FEM meets Differential Geometry: Curvature approximation with distributional finite elements

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Project J 4824-N

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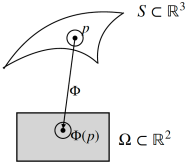
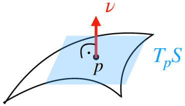
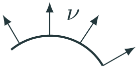
Extrinsic & intrinsic curvature



- Surface $\mathcal{S}$ embedded in $\mathbb{R}^3$
- Normal vector $\nu : \mathcal{S} \to \mathbb{S}^2$
- Shape operator, Weingarten tensor, second fundamental form $\nabla\nu$
- Eigenvalues 0, κ_1, κ_2





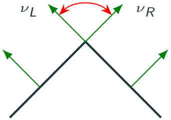
Mean curvature $H = 0.5(\kappa_1 + \kappa_2) = 0.5\,\mathrm{tr}\,(\nabla\nu)$ $\Rightarrow$ extrinsic curvature

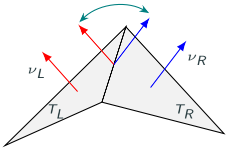
Gauss curvature $K = \kappa_1\kappa_2 = \det(\nabla\nu + \nu \otimes \nu)$ $\Rightarrow$ intrinsic curvature

Intrinsic curvature is independent of the embedding (surrounding space)

Extrinsic curvature

- Weingarten tensor $\nabla \nu$ is classically well-defined for C^2 surfaces (C^1 and pw C^2)
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps





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- How to define and approximate the Weingarten tensor?

Weingarten tensor

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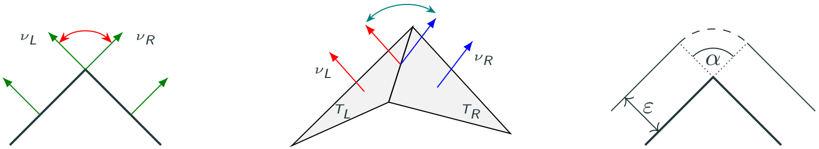

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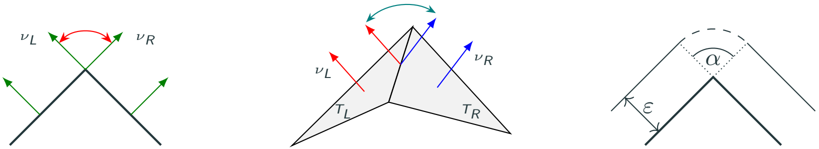
- Consider piecewise affine surface
- Normal vector ν is piecewise constant and jumps
- How to define and approximate the Weingarten tensor? $\Rightarrow$ Discrete differential geometry
- Steiner's offset formula: $\mathrm{Vol}(B_\varepsilon(X)) = \mathrm{Vol}(X) + \varepsilon\,\mathrm{Area}(\partial X) + \frac{\varepsilon^2}{2}\int_{\partial X} H\,dA + \frac{\varepsilon^3}{3}\int_{\partial X} K\,dA$

  Dihedral angle formula: $\sum_{E\in\mathring{\mathscr{E}}} \alpha_E\,|E|$

STEINER: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

Weingarten tensor



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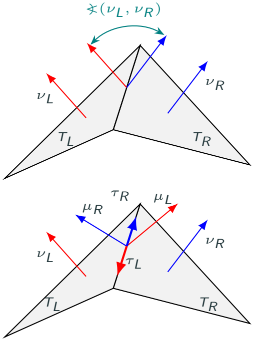
  How to define a generalized Weingarten tensor object? Combine FEM & DDG!

STEINER: Über parallele Flächen, *Preuss. Akad. Wiss.* (1840)

- Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
- $\nabla\nu \notin L^2$, it is a distribution (or measure)
- Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\langle \nabla\nu, \boldsymbol{\Psi}\rangle_{\mathscr{T}} = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, da + \sum_{E\in\mathring{\mathscr{E}}} \int_E \measuredangle(\nu_L,\nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, dl$$



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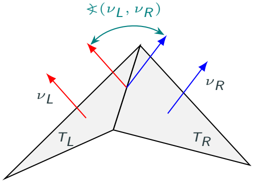
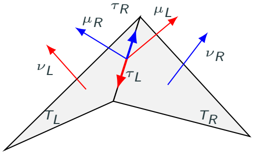


- Test function space

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T}, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \,:\, (\boldsymbol{\sigma}\nu)|_T = 0,\, (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$



- Motivation: TDNNS method: $\nabla H(\mathrm{curl}) \subset H(\mathrm{div}\,\mathrm{div})^*$

  $\Sigma \ldots$ Hellan–Herrmann–Johnson space

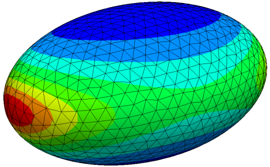
PECHSTEIN, SCHÖBERL: Tangential-Displacement and Normal-Normal-Stress Continuous Mixed Finite Elements for Elasticity, *M3AS* (2011)

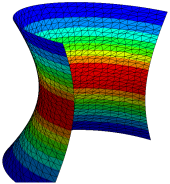
Example curvature approximation

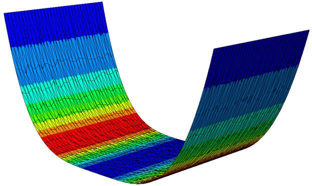
Lifting of distributional Weingarten tensor

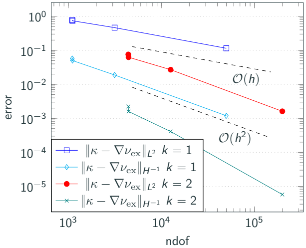
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

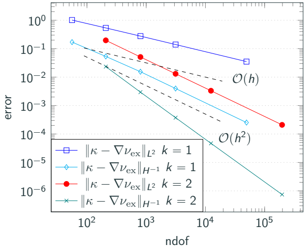
$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, da = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} .$$

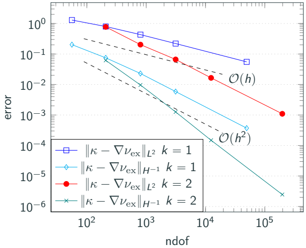












Error analysis (integral representation)

- If $\mathscr{T} \to \mathcal{S}$, does $\boldsymbol{\kappa} \to \nabla \nu$?
- Dihedral angle $\measuredangle(\nu_L, \nu_R)$ is highly nonlinear
- Approach: Parameterize $\Phi(t) = \bar{\Phi} + t(\Phi_h - \bar{\Phi})$ and use integral representation of the error

$$\langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} - \int_{\mathcal{S}} \nabla \nu : \boldsymbol{\sigma} \, da = \int_0^1 \frac{d}{dt} \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} \, dt$$

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- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T}, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \; : \; (\boldsymbol{\sigma}\nu)|_T = 0, \\ (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

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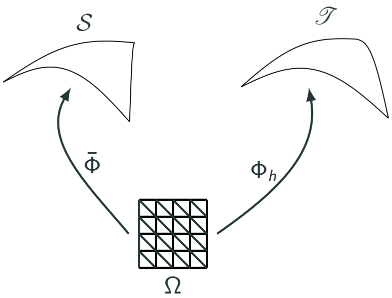


- Problem: Test function $\boldsymbol{\sigma}$ depends on embedding Φ
- Solution: Use fixed reference domain (Uhlenbeck trick)
- Then estimate integrand

$$\Sigma = \{\boldsymbol{\sigma} \in L^2(\mathscr{T}, \mathbb{R}^{3\times 3}_{\text{sym}}) : (\boldsymbol{\sigma}\nu)|_T = 0, \\ (\boldsymbol{\sigma}_{\mu\mu})|_{T_L} = (\boldsymbol{\sigma}_{\mu\mu})|_{T_R}\}$$

Theorem (Gopalakrishnan, N.)

There holds for $\boldsymbol{\sigma} \in \Sigma$ and $X = \dot{\Phi}$

$$\frac{d}{dt}\langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} = a(\Phi; \boldsymbol{\sigma}, X) + b(\Phi; \boldsymbol{\sigma}, X),$$

where with $\mathcal{H}_\nu = \mathrm{hesse}(X)_i \nu_i$

$$a(\Phi; \boldsymbol{\sigma}, X) = \sum_{T \in \mathscr{T}} \int_T -\mathrm{div}(X) \nabla \nu : \boldsymbol{\sigma} - \sum_{E \in \mathscr{E}} \int_E (\nabla X)_{\tau\tau} \sphericalangle(\nu_L, \nu_R) \boldsymbol{\sigma}_{\mu\mu},$$

$$b(\Phi; \boldsymbol{\sigma}, X) = \sum_{T \in \mathscr{T}} \int_T -\mathcal{H}_\nu : \boldsymbol{\sigma} + \sum_{E \in \mathscr{E}} \int_E [\![(\nabla X)_{\nu\mu}]\!]_E \boldsymbol{\sigma}_{\mu\mu}.$$

Bilinear form $b(\Phi; \boldsymbol{\sigma}, X)$ is closely related to the surface Hellan–Herrmann–Johnson method

WALKER: The Kirchhoff plate equation on surfaces: the surface Hellan–Herrmann–Johnson method, *IMA J. Numer. Anal.* (2021)

1. $\langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} - \int_{\mathcal{S}} \nabla \nu : \boldsymbol{\sigma} \, da = \int_0^1 \frac{d}{dt} \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}(\Phi(t))} \, dt$ with $\Phi(t) = \bar{\Phi} + t(\Phi_h - \bar{\Phi})$.
2. $\frac{d}{dt} \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} = a(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t)) + b(\Phi; \boldsymbol{\sigma}, \dot{\Phi}(t))$ sum of the bilinear forms a and b.
3. Estimate $a(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$ and $b(\Phi(t); \boldsymbol{\sigma}, \dot{\Phi}(t))$

Theorem (Gopalakrishnan, N.)

Suppose Φ_h is a collection of embeddings such that $\Phi_h = \mathcal{I}_h^{\mathrm{Lag}^k} \bar{\Phi}$ for $k \geqslant 1$. Let $\boldsymbol{\kappa}_h$ be the lifted Weingarten tensor. Then

$$\|\boldsymbol{\kappa}_h - \nabla \nu\|_{L^2} \leqslant C\, h^k$$

Same convergence rate, but less computation/memory requirements as in

Walker: Approximating the Shape Operator with the Surface Hellan–Herrmann–Johnson Element, *SIAM J. Sci. Comput.* (2024)

Application (Koiter/Kirchhoff–Love shell)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu \circ \phi) - \nabla\hat{\nu}\|^2_{\mathcal{M}}$$

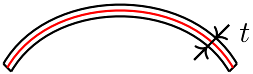
u ... displacement of mid-surface

t ... thickness

$\mathcal{M}$ ... material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu} \otimes \hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top \nabla u + \nabla u^\top \boldsymbol{P} + \boldsymbol{P}\nabla u)$



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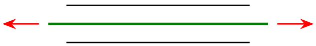

u . . . displacement of mid-surface

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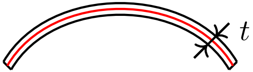

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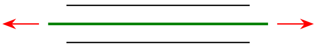

membrane energy

u . . . displacement of mid-surface

t . . . thickness

$\mathcal{M}$ . . . material tensor

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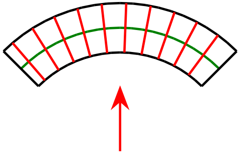
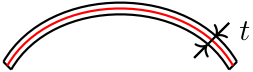


bending energy



- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

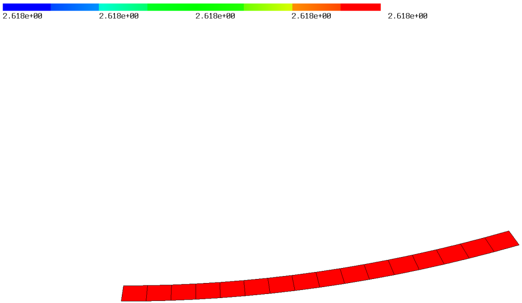
$$\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathcal{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\mathcal{M}} - \langle f, u\rangle$$
$$+ \sum_{T\in\mathscr{T}} \int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathscr{E}} \int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

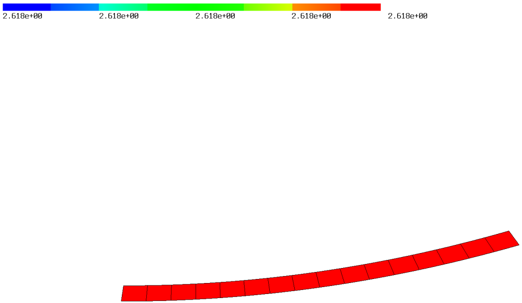
- Lagrange parameter $\boldsymbol{\sigma} \in \Sigma_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}}$ $\rightarrow$ two-field formulation in $(u, \boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

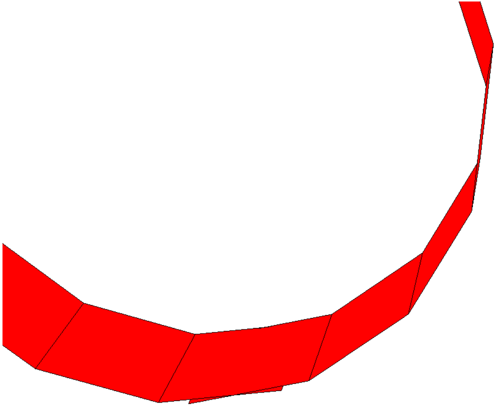
Application (Koiter/Kirchhoff–Love shell)

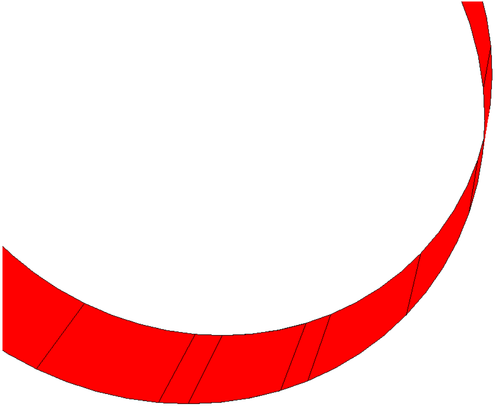
NGSolve





Application (Koiter/Kirchhoff–Love shell)





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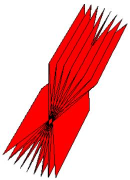
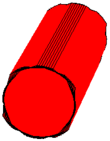
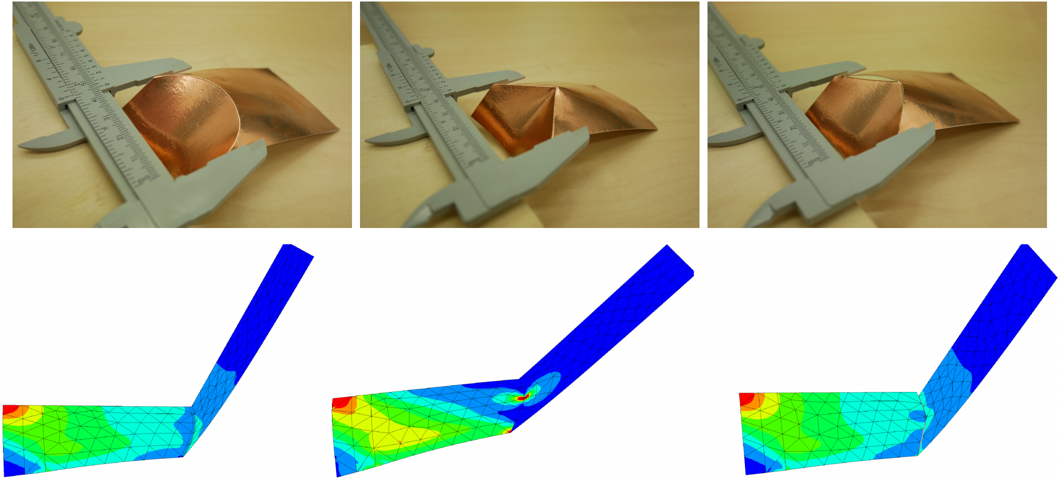


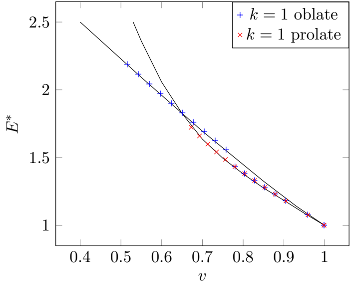


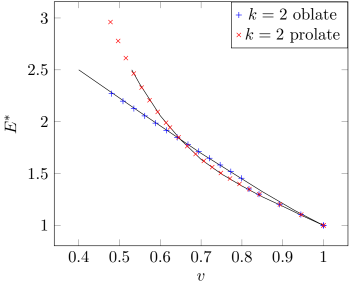
Application (bending-folding model)

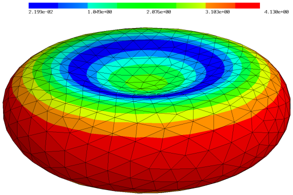


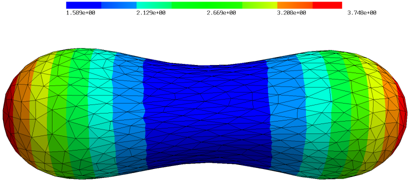
Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending model, *arXiv:2503.17190*

Application (cell membrane)









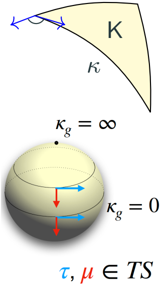
N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *JoCP* (2023)

Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)

Intrinsic curvature

Gauss-Bonnet

$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds + \sum_i \alpha_i = 2\pi\chi(M)$$

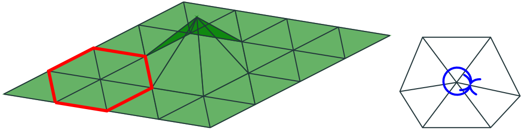


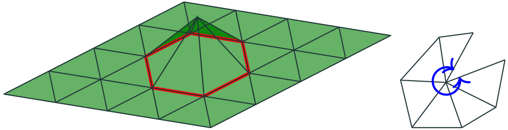
- Gauss curvature K
- Geodesic curvature $\kappa_g = g(\nabla_\tau \tau, \mu)$
- External angle α_i at corner points
- $\chi(M) = \#V - \#E + \#F$ Euler characteristic of M

For affine triangulation of surface: Angle defect

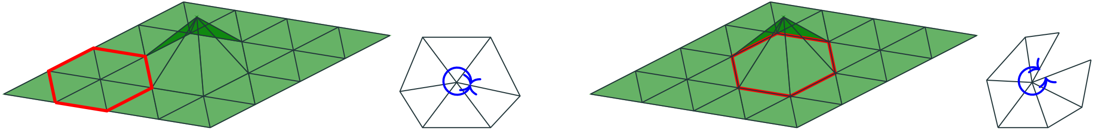
$$\int_{\mathscr{T}} K\,dA = \sum_{V \in \mathscr{T}} \Theta_V, \qquad \Theta_V = 2\pi - \sum_{T \supset V} \sphericalangle_V^T$$

Angle defect





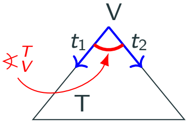
Angle defect



- In 2D, the angle defect Θ_V at vertex V is given by

$$\Theta_V = 2\pi - \sum_{T \supset V} \sphericalangle_V^T,$$

with the interior angle $\sphericalangle_V^T$ is measured with $g|_T$



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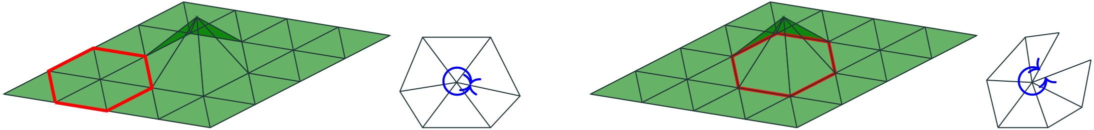
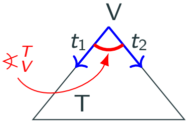


- In 2D, the angle defect Θ_V at vertex V is given by

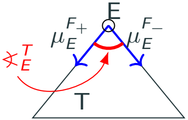
$$\Theta_V = 2\pi - \sum_{T \supset V} \measuredangle_V^T,$$



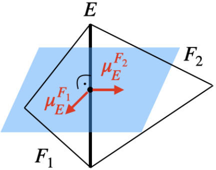
with the interior angle $\measuredangle_V^T$ is measured with $g|_T$

- N dimensions: generalized angle defect at $\mathring{\mathscr{E}} = \{\text{interior subsimplices of codimension } 2\}$

$$\Theta_E = 2\pi - \sum_{T \supset E} \underbrace{\arccos(g(\mu_E^{F_+}, \mu_E^{F_-}))}_{\measuredangle_E^T}$$

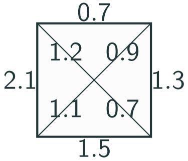


Plane g-perpendicular to E



Regge space and finite elements

- Regge's idea: Approximate metric by assigning squared lengths to edges

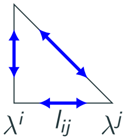


REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), (1961).

Regge space and finite elements

- Regge's idea: Approximate metric by assigning squared lengths to edges
- With barycentric coordinates λ^i

$$g = -\sum_{i \neq j} l_{ij}^2 \nabla\lambda^i \odot \nabla\lambda^j$$



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), (1961).

SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D 12* (1975).

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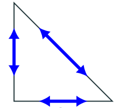

$$g = -\sum_{i \neq j} l_{ij}^2 \nabla\lambda^i \odot \nabla\lambda^j$$

  - g is piecewise constant and tangential-tangential continuous: for all interior facets F the value $g(X, Y)$ coincides from both elements for all tangential $X, Y \in \mathfrak{X}(F)$
  - Regge finite element space

$$\mathcal{R}_h^k = \{g \in L^2(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \;:\; g_{ij} \in \mathcal{P}^k(\mathscr{T}),\, g \text{ is tt-continuous}\}$$

Christiansen: On the linearization of Regge calculus, *Numerische Mathematik, (2011).*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

Regge space and finite elements

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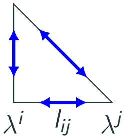

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N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*

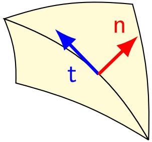
- volume forms $\omega_T = \sqrt{\det g}$

$$(K\omega)_{\mathrm{distr}} = \sum_T K|_T\, \omega_T + \qquad\qquad + \sum_V \Theta_V\, \delta_V$$

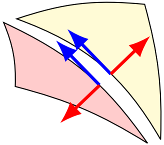
Distributional Gauss curvature

- Compute the geodesic curvature from both elements

$$\kappa_g = g(\nabla_t t, n)$$



- n changes sign
- g smooth $\Rightarrow [\![\kappa_g]\!] = 0$
- g tt-continuous $\Rightarrow$ jump at edge $[\![\kappa_g]\!]\,\delta_E$
- volume forms $\omega_T = \sqrt{\det g}$, $\omega_E = \sqrt{g(t,t)}$



$$(K\omega)_{\mathrm{distr}} = \sum_T K|_T\,\omega_T + \sum_E [\![\kappa_g]\!]\,\omega_E\,\delta_E + \sum_V \Theta_V\,\delta_V$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

Extension to generalized Riemann curvature tensor

$$\mathcal{R}(X,Y,Z,W) = g(\mathcal{R}_{X,Y}Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$$

- Second fundamental form: For hypersurface F with g-normal ν

$$I\!I^{\nu}(X,Y) = -g(\nabla_X \nu, Y), \qquad X,Y \in \mathfrak{X}(F)$$

- Since the metric g and the g-normal ν jumps across interior facets $F \in \mathring{\mathscr{F}}$, the second fundamental form jumps as well
- Facet contribution: Jump of second fundamental form $[\![I\!I]\!]$
- Motivation via Gauss-Bonnet theorem or mollification argument

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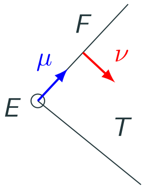
Summary: The generalized Riemann curvature tensor has the following contributions

$$\mathcal{R}(g) = \begin{cases} \mathcal{R}|_T & \text{on each } T \in \mathscr{T} \\ [\![I\!I]\!] & \text{on each } F \in \mathring{\mathscr{F}} \\ \Theta_E & \text{on each } E \in \mathring{\mathscr{E}} \end{cases}$$

Generalized Riemann curvature (Gopalakrishnan, N., Schöberl, Wardetzky)

$$\widetilde{\mathcal{R}\omega}(A) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, A\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![I\!I]\!], A_{.\nu\nu.}|_F\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\mu\nu\nu\mu}\, \omega_E$$

$\widetilde{\mathcal{R}\omega}$ is acting on $A \in \mathring{\mathcal{A}}$, ω_D volume form on D



$$A_{.\nu\nu.}(X,Y) = A(X,\nu,\nu,Y), \qquad A_{\mu\nu\nu\mu} = A(\mu,\nu,\nu,\mu).$$

Test space $\mathring{\mathcal{A}}$ (has Riemann curvature tensor symmetries)

$$\mathcal{A} = \{A \in \mathcal{T}^4(\mathscr{T}) \,:\, A_{.\nu\nu.}|_F \text{ is single-valued for all } F \in \mathring{\mathscr{F}}, \text{ and}$$
$$A(X,Y,Z,W) = -A(Y,X,Z,W) = -A(X,Y,W,Z) = A(Z,W,X,Y)\}$$
$$\mathring{\mathcal{A}} = \{A \in \mathcal{A} \,:\, A_{.\nu\nu.}|_F = 0 \text{ on } \partial\Omega\}$$

GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Generalizing Riemann curvature to Regge metrics, *arXiv:2311.01603*.

Specialization to other curvature quantities

Gauss curvature: $$\widetilde{K\omega}(v) = \sum_{T\in\mathscr{T}}\int_T K|_T\, v\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F [\![\kappa]\!]\, v\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, v\,\omega_E$$

Scalar curvature: $$\widetilde{S\omega}(v) = \sum_{T\in\mathscr{T}}\int_T S|_T\, v\,\omega_T + 2\sum_{F\in\mathring{\mathscr{F}}}\int_F [\![H]\!]\, v\,\omega_F + 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, v\,\omega_E$$

Ricci curvature:

$$\widetilde{\mathrm{Ric}\,\omega}(\sigma) = \sum_{T\in\mathscr{T}}\int_T \langle \mathrm{Ric}|_T, \sigma\rangle\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![I\!I]\!], \sigma|_F + \sigma_{\nu\nu} g|_F\rangle\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\int_E (\sigma_{\nu\nu}+\sigma_{\mu\mu})\,\Theta_E\omega_E$$

Einstein tensor: $$\widetilde{G\omega}(\sigma) = \sum_{T\in\mathscr{T}}\int_T \langle G|_T, \sigma\rangle\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\overline{I\!I}]\!], \sigma|_F\rangle\omega_F - \sum_{E\in\mathring{\mathscr{E}}}\int_E \mathrm{tr}(\sigma|_E)\,\Theta_E\,\omega_E$$

- Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *SMAI J. Comput. Math.* (2023).
- Gawlik, N.: Finite element approximation of scalar curvature in arbitrary dimension, *Math. Comp.* (2024).
- Gawlik, N.: Finite element approximation of the Einstein tensor, *IMA J. Numer. Anal.* (2025).

Roadmap of the analysis

Portland State University

1. $\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h}(U) - (\mathcal{R}\omega\mathbb{A})_{\bar{g}}(U) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U)\,dt$ with $g(t) = \bar{g} + t(g_h - \bar{g})$.

2. $\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U) = a(g;\dot{g},U) + b(g;\dot{g},U)$ sum of the bilinear forms a and b.

3. $b(g;\dot{g},U) = -2\,\widetilde{\mathrm{Inc}\,\dot{g}}(A)$, with $\dot{g} = g_h - \bar{g}$ and $A = \mathbb{A}_g(U)$.
   - Analyze the adjoint: $\widetilde{\mathrm{Inc}\,\dot{g}}(A) = (\widetilde{\mathrm{Inc}^*A})(\dot{g})$
     Then all spatial derivatives are applied on the test function A, not $\dot{g}$.
   - $b(g;\dot{g},U) \leqslant C_{g_h,\bar{g}} |||g_h - \bar{g}|||_2 \|U\|_{H^2}$

4. $a(g;\dot{g},U)$ has no spatial derivatives of $\dot{g}$
   - $a = 0$ in 2D, but $a \neq 0$ in higher dimensions
   - $a(g;\dot{g},U) \leqslant C_{g_h,\bar{g}} |||g_h - \bar{g}|||_2 \|U\|_{H^2}$

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Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)

Suppose g_h is a collection of Regge metrics such that $g_h \to \bar{g}$ in L^∞ and g_h is uniformly bounded in $W^{2,\infty}$. Then

$$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \leqslant C_{\bar{g},g_h} \|\!|g_h - \bar{g}|\!\|_2.$$

Here,

$$\|\!|\sigma|\!\|_2^2 = \sum_{T \in \mathscr{T}} \left(\|\sigma\|_{L^2(T)}^2 + h^2 \|\sigma\|_{H^1(T)}^2 + h^4 \|\sigma\|_{H^2(T)}^2 \right)$$

$$C_{\bar{g},g_h} = C \left(1 + \max_{T \in \mathscr{T}} h_T^{-2+\delta_2^N} \|g_h - \bar{g}\|_{L^\infty(T)} + \max_{T \in \mathscr{T}} h_T^{-1} \|g_h - \bar{g}\|_{W^{1,\infty}(T)} \right)$$

Corollary

If additionally $\|g_h - \bar{g}\|_{W^{t,\infty}} \lesssim h^{s-t} \|g\|_{W^{s,\infty}}$ for $0 \leqslant t \leqslant s \leqslant k+1$ for some $k \geqslant 1 - \delta_2^N$, then

$$\|\widetilde{\mathcal{R}\omega\mathbb{A}}_{g_h} - \mathcal{R}\omega\mathbb{A}_{\bar{g}}\|_{H^{-2}} \lesssim \mathcal{O}(h^{k+1}).$$

Application (Gauss curvature approximation)

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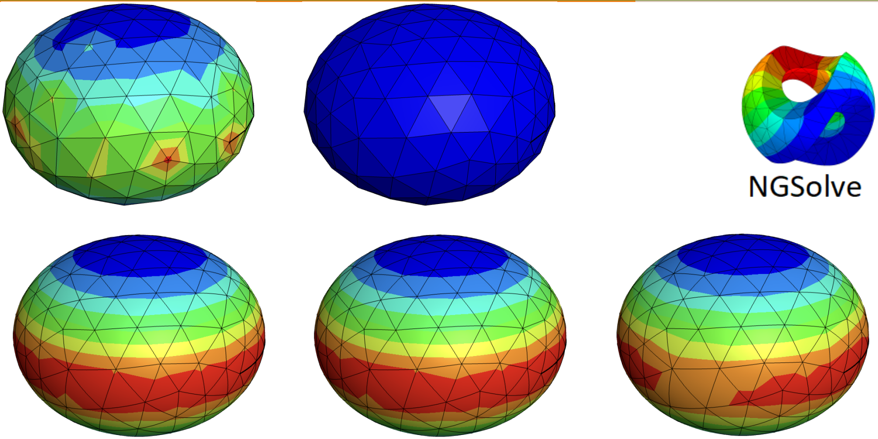
$k = 0$

$k = 1$



NGSolve

all terms | no angle defect | no geodesic curvature

$$\widetilde{K\omega}(v) = \sum_{T\in\mathscr{T}} \int_T K|_T \, v \, \omega_T \quad + \quad \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \, v \, \omega_E \quad + \quad \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\kappa]\!] \, v \, \omega_F$$

$$\mathcal{W}(u) = t\, E_{\text{mem}}(u) + t^3\, E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Application (membrane locking in shells)

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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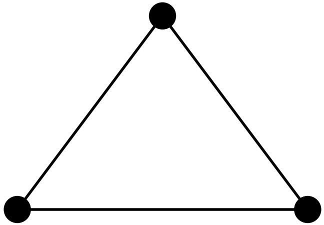
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



$$\mathcal{L}_h(\mathscr{T}_h) = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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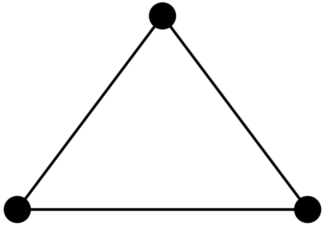

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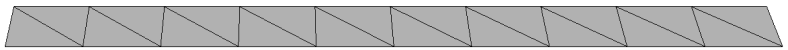

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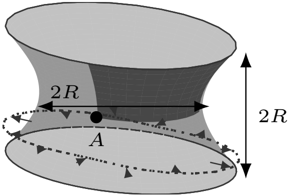
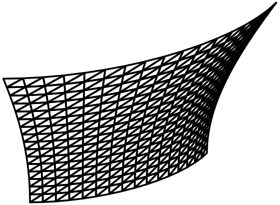
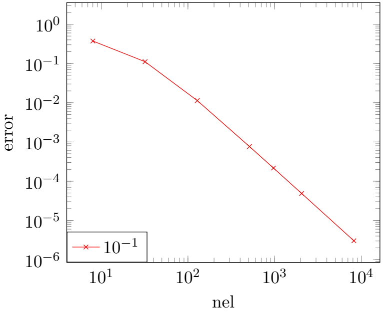
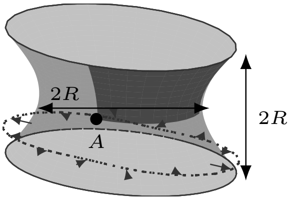
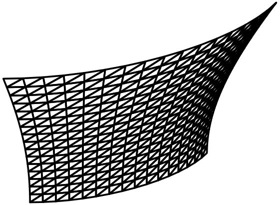
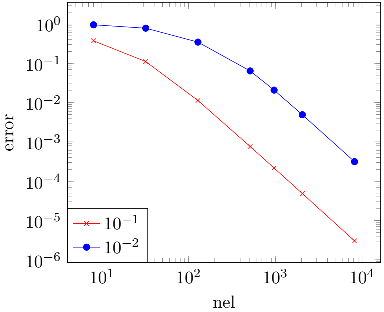






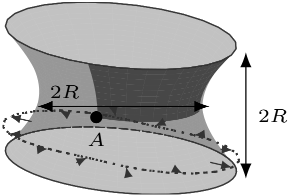


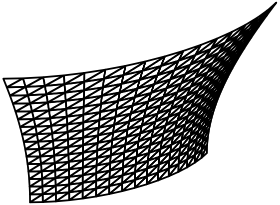


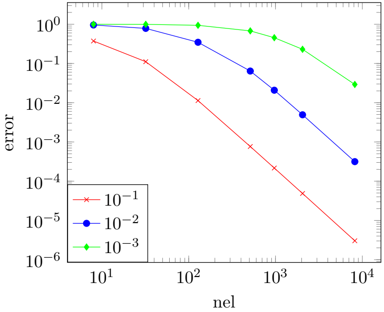


- Pre-asymptotic regime

Application (membrane locking in shells)







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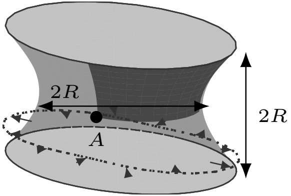
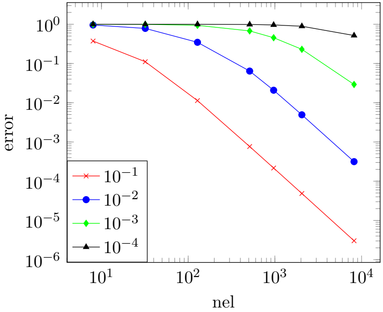
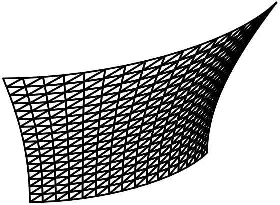






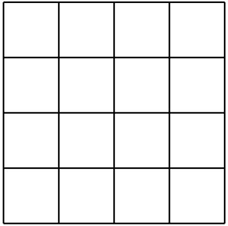
- Pre-asymptotic regime

Application (membrane locking in shells)

$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

Application (membrane locking in shells)

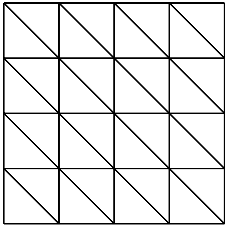
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$

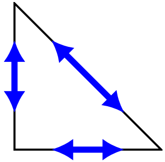


- Reduced integration for quadrilateral meshes

Application (membrane locking in shells)

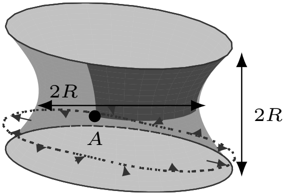
$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$





- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*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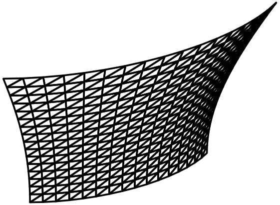

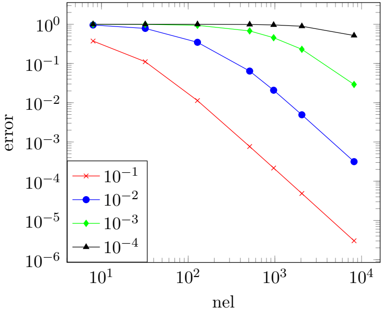
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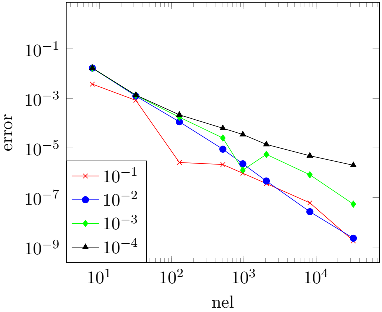
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NGSolve add-on package for differential geometry support

- Co-/contravariant coordinate expressions and Christoffel symbols are error prone
- Framework to handle differential geometry objects in efficient manner
- GitHub (WIP): `https://github.com/MichaelNeunteufel/NGSDiffGeo`
- Documentation & tutorials: `https://michaelneunteufel.github.io/NGSDiffGeo/`

```
import ngsdiffgeo as dg
mf = dg.RiemannianManifold(metric)

X = dg.VectorField(CF((x*y, y)))
alpha = dg.OneForm(CF((sin(x), y**2)))
A = dg.TensorField(CF((y, x, sin(x), y**2), dims=(2, 2)), "00")
B = dg.TensorProduct(alpha, X)

mf.InnerProduct(A, B)
mf.CovDeriv(A)
```

Summary

- Combining DDG and distributional FEM for extrinsic & intrinsic curvature
- Definition of generalized Weingarten tensor
- Definition of generalized Riemann curvature tensor (Gauss, scalar, Ricci, Einstein)
- Numerical analysis with integral representation
- Uhlenbeck trick for test functions

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- Applications to geometric flows and numerical relativity

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Thank You for Your attention!

Literature

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Metric independent test space

We use an approach inspired by the Uhlenbeck trick: Define the metric independent test space

$$\mathcal{U} = \{U \in \Lambda^{N-2}(\mathscr{T}) \odot \Lambda^{N-2}(\mathscr{T}) \;:\; U(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \text{ is single-valued}$$
$$\text{on all } F \in \mathring{\mathscr{F}} \text{ for all } X_1, \ldots X_{N-2}, Y_1, \ldots, Y_{N-2} \in \mathfrak{X}(F)\}$$

Lemma

The map $\mathbb{A}_g : \mathcal{U} \to \mathcal{A}$, $U \mapsto -\star^{\odot^2} U$ is a bijection.

Define $\widetilde{\mathcal{R}\omega\mathbb{A}}_g(U) = \widetilde{\mathcal{R}\omega}(\mathbb{A}_g(U))$ for g-independent $U \in \mathcal{U}$. Then we have

$$\widetilde{\mathcal{R}\omega\mathbb{A}}_g(U) - (\mathcal{R}\omega\mathbb{A})_{\bar{g}}(U) = \int_0^1 \frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}}_{g(t)}(U)\, dt.$$

We can proceed computing and estimating the right-hand side.

Incompatibility operator

Define for smooth 2-tensor σ the incompatibility operator $\mathrm{Inc} : \mathcal{T}^2 \to \mathcal{T}^4$ by

$$(\mathrm{Inc}\,\sigma)(X,Y,Z,W) := \frac{1}{4}\big[(\nabla^2_{Y,Z}\sigma)(X,W) + (\nabla^2_{X,W}\sigma)(Y,Z) - (\nabla^2_{X,Y}\sigma)(Z,W) - (\nabla^2_{Y,W}\sigma)(X,Z)\big].$$

In 2D and 3D Inc can be related to the standard incompatibility operator $\mathrm{inc} = \mathrm{curl}^T\mathrm{curl}$.

Lemma (linearization Riemann curvature tensor)

For t-independent vector fields $X, Y, Z, W \in \mathfrak{X}(T)$ there holds

$$\dot{\mathcal{R}}(X,Y,Z,W) = -\frac{1}{2}(\mathrm{Inc}\,\dot{g})(X,Y,Z,W) + \frac{1}{2}\left[\dot{g}(\mathcal{R}_{X,Y}Z,W) - \dot{g}(\mathcal{R}_{X,Y}W,Z)\right].$$

Generalized incompatibility operator

For tt-continuous σ, a generalized Inc can be defined as

$$\widetilde{\mathrm{Inc}\,\sigma}(A) = \sum_{T\in\mathscr{T}}\int_T \langle \mathrm{Inc}\,\sigma, A\rangle\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}} \dots + \sum_{E\in\mathring{\mathscr{E}}} \dots$$

Lemma

Let $\sigma := \dot{g}(t)$, $A \in \mathcal{A}$, $(SA)(X,Y,Z,W) = A(X,Z,Y,W)$ swaps second with third argument. There holds

$$\dot{A}(X,Y,Z,W) = -\mathrm{tr}(\sigma)A(X,Y,Z,W) + A(\sigma(X,\cdot)^\sharp,Y,Z,W) + A(X,\sigma(Y,\cdot)^\sharp,Z,W) \\ + A(X,Y,\sigma(Z,\cdot)^\sharp,W) + A(X,Y,Z,\sigma(W,\cdot)^\sharp),$$

Lemma

Let $\sigma := \dot{g}(t)$, $A \in \mathcal{A}$, $(SA)(X,Y,Z,W) = A(X,Z,Y,W)$ swaps second with third argument. There holds

$$
\begin{aligned}
\dot{A}(X,Y,Z,W) &= -\mathrm{tr}(\sigma)A(X,Y,Z,W) + A(\sigma(X,\cdot)^\sharp, Y, Z, W) + A(X, \sigma(Y,\cdot)^\sharp, Z, W) \\
&\quad + A(X,Y,\sigma(Z,\cdot)^\sharp, W) + A(X,Y,Z,\sigma(W,\cdot)^\sharp), \\
\frac{d}{dt}\big(\langle \mathcal{R}, A\rangle \omega_T\big) &= \big(2\langle \nabla^2\sigma, S(A)\rangle + \langle \mathcal{R}(\sigma(\cdot,\cdot)^\sharp, \cdot,\cdot,\cdot), A\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle \mathcal{R}, A\rangle\big)\omega_T, \\
\frac{d}{dt}\big(\langle [\![I\!I]\!], A_{\cdot\nu\nu\cdot}|_F\rangle \omega_F\big) &= \frac{1}{2}\langle [\![(\sigma(\nu,\nu) - \mathrm{tr}(\sigma|_F)) I\!I + 2(\nabla_F\sigma)(\nu,\cdot)|_F - (\nabla_\nu \sigma)|_F]\!], A_{\cdot\nu\nu\cdot}|_F\rangle\, \omega_F, \\
\frac{d}{dt}\big(\Theta_E A_{\mu\nu\nu\mu}\,\omega_E\big) &= -\frac{1}{2}\Big(\sum_{F\supset E} [\![\sigma(\nu,\mu)]\!]_F^E + \mathrm{tr}(\sigma|_E)\Theta_E\Big) A_{\mu\nu\nu\mu}\,\omega_E.
\end{aligned}
$$

Proposition

Let $\sigma := \dot{g}$ and $A \in \mathring{\mathcal{A}}$ with corresponding $U = \mathbb{A}_g^{-1} A \in \mathring{\mathcal{U}}$. Then there holds

$$\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}_g}(U) = a(g;\sigma,U) + b(g;\sigma,U),$$

$$\begin{aligned}
a(g;\sigma,U) = &\sum_{T\in\mathscr{T}}\int_T \left(\langle \mathcal{R}(\sigma(\cdot,\cdot)^\sharp,\cdot,\cdot,\cdot),\mathbb{A}_g U\rangle - \frac{1}{2}\mathrm{tr}(\sigma)\langle\mathcal{R},\mathbb{A}_g U\rangle\right)\omega_T\\
&- 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \Big(\mathrm{tr}(\sigma|_F)\langle [\![II]\!], (\mathbb{A}_g U)_{\cdot\nu\nu\cdot}|_F\rangle - [\![II]\!] : \sigma|_F : (\mathbb{A}_g U)_{\cdot\nu\nu\cdot}|_F\Big)\omega_F\\
&- 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \mathrm{tr}(\sigma|_E)\Theta_E(\mathbb{A}_g U)_{\mu\nu\nu\mu}\,\omega_E
\end{aligned}$$

$[\![II]\!] : \sigma|_F : (\mathbb{A}_g U)_{\cdot\nu\nu\cdot} = [\![II]\!]_{ij}(\sigma|_F)^{jk}((\mathbb{A}_g U)_{\cdot\nu\nu\cdot})_k{}^i.$

Proposition

Let $\sigma := \dot{g}$ and $A \in \mathring{\mathcal{A}}$ with corresponding $U = \mathbb{A}_g^{-1} A \in \mathring{\mathcal{U}}$. Then there holds

$$\frac{d}{dt}\widetilde{\mathcal{R}\omega\mathbb{A}_g}(U) = a(g;\sigma,U) + b(g;\sigma,U),$$

$$\begin{aligned} b(g;\sigma,U) = 2\sum_{T\in\mathscr{T}}\int_T \langle \nabla^2\sigma, S\mathbb{A}_g U\rangle\,\omega_T \\ + 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\sigma(\nu,\nu)I\!I + (\nabla_F\sigma)(\nu,\cdot)|_F + \nabla_F(\sigma(\nu,\cdot))|_F - (\nabla_\nu\sigma)|_F]\!], (\mathbb{A}_g U)_{\cdot\nu\nu\cdot}|_F\rangle\,\omega_F \\ - 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \sum_{F\supset E} [\![\sigma(\nu,\mu)]\!]_F^E (\mathbb{A}_g U)_{\mu\nu\nu\mu}\,\omega_E. \end{aligned}$$