{v})}_{=(v-\hat{v})_\tau}
\end{aligned}
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Lehrenfeld *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

- Polynomials on each triangle T

$$-\int_T \Delta u v$$
$$= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} \underbrace{(v - \bar{v})}_{=(v-\hat{v})_\tau}$$
$$= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v - \hat{v})_\tau - \frac{\partial v}{\partial n}(u - \hat{u})_\tau$$

- Consistency term for symmetry

LEHRENFELD *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

H(div)-conforming Laplace

- Polynomials on each triangle T

$$
\begin{aligned}
&-\int_T \Delta u v \\
&= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} \underbrace{(v - \bar{v})}_{=(v-\hat{v})_\tau} \\
&= \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v - \hat{v})_\tau - \frac{\partial v}{\partial n}(u - \hat{u})_\tau + \alpha (v - \hat{v})_\tau (u - \hat{u})_\tau
\end{aligned}
$$

- Consistency term for symmetry
- Stability term, $\alpha = c(\Omega)\frac{k^2}{h}$

LEHRENFELD *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

H(div)-conforming HDG Stokes

- Incompressibility constraint

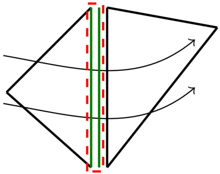
$$\int_\Omega \mathrm{div}(u) q \, dx = 0$$

- exact divergence-free solutions

$$\int_\Omega \mathrm{div}(u) q \, dx = 0 \quad \forall q \in Q \Rightarrow \mathrm{div}(u) = 0$$

H(div)-conforming HDG convection

- Up-winding technique for convection
- Glueing facet variable to upwind triangle



H(div)-conforming HDG convection

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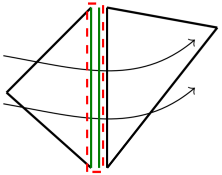
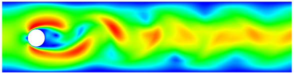


$$c(u, \hat{u}, v, \hat{v}) = \sum_T - \int_T (\nabla v u) u + \int_{\partial T} (un) u^{up} v + \int_{\partial T_{out}} (un)(\hat{u} - u)_\tau \hat{v}$$

$$u^{up} = u_n + \begin{cases} \hat{u}_\tau, & u_n < 0, \\ u_\tau, & u_n \geq 0 \end{cases}$$

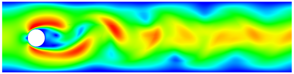
Arbitrary Lagrangian Eulerian (ALE) description

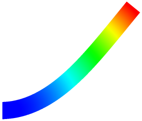
- Fluid problems in Eulerian form
  - Fix mesh, particles move



Arbitrary Lagrangian Eulerian (ALE) description

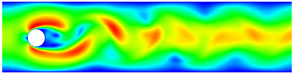
- Fluid problems in Eulerian form
  - Fix mesh, particles move
- Elasticity problems in Lagrangian form
  - Identify mesh nodes with particles, mesh "moves"

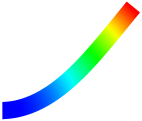


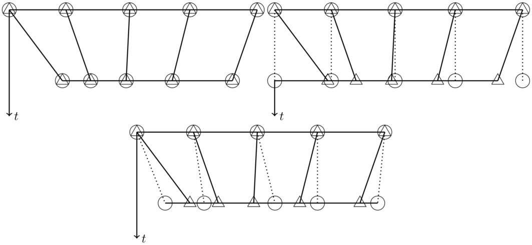


Arbitrary Lagrangian Eulerian (ALE) description

- Fluid problems in Eulerian form
  - Fix mesh, particles move
- Elasticity problems in Lagrangian form
  - Identify mesh nodes with particles, mesh “moves”
- ALE combines both

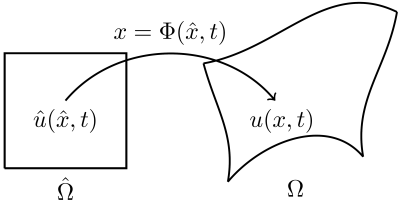






- $\Phi = \text{id} + d$ describes the movement of the domain
- Use chain rule and transformation theorem

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v + (\nabla u u) \cdot v + \nu \nabla u : \nabla v - \operatorname{div}(v) p \, dx = 0$$
$$\int_\Omega \operatorname{div}(u) q \, dx = 0$$

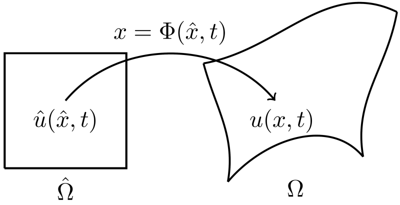


ALE for Taylor-Hood elements

- $\Phi = \mathrm{id} + d$ describes the movement of the domain
- Use chain rule and transformation theorem

$$\int_{\hat\Omega} J\left(\frac{\partial u}{\partial t} \cdot v + (\nabla u u) \cdot v + \nu \nabla u : \nabla v - \mathrm{tr}(\nabla v) p\right) \circ \Phi \, dx = 0$$

$$\int_{\hat\Omega} J(\mathrm{tr}(\nabla u) q) \circ \Phi \, dx = 0$$



ALE for Taylor-Hood elements

- $\Phi = \mathrm{id} + d$ describes the movement of the domain
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$$\int_{\hat{\Omega}} J(\frac{\partial \hat{u}}{\partial t}\hat{v} + \nabla \hat{u} F^{-1}(\hat{u} - \dot{d})\hat{v} + \nu \nabla \hat{u} F^{-1} \nabla \hat{v} F^{-1} - \mathrm{tr}(\nabla \hat{v} F^{-1})\hat{p}) = 0$$

$$\int_{\hat{\Omega}} J \, \mathrm{tr}(\nabla \hat{u} F^{-1})\hat{q} = 0$$

$$u \circ \Phi = \hat{u}, \; v \circ \Phi = \hat{v}, \; \ldots$$

ALE for Taylor-Hood elements

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$$u \circ \Phi = \hat{u},\ v \circ \Phi = \hat{v},\ \ldots$$

$$(\nabla_x u) \circ \Phi = \nabla_{\hat{x}}\hat{u}F^{-1}, \qquad F = I + \nabla d$$

ALE for Taylor-Hood elements

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$$(\nabla_x u) \circ \Phi = \nabla_{\hat{x}}\hat{u}F^{-1}, \qquad F = I + \nabla d$$

$$(\frac{\partial u}{\partial t}) \circ \Phi(\hat{x}, t) = \frac{\partial \hat{u}}{\partial t} - \nabla_{\hat{x}}\hat{u}F^{-1}\dot{d}$$

- Piola-transformation to preserve normal-continuity

$$P[\hat{u}] := \frac{1}{J} F \hat{u}, \quad F = I + \nabla d,\ J = \det(F)$$

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$$P[\hat{u}] := \frac{1}{J} F \hat{u}, \quad F = I + \nabla d, \; J = \det(F)$$

$$\mathrm{div}(\frac{1}{J} F \hat{u}) = \frac{1}{J} \mathrm{div}(\hat{u})$$

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- Second additional term from differentiating $u \circ \Phi = \frac{1}{J} F \hat{u}$

ALE for H(div)-conforming HDG

- Piola-transformation to preserve normal-continuity

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$$\mathrm{div}(\frac{1}{J} F \hat{u}) = \frac{1}{J} \mathrm{div}(\hat{u})$$

- Second additional term from differentiating $u \circ \Phi = \frac{1}{J} F \hat{u}$

$$(\nabla \dot{d} F^{-1} - \mathrm{tr}(\nabla \dot{d} F^{-1})) P[\hat{u}]$$

ALE for H(div)-conforming HDG

$$\int_{\bar{T}} \nabla u : \nabla v - \int_{\partial \bar{T}} \frac{\partial u}{\partial n}(v - \hat{v})_\tau$$

ALE for H(div)-conforming HDG

$$\int_{\bar{T}} \nabla u : \nabla v - \int_{\partial \bar{T}} \nabla u n (v - \hat{v})_\tau$$

ALE for H(div)-conforming HDG

$$\int_{\bar{T}} \nabla u : \nabla v - \int_{\partial \bar{T}} \nabla u n (v - \hat{v})_\tau$$

$$\int_T J \nabla P[u] F^{-1} \nabla P[v] F^{-1} - \int_{\partial T} J_{bnd} \nabla P[u] F^{-T} n (P[v] - F^{-T} \hat{v})_{F\tau}$$

ALE for H(div)-conforming HDG

$$\int_{\bar{T}} \nabla u : \nabla v - \int_{\partial \bar{T}} \nabla u n (v - \hat{v})_\tau$$

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- *SetDeformation*() function in NGS-Py

NGSolve

Time discretization first approach

- Stokes operator D, convection C, mass M

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- Neglect pressure and facet variables

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$$M(d)\frac{\partial u}{\partial t} + D(d)u + C(d, \dot{d}, u) = 0$$

Time discretization first approach

- Stokes operator D, convection C, mass M
- Neglect pressure and facet variables

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- Nonlinear displacement dependency

Time discretization first approach

- Stokes operator D, convection C, mass M
- Neglect pressure and facet variables

$$M(d)\frac{\partial u}{\partial t} + D(d)u + C(d, \dot{d}, u) = 0$$

- Nonlinear displacement dependency
- Use product rule to rewrite

$$\frac{\partial}{\partial t}\left(M(d)u\right) - \frac{\partial}{\partial t}\left(M(d)\right)u + D(d)u + C(d, \dot{d}, u)$$

Time discretization first approach

- Stokes operator D, convection C, mass M
- Neglect pressure and facet variables

$$M(d)\frac{\partial u}{\partial t} + D(d)u + C(d,\dot{d},u) = 0$$

- Nonlinear displacement dependency
- Use product rule to rewrite

$$\frac{\partial}{\partial t}\left(M(d)u\right) - \frac{\partial}{\partial t}\left(M(d)\right)u + D(d)u + C(d,\dot{d},u)$$

- Using RK-Methods (IMEX)

Time discretization (IMEX)

- Stokes operator D implicit, convection C explicit

Time discretization (IMEX)

- Stokes operator D implicit, convection C explicit
- Different schemes depending on choice of integration rule and difference quotient of

$$\int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t} (M(d))\, u \, dt$$

Time discretization (IMEX)

- Stokes operator D implicit, convection C explicit
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$$\int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t}\left(M(d)\right) u \, dt$$

- For second order scheme central difference quotient

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$$\int_{t^n}^{t^{n+1}} \frac{\partial}{\partial t}\left(M(d)\right) u \, dt$$

- For second order scheme central difference quotient

$$\left(M_{n+1} + \tau D_{n+1}\right)\left(u^{n+1} - u^n\right) = -\tau \left(D_{n+1} u^{n+1} + C_n\left(\frac{d^n - d^{n-1}}{\tau}, u^n\right) \right)$$

Time discretization (implementation)

- *SetDeformation*() only for known deformation

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- For second order schemes quadratic extrapolation needed

Numerical experiment

- Given displacement

$$d_2(x, y, t) = t \sin(\pi t) x(2 - x) y(1 - y) \sin(\frac{5\pi x}{2})$$

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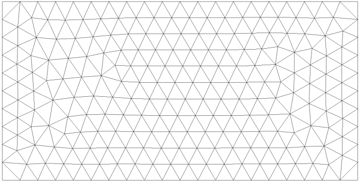

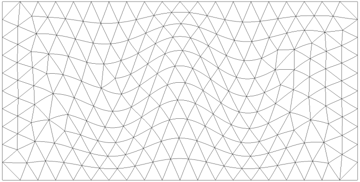

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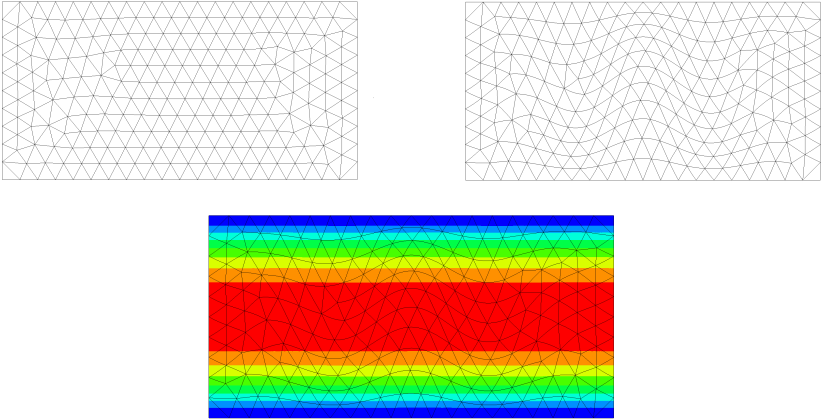

Numerical experiment

$$\text{err} = \| u_h(T_{end}) - u_{exact} \|_{L^2(\Omega)}$$

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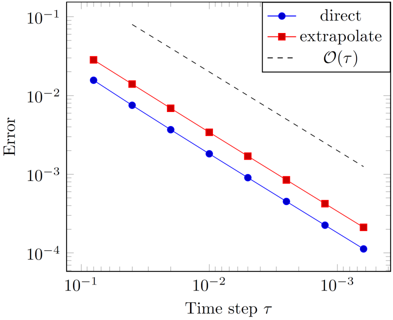

Numerical experiment

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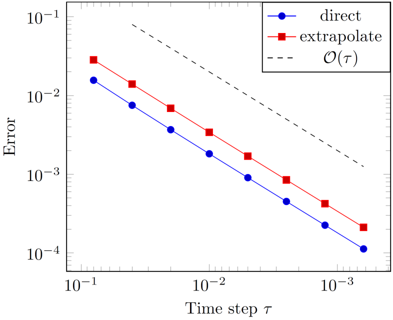

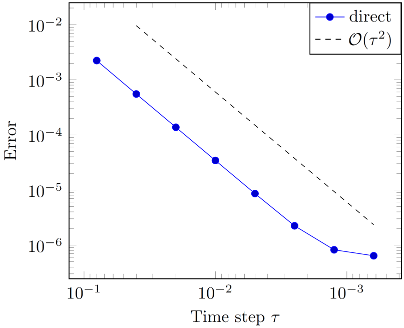

Time discretization part 2

- Crank-Nicolson and Newton

$$M(u^{n+1} - u^n) + \frac{\tau}{2} A(u^{n+1} + u^n) + \tau B p^{n+1} + \frac{\tau}{2}(C(u^{n+1}) + C(u^n)) = 0$$

$$\tau B^T u^{n+1} = 0$$

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- shifted Crank-Nicolson ($\theta := \frac{1}{2} + \varepsilon$)

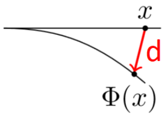
$$A(\theta u^{n+1} + (1 - \theta)u^n)$$

Elastic wave equation

Deformation $\Phi : \Omega \to \mathbb{R}^3$

Elasticity

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$d := \Phi - id$



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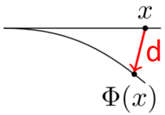

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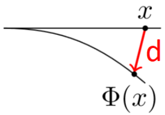
Cauchy-Green strain tensor	$C := F^T F$

$$\frac{||\Phi(x + \Delta x) - \Phi(x)||^2}{||\Delta x||^2} = \frac{\Delta x^T F^T F \Delta x}{||\Delta x||^2} + \mathcal{O}(||\Delta x||)$$



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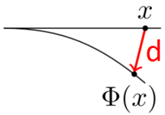

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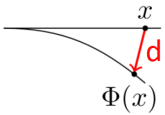

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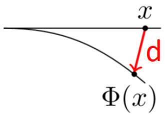
Elasticity

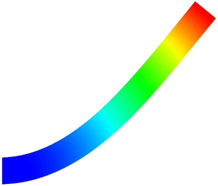
$$-\mathrm{div}(F\Sigma) = g$$



$$F = I + \nabla d \qquad E = \tfrac{1}{2}(C - I)$$
$$C = F^T F \qquad \Sigma = 2\mu E + \lambda \mathrm{tr}(E) I$$

Elastic wave

$$\rho \frac{\partial^2 d}{\partial t^2} - \mathrm{div}(F\Sigma) = g$$



$$F = I + \nabla d \qquad E = \tfrac{1}{2}(C - I)$$
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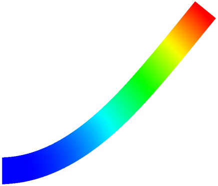
Elastic wave

$$\begin{aligned} \dot{d} &= u \\ \rho \dot{u} - \mathrm{div}(F\Sigma) &= g \end{aligned}$$



First spatial discretization

- H^1 elements for displacement and velocity
- Same polynomial order, $V := [\Pi^k(\mathcal{T}_h)]^n \cap [C^0(\Omega)]^n$

Find $(d, u) \in V \times V$ such that

$$\begin{aligned} \int_\Omega \frac{\partial d}{\partial t} \cdot w \, dx &= \int_\Omega u \cdot w \, dx && \forall w \in V \\ \int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= -\int_\Omega (F\Sigma) : \nabla v \, dx && \forall v \in V \end{aligned}$$

- Elasticity operator K and mass M

$$
\begin{aligned}
\frac{d^{n+1} - d^n}{\tau} &= \frac{1}{2}\left(u^n + u^{n+1}\right) \\
M \frac{u^{n+1} - u^n}{\tau} &= -K\left(\frac{d^{n+1} + d^n}{2}\right)
\end{aligned}
$$

Newmark time discretization

- Elasticity operator K and mass M

$$
\begin{aligned}
d^{n+1} &= d^n + \frac{\tau}{2}\left(u^n + u^{n+1}\right) \\
M\frac{u^{n+1} - u^n}{\tau} &= -K\left(\frac{d^{n+1} + d^n}{2}\right)
\end{aligned}
$$

- Eliminate d^{n+1} with first equation

$$
u^{n+1} = u^n - \tau M^{-1} K\left(d^n + \frac{\tau}{4}\left(u^n + u^{n+1}\right)\right)
$$

Newmark time discretization

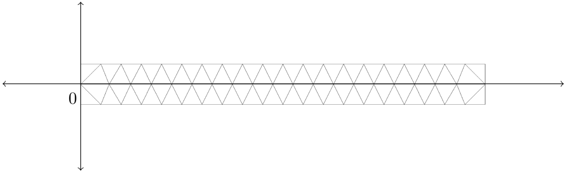
- Elasticity operator K and mass M

$$
\begin{aligned}
d^{n+1} &= d^n + \frac{\tau}{2}\left(u^n + u^{n+1}\right) \\
M\frac{u^{n+1} - u^n}{\tau} &= -\frac{K(d^{n+1}) + K(d^n)}{2}
\end{aligned}
$$

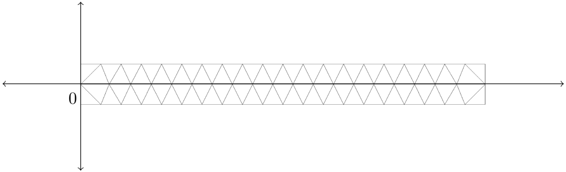
- Eliminate d^{n+1} with first equation

$$
u^{n+1} = u^n - \tau M^{-1}\frac{K(d^n + \frac{\tau}{2}\left(u^n + u^{n+1}\right)) + K(d^n)}{2}
$$

Numerical example

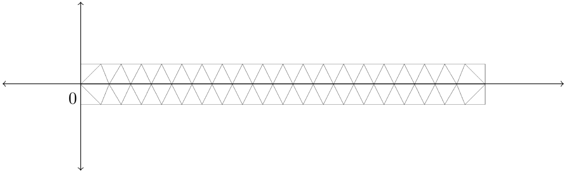


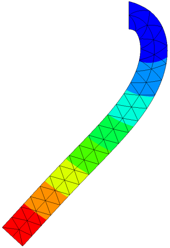
Numerical example



- Energy conservation important

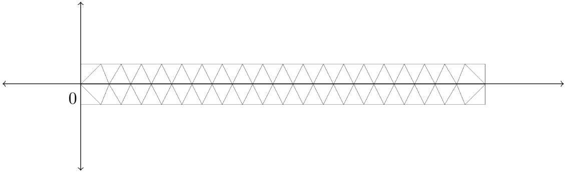
Numerical example

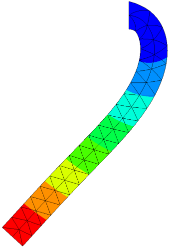




- Energy conservation important
- Neo-Hook's law
  $\Sigma := \frac{\mu}{2}(I - \det(C)^{-\frac{\lambda}{2\mu}})C^{-1}$

Numerical example





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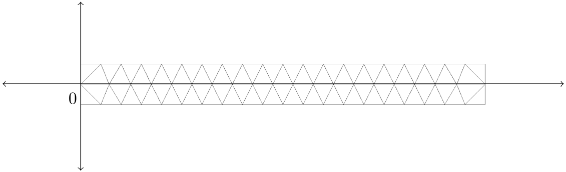
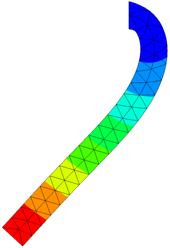
$E_{tot} = E_{kin} + E_{pot}$

Numerical example





- Energy conservation important
- Neo-Hook's law
  $\Sigma := \frac{\mu}{2}(I - \det(C)^{-\frac{\lambda}{2\mu}})C^{-1}$

$$E_{tot} = E_{kin} + E_{pot} = \int_\Omega \frac{|u|^2}{2} + f \cdot d + \Sigma(d)\,dx$$

Numerical example

$$\mathrm{err} := \frac{1}{N}\sum_{i=1}^{N} |E_h(t_i)|$$

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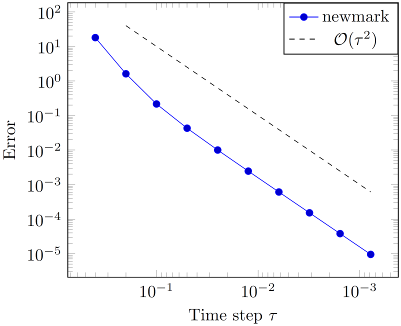

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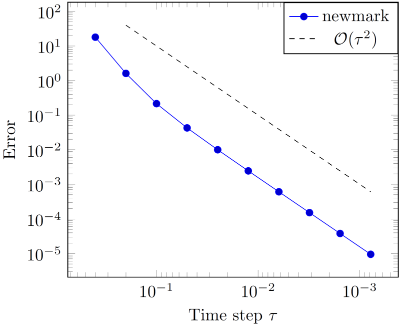

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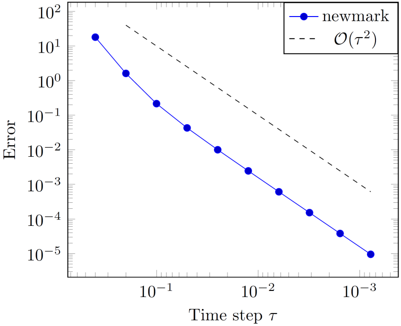

- Quadratic convergence rate

- Displacement again in H^1, $H := [\Pi^k(\Omega)]^n \cap [C^0(\Omega)]^n$

H(curl)-conforming discretization for elastic waves

- Displacement again in H^1, $H := [\Pi^k(\Omega)]^n \cap [C^0(\Omega)]^n$
- Velocity in H(curl)-conforming space V

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Find $(d, u) \in H \times V$ such that

$$\begin{aligned} \int_\Omega \frac{\partial d}{\partial t} \cdot v \, dx &= \int_\Omega u \cdot v \, dx && \forall v \in V \\ \int_\Omega \rho \frac{\partial u}{\partial t} \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in H \end{aligned}$$

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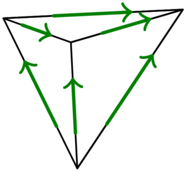
\end{aligned}$$

Motivation for H(curl)

- Velocity is an one-form (Whitney forms)
- H(curl) natural space for one-forms



The space H(curl)

- $H(\mathsf{curl}) := \{u \in [L^2(\Omega)]^n : \mathrm{curl}(u) \in [L^2(\Omega)]^n\}$

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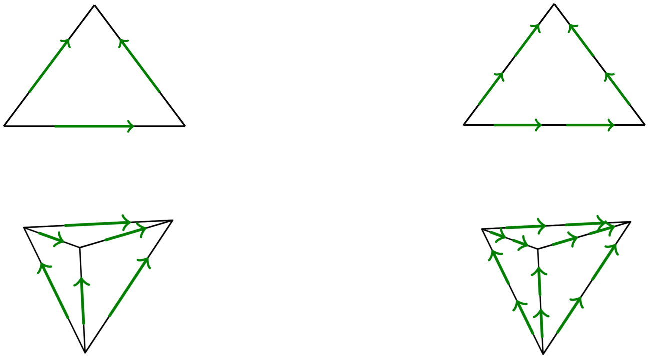
- Nédélec elements 1st and 2nd kind



De Rham complex part 2

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\
 & & & & \bigcup & & \bigcup \\
 & & & & W_h & \xrightarrow{\mathrm{div}_h} & S_h
\end{array}
$$

- (Exact) sequence in continuous setting
- Simply connected domains
- Mimic this sequence in discrete setting

De Rham complex part 2

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\bigcup & & \bigcup & & \bigcup & & \bigcup \\
Q_h & \xrightarrow{\nabla_h} & V_h & \xrightarrow{\mathrm{curl}_h} & W_h & \xrightarrow{\mathrm{div}_h} & S_h
\end{array}$$

- (Exact) sequence in continuous setting
- Simply connected domains
- Mimic this sequence in discrete setting

Time discretization

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$$\int_{\mathcal{T}_h} \frac{1}{\tau}(d^{n+1} - d^n) \cdot v \, dx = \int_{\mathcal{T}_h} \frac{1}{2}(u^{n+1} + u^n) \cdot v \, dx \qquad \forall v \in V$$

$$\int_{\mathcal{T}_h} \frac{1}{\tau}(u^{n+1} - u^n) \cdot w \, dx = -\int_{\mathcal{T}_h} (F^{n+\frac{1}{2}} \Sigma(C^{n+\frac{1}{2}})) : \nabla w \, dx \qquad \forall w \in H$$

Time discretization

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Time discretization

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$$C^{n+\frac{1}{2}} := \frac{1}{2}(C(d^{n+1}) + C(d^n)) = \frac{1}{2}(F^T(d^{n+1})F(d^{n+1}) + F^T(d^n)F(d^n))$$

Time discretization

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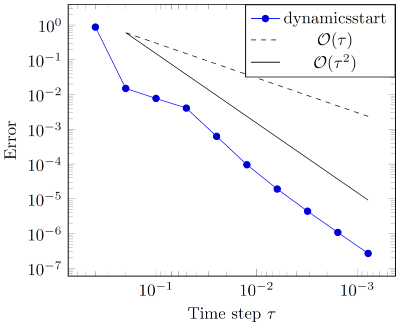
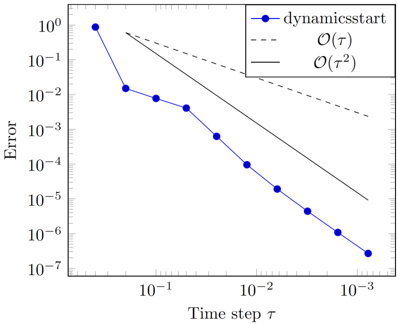
- Mix of midpoint and Crank-Nicolson

Numerical experiment

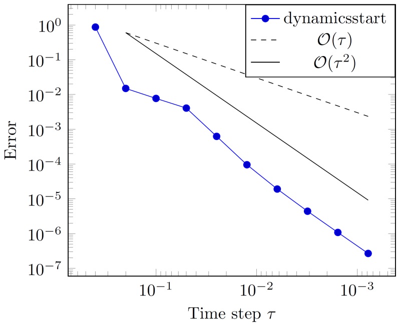


Numerical experiment



- Optimal convergence rate

Numerical experiment



- Optimal convergence rate
- Better than Newmark

Coupling problem

$$
\begin{aligned}
\int_{\Gamma_I} \frac{\partial d}{\partial t} \cdot v \, ds &= \int_{\Gamma_I} u \cdot v \, ds \quad \forall v \in V \\
- \int_{\Gamma_I} \frac{\partial u}{\partial n}(v - \hat{v})_\tau + \alpha (v - \hat{v})_\tau (u - \hat{u})_\tau \, ds
\end{aligned}
$$

Coupling problem

$$
\begin{aligned}
\int_{\Gamma_I} \frac{\partial d}{\partial t} \cdot v \, ds &= \int_{\Gamma_I} u \cdot v \, ds && \forall v \in V \\
\int_{\Gamma_I} \frac{\partial u}{\partial n} \hat{v}_\tau - \alpha \hat{v}_\tau (u - \hat{u})_\tau \, ds &= 0 && \forall \hat{v} \in V
\end{aligned}
$$

Coupling problem

$$\int_{\Gamma_I} \frac{\partial d}{\partial t} \cdot v \, ds = \int_{\Gamma_I} u \cdot v \, ds \quad \forall v \in V$$

$$\int_{\Gamma_I} \frac{\partial u}{\partial n} \hat{v}_\tau - \alpha \hat{v}_\tau (u - \hat{u})_\tau \, ds = 0 \quad \forall \hat{v} \in V$$

- H(curl) testfunctions on solid and fluid interact!
- Wrong behavior on the interface, equations are not fulfilled

Find $(d, u) \in H \times V$ such that

$$\int_\Omega \frac{\partial d}{\partial t} \cdot v \, dx \quad = \int_\Omega u \cdot v \, dx \qquad\qquad \forall v \in V$$

$$\int_\Omega \rho \frac{\partial u}{\partial t} \cdot w \, dx = - \int_\Omega (F\Sigma) : \nabla w \, dx \quad \forall w \in H$$

H(curl)-conforming discretization for elastic waves

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Find $(d, u, p) \in H \times V \times P$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega \frac{\partial p}{\partial t} \cdot v \, dx && \forall v \in V \\
\int_\Omega \frac{\partial p}{\partial t} \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in H
\end{aligned}
$$

H(curl)-conforming discretization for elastic waves

TECHNISCHE UNIVERSITÄT WIEN

Find $(d, u, p) \in H \times V \times P$ such that

$$\begin{aligned}
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\end{aligned}$$

H(curl)-conforming discretization for elastic waves

Find $(d, u, p) \in H \times V \times P$ such that

$$
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\int_\Omega p \cdot w \, dx &= - \int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in H
\end{aligned}
$$

$$P = ?$$

Find $(d, u, p) \in H \times V \times P$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega p \cdot v \, dx && \forall v \in V \\
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\end{aligned}$$

$$P = H(\mathrm{curl}, \Omega)^*$$

H(curl)-conforming discretization for elastic waves

Find $(d, u, p) \in H \times V \times P$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
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\int_\Omega p \cdot w \, dx &= - \int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in H
\end{aligned}
$$

$$P = H(\mathrm{curl}, \Omega)^*$$

$p = \mathrm{H(curl)\text{-}TrialFunction()}$

$p = p.Operator(''dual'')$

NGSolve

The dual space $H(\mathrm{curl})^*$

- Facet and inner components

The dual space H(curl)*

- Facet and inner components
- Facet ($\mathcal{N}^{II}$):

$$v \mapsto \int_E v_\tau \cdot q_k \, ds \qquad \{q_k\} \dots \text{ basis of } \Pi^k(E)$$

$$SymbolicBFI(\, p \cdot v,\ element\_boundary = True\,)$$

The dual space H(curl)*

- Facet and inner components
- Facet ($\mathcal{N}^{II}$):

$$v \mapsto \int_E v_\tau \cdot q_k \, ds \qquad \{q_k\} \ldots \text{ basis of } \Pi^k(E)$$

- Inner ($\mathcal{N}^{II}$):

$$v \mapsto \int_T v \cdot q_k \, dx \qquad \{q_k\} \ldots \text{ basis of } RT_{k-2}(T)$$

$$SymbolicBFI(\, p \cdot v,\ element\_boundary = True\,)$$
$$SymbolicBFI(\, p \cdot v,\ element\_boundary = False\,)$$

Coupling part 2

$$
\begin{aligned}
\int_{\Gamma_I} \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_{\Gamma_I} p \cdot v \, dx \quad && \forall v \in V \\
\int_{\Gamma_I} \frac{\partial u}{\partial n} \hat{v}_\tau - \alpha \hat{v}_\tau (u - \hat{u})_\tau \, ds = 0 & && \forall \hat{v} \in V
\end{aligned}
$$

Coupling part 2

$$\int_{\Gamma_I} \rho \frac{\partial u}{\partial t} \cdot v \, dx = \int_{\Gamma_I} p \cdot v \, dx \quad \forall v \in V$$

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- Forces are interchanged over the interface
- Correct behavior

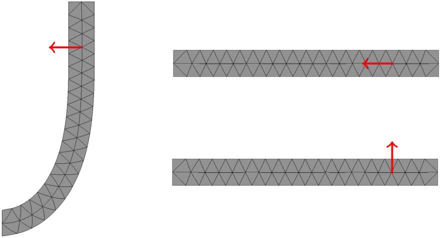
Coupling part 2

$$\int_{\Gamma_I} \rho \frac{\partial u}{\partial t} \cdot v \, dx = \int_{\Gamma_I} p \cdot v \, dx \quad \forall v \in V$$

$$\int_{\Gamma_I} \frac{\partial u}{\partial n} F^{-T} \hat{v}_\tau - \alpha F^{-T} \hat{v}_\tau (u - \hat{u})_\tau \, ds = 0 \quad \forall \hat{v} \in V$$

- Forces are interchanged over the interface
- Correct behavior
- Velocity in fluid in material coordinates

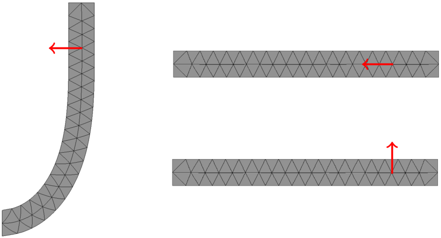
Transformation to material coordinates



Transformation to material coordinates

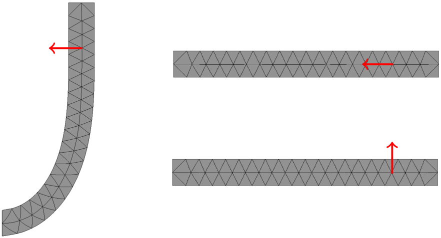
- Covariant transformation from global to material velocity

$$u = F^{-T}\,\hat{u}$$



- Covariant transformation from global to material velocity

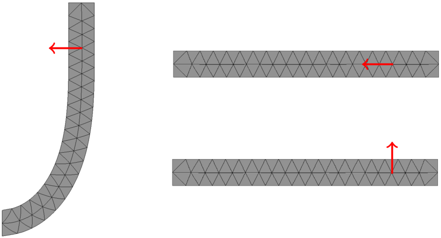
$$u = F^{-T}\hat{u}, \qquad \mathrm{curl}(F^{-T}\hat{u}) = J^{-1}F\,\mathrm{curl}(\hat{u})$$



Transformation to material coordinates

- Covariant transformation from global to material velocity

$$u = F^{-T} \hat{u}, \qquad \mathrm{curl}(F^{-T} \hat{u}) = P[\mathrm{curl}(\hat{u})]$$



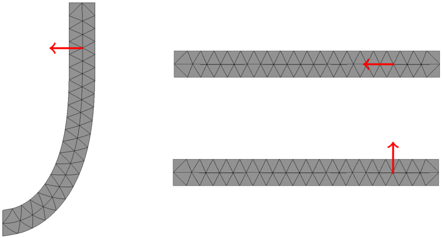
Transformation to material coordinates

- Covariant transformation from global to material velocity

$$u = F^{-T}\hat{u}, \qquad \mathrm{curl}(F^{-T}\hat{u}) = P[\mathrm{curl}(\hat{u})]$$

- Dual transformation for p

$$p = F\hat{p}$$



H(curl)-conforming discretization for elastic waves

Find $(d, \hat{u}, \hat{p}) \in H \times V \times V^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx & \forall q \in V^* \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T}\hat{u}) \cdot (F^{-T}v)\, dx &= \int_\Omega \hat{p} \cdot v\, dx & \forall v \in V \\
\int_\Omega (F\hat{p}) \cdot w\, dx &= -\int_\Omega (F\Sigma) : \nabla w\, dx & \forall w \in H
\end{aligned}
$$

Find $(d, \hat{u}, \hat{p}) \in H \times V^{dc} \times V^{*,dc}$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx && \forall q \in V^{*,dc} \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T}\hat{u}) \cdot (F^{-T} v)\, dx &= \int_\Omega \hat{p} \cdot v\, dx && \forall v \in V^{dc} \\
\int_\Omega (F\hat{p}) \cdot w\, dx &= -\int_\Omega (F\Sigma) : \nabla w\, dx && \forall w \in H
\end{aligned}
$$

- Static condensation for discontinuous $\hat{u}$ and $\hat{p}$
- Further discretisation in 2d and 3d

H(curl)-conforming discretization 3d

Find $(d, \hat{u}, \hat{p}) \in H \times V \times V^*$ such that

$$\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx = \int_\Omega \hat{u} \cdot q\, dx \quad \forall q \in V^*$$

$$\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T}v + \dot{F}^{-T}\hat{u} \cdot F^{-T}v)\, dx = \int_\Omega \hat{p} \cdot v\, dx \quad \forall v \in V$$

$$\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx = 0 \qquad \forall w \in H$$

H(curl)-conforming discretization 3d

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Find $(d, \hat{u}, \hat{p}) \in H \times V \times V^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx && \forall q \in V^* \\
\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T} v + \mathrm{sym}(\dot{F}^{-T}\hat{u} \cdot F^{-T} v) & \\
+ \mathrm{skew}(\dot{F}^{-T}\hat{u} \cdot F^{-T} v))\, dx &= \int_\Omega \hat{p} \cdot v\, dx && \forall v \in V \\
\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx &= 0 && \forall w \in H
\end{aligned}
$$

Find $(d, \hat{u}, \hat{p}) \in H \times V \times V^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx \quad && \forall q \in V^* \\
\int_\Omega \rho\Big(F^{-T}\dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1} \dot{C} C^{-1} \hat{u} \cdot v & \\
+ \frac{1}{2J} \mathrm{curl}(\hat{u}) \times (F^{-T}\hat{u}) \cdot v\Big)\, dx &= \int_\Omega \hat{p} \cdot v\, dx \quad && \forall v \in V \\
\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx &= 0 && \forall w \in H
\end{aligned}
$$

H(curl)-conforming discretization 2d

Find $(d, \hat{u}, \hat{p}) \in H \times V \times V^*$ such that

$$
\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx \quad \forall q \in V^* \\
\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1}\dot{C} C^{-1} \hat{u} \cdot v \\
- \frac{1}{2J^2} \mathrm{curl}(\hat{u}) \mathrm{rot}(\hat{u}) \cdot v)\, dx &= \int_\Omega \hat{p} \cdot v\, dx \quad \forall v \in V \\
\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx &= 0 \qquad \forall w \in H
\end{aligned}
$$

Time discretization

$$
\begin{aligned}
\int_{\mathcal{T}_h} \frac{1}{\tau}(d^{n+1} - d^n) \cdot F_m q &= \int_{\mathcal{T}_h} u_m \cdot q \quad && \forall q \\
\int_{\mathcal{T}_h} C_m^{-1} \frac{u^{n+1} - u^n}{\tau} \cdot v - C_m^{-1} \frac{C^{n+1} - C^n}{2\tau} C_m^{-1} u_m \cdot v & && \\
- \frac{1}{2\det(C_m)} \mathrm{curl}(u_m) \mathrm{rot}(u_m) \cdot v &= \int_{\mathcal{T}_h} p_m \cdot v \quad && \forall v \\
\int_{\mathcal{T}_h} F_m p_m \cdot w + (F_m \Sigma_m) : \nabla w &= 0 && \forall w
\end{aligned}
$$

- $u_m := u^{n+\frac{1}{2}}$

Time discretization

$$
\begin{aligned}
\int_{\mathcal{T}_h} \frac{1}{\tau}(d^{n+1} - d^n) \cdot F_m q &= \int_{\mathcal{T}_h} u_m \cdot q \quad && \forall q \\
\int_{\mathcal{T}_h} C_m^{-1} \frac{u^{n+1} - u^n}{\tau} \cdot v - C_m^{-1} \frac{C^{n+1} - C^n}{2\tau} C_m^{-1} u_m \cdot v & && \\
- \frac{1}{2\det(C_m)} \operatorname{curl}(u_m) \operatorname{rot}(u_m) \cdot v &= \int_{\mathcal{T}_h} p_m \cdot v \quad && \forall v \\
\int_{\mathcal{T}_h} F_m p_m \cdot w + (F_m \Sigma_m) : \nabla w &= 0 && \forall w
\end{aligned}
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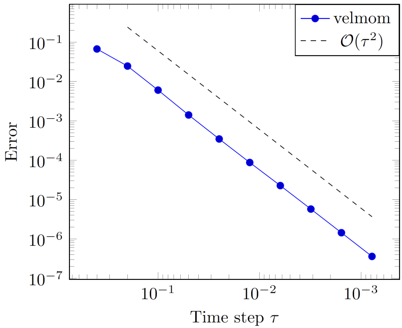
Time discretization

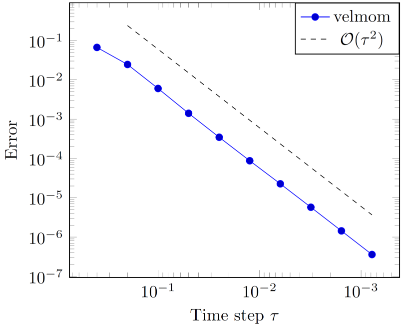
$$
\begin{aligned}
\int_{\mathcal{T}_h} \frac{1}{\tau}(d^{n+1} - d^n) \cdot F_m q &= \int_{\mathcal{T}_h} u_m \cdot q && \forall q \\
\int_{\mathcal{T}_h} C_m^{-1} \frac{u^{n+1} - u^n}{\tau} \cdot v - C_m^{-1} \frac{C^{n+1} - C^n}{2\tau} C_m^{-1} u_m \cdot v & && \\
- \frac{1}{2\det(C_m)} \mathrm{curl}(u_m) \mathrm{rot}(u_m) \cdot v &= \int_{\mathcal{T}_h} p^{n+1} \cdot v && \forall v \\
\int_{\mathcal{T}_h} F_m p^{n+1} \cdot w + (F_m \Sigma_m) : \nabla w &= 0 && \forall w
\end{aligned}
$$

- $u_m := u^{n+\frac{1}{2}}$

Numerical experiment

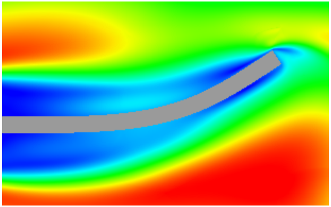


Numerical experiment



- Again quadratic convergence rate in time

- Deformations are not expected to be enormous
- Lagrangian form is satisfying
- No further transformations needed



Coupling

Interface conditions

- Velocity and displacement continuous over Γ_I
- Forces are in equilibrium

Interface conditions

- Velocity and displacement continuous over Γ_I
- Forces are in equilibrium

$$M\frac{\partial u}{\partial t} + Du + C(u) = \int_{\Gamma_I} \sigma_n^f \, ds$$

$$M\frac{\partial^2 d}{\partial t^2} + K(d) = \int_{\Gamma_I} \sigma_n^s \, ds$$

Interface conditions

- Velocity and displacement continuous over Γ_I
- Forces are in equilibrium

$$M\frac{\partial u}{\partial t} + Du + C(u) = \int_{\Gamma_I} \sigma_n^f \, ds$$

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$$\ldots = \int_{\Gamma_I} \sigma_n^f \, ds + \int_{\Gamma_I} \sigma_n^s \, ds$$

Interface conditions

- Velocity and displacement continuous over Γ_I
- Forces are in equilibrium

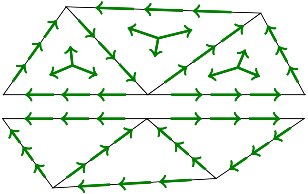
$$M\frac{\partial u}{\partial t} + Du + C(u) = \int_{\Gamma_I} \sigma_n^f \, ds$$

$$M\frac{\partial^2 d}{\partial t^2} + K(d) = \int_{\Gamma_I} \sigma_n^s \, ds$$

$$\ldots = \int_{\Gamma_I} \sigma_n^f \, ds + \int_{\Gamma_I} \sigma_n^s \, ds \overset{!}{=} 0$$

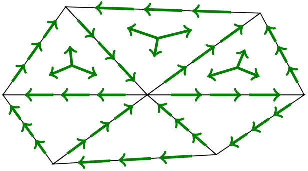
- Monolithic approach

Interface conditions



- Continuity of displacement and tangential continuity of velocity fulfilled

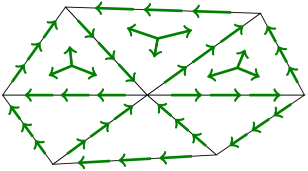
$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

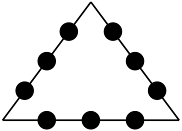


- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u_\tau^s = u_\tau^f \qquad \text{on } \Gamma_I$$

Interface conditions



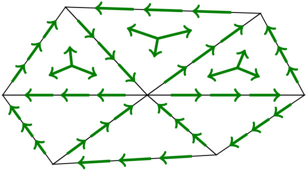


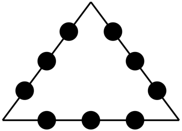
- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

- Normal continuity by Lagrange multiplier

$$\int_{\Gamma_I} (u^f - u^s)_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$





- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

- Normal continuity by Lagrange multiplier

$$\int_{\Gamma_I} (u^f - \frac{\partial d^s}{\partial t})_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$

Deformation extension

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$$\int_{\hat{\Omega}} J(\frac{\partial \hat{u}}{\partial t}\hat{v} + \nabla\hat{u}F^{-1}(\hat{u} - \dot{d})\hat{v} + \nu\nabla\hat{u}F^{-1}\nabla\hat{v}F^{-1} - \mathrm{tr}(\nabla\hat{v}F^{-1})\hat{p}) = 0$$

$$\int_{\hat{\Omega}} J\,\mathrm{tr}(\nabla\hat{u}F^{-1})\hat{q} = 0$$

Deformation extension

$$\int_{\hat{\Omega}} J(\frac{\partial \hat{u}}{\partial t}\hat{v} + \nabla \hat{u} F^{-1}(\hat{u} - \dot{d})\hat{v} + \nu \nabla \hat{u} F^{-1} \nabla \hat{v} F^{-1} - \mathrm{tr}(\nabla \hat{v} F^{-1})\hat{p}) = 0$$

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- Extend the information d^s from the interface to the fluid domain

Deformation extension

$$\int_{\hat{\Omega}} J(\frac{\partial \hat{u}}{\partial t}\hat{v} + \nabla \hat{u} F^{-1}(\hat{u} - \dot{d})\hat{v} + \nu \nabla \hat{u} F^{-1} \nabla \hat{v} F^{-1} - \mathrm{tr}(\nabla \hat{v} F^{-1})\hat{p}) = 0$$

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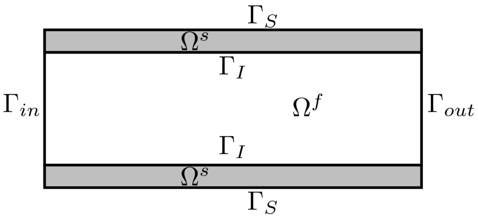


Deformation extension, first approach

- Solve a simple Poisson problem for a given d^s

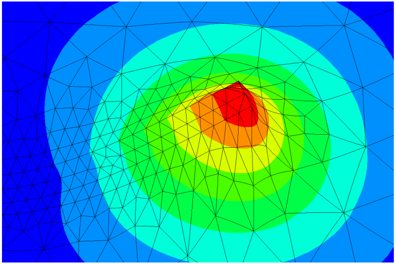
$$\begin{aligned} -\Delta d^f &= 0, && \text{in } \Omega^f \\ d^f &= 0, && \text{on } \partial\Omega \\ d^f &= d^s, && \text{on } \Gamma_I \end{aligned}$$

$$\begin{aligned} -\mathrm{div}(\varepsilon(d^f)) &= 0, && \text{in } \Omega^f \\ d^f &= 0, && \text{on } \partial\Omega \\ d^f &= d^s, && \text{on } \Gamma_I \end{aligned}$$

Deformation extension, first approach

- Solve a simple Poisson problem for a given d^s

$$
\begin{aligned}
-\Delta d^f &= 0, \quad \text{in } \Omega^f \\
d^f &= 0, \quad \text{on } \partial\Omega \\
d^f &= d^s, \quad \text{on } \Gamma_I
\end{aligned}
$$



$$
\begin{aligned}
-\mathrm{div}(\varepsilon(d^f)) &= 0, \quad \text{in } \Omega^f \\
d^f &= 0, \quad \text{on } \partial\Omega \\
d^f &= d^s, \quad \text{on } \Gamma_I
\end{aligned}
$$

- Elements get pressed through
- Equations not “stiff” enough

Deformation extension, first approach

- Solve a simple Poisson problem for a given d^s

$$
\begin{aligned}
-\Delta d^f &= 0, \quad && \text{in } \Omega^f \\
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-\operatorname{div}(\varepsilon(d^f)) &= 0, \quad && \text{in } \Omega^f \\
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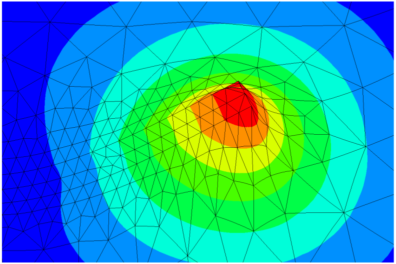
$$



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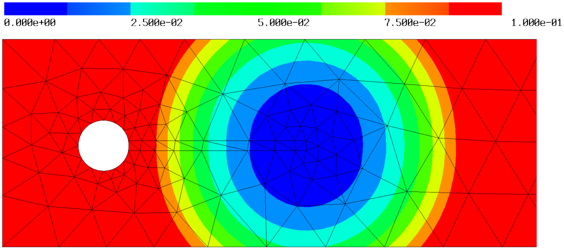
Deformation extension, second approach

- Use space-dependent coefficients

$$h(\vec{x})^{-1} = \sqrt{|\text{dist}(\vec{x})|^2 + \varepsilon}$$



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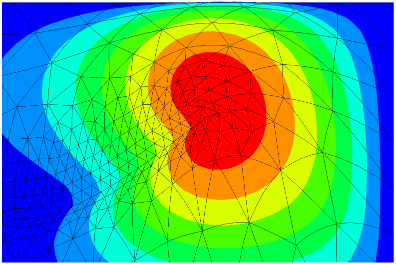

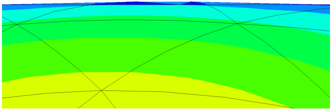

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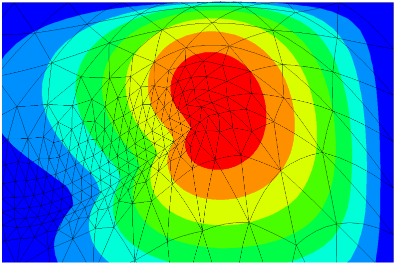

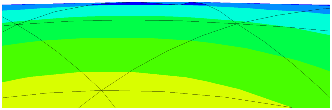

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Deformation extension, third approach

- Penalize the volume, $J := \det(I + \nabla d^f)$

$$\int_{\Omega^f} \frac{1}{J} |\nabla d^f|^2 \, dx \to \min!$$

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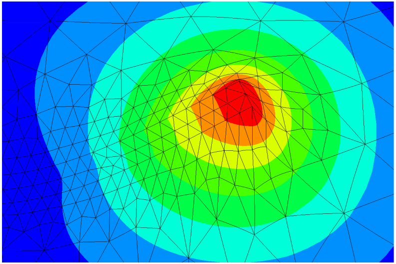

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- Sometimes compressed very strong (integration points)

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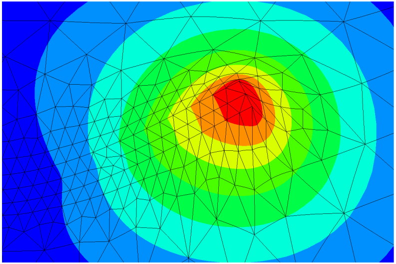

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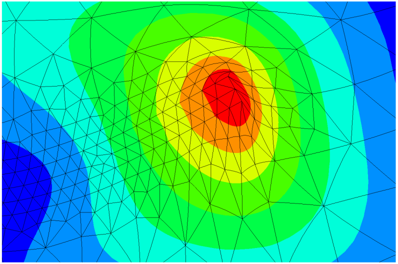
Deformation extension

- Combination of both gives the best results
- Important for stability



$$N(u, \vec{x}) = h(\vec{x})\mu(E + \frac{\mu}{\lambda}\det(C)^{-\frac{\lambda}{2\mu}} - 1)$$

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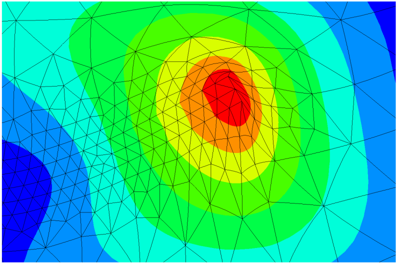

$$N(u, \vec{x}) = h(\vec{x})(E + \det(C)^{-\frac{1}{2}} - 1)$$

Solving the system

- Put everything together in huge “bilinear” form
- Extrapolate deformation for *SetDeformation*()
- Solve with Newton’s method

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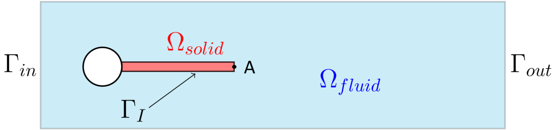
- At the moment direct solver ☹

Numerical example

Benchmark (Turek/Hron)

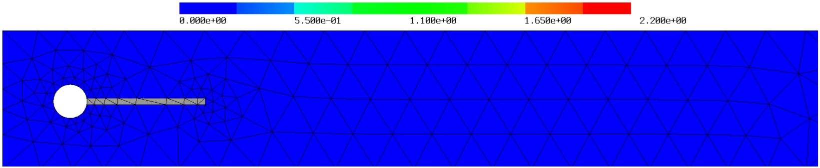


- Parabolic inflow
- Quantity of interest: y-displacement of A

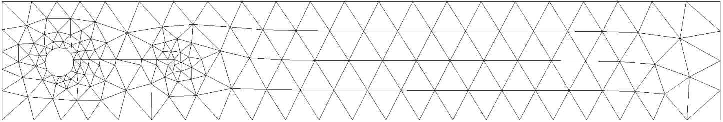
Turek and Hron *Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow*. 2006

Benchmark (Turek/Hron)

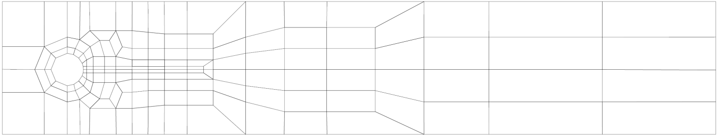
Video



Benchmark (Turek/Hron)

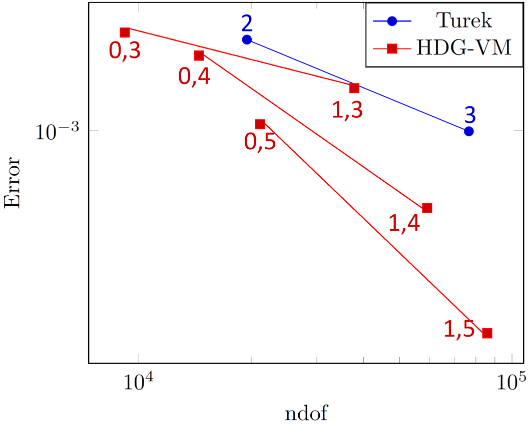


Coarsest mesh level



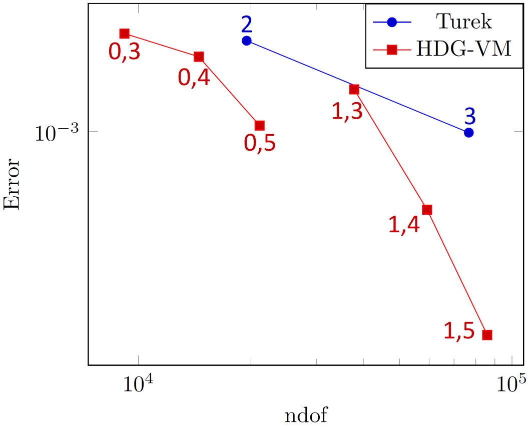
Coarsest mesh level in benchmark

Benchmark (Turek/Hron)



- Uniform h refinement

Benchmark (Turek/Hron)



- Uniform h refinement
- Faster convergence with p refinement

Summary

- ALE for H(div)-conforming HDG Navier-Stokes

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- Appropriate time discretization for elastic wave equation
- Preconditioner, faster solvers
- Improved deformation extension
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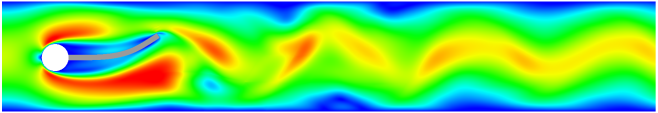
THANK YOU FOR YOUR ATTENTION!



References

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- TUREK AND HRON *Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow*. 2006
- NEUNTEUFEL *Advanced numerical methods for fluid-structure interaction*. 2017