Locking-free mixed finite element methods for nonlinear shells

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Project J 4824-N

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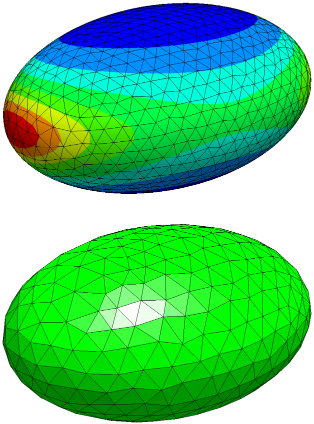
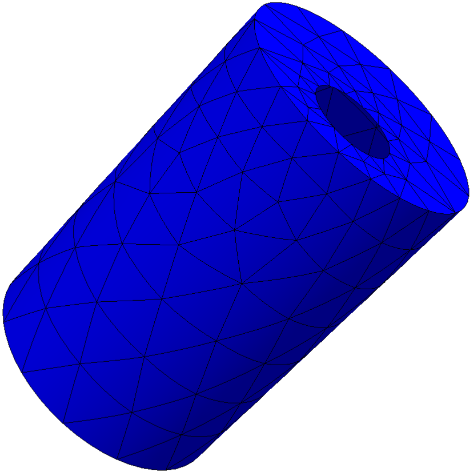
Approximate curvature of non-smooth surfaces

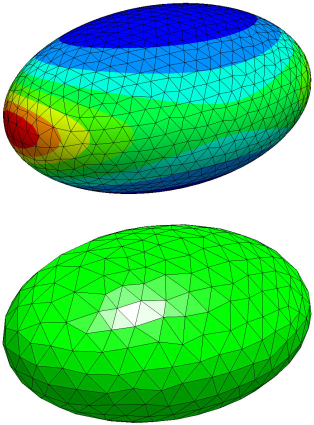


Thin-walled structures (shells)

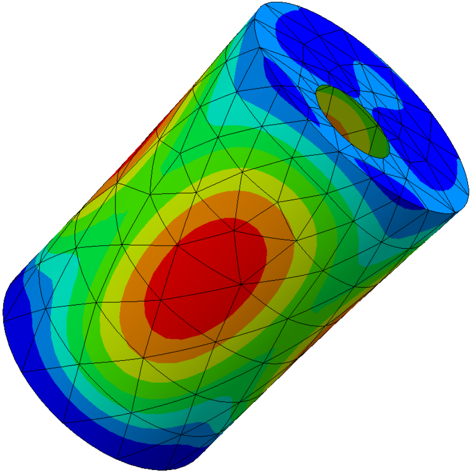


Motivation

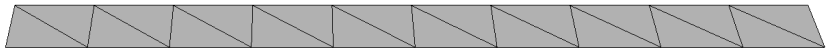
Approximate curvature of non-smooth surfaces

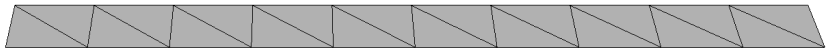


Thin-walled structures (shells)



Motivation (membrane locking)





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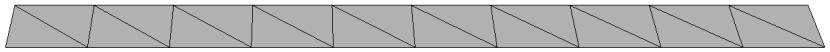
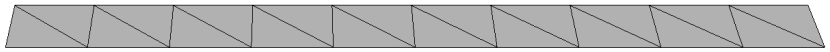
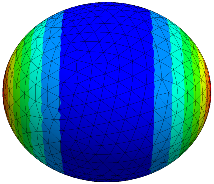
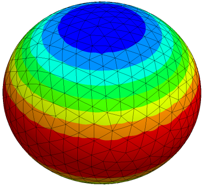
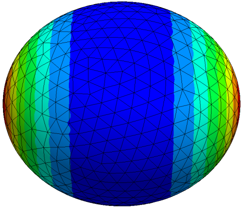


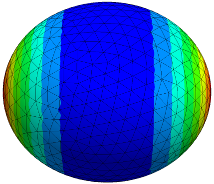


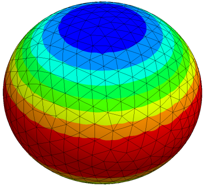
Motivation (cell membranes)

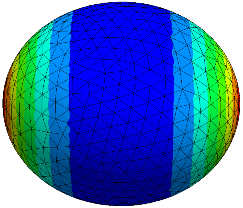








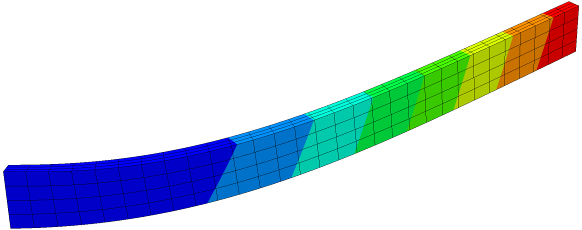




Contents

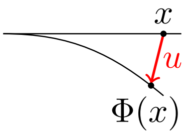
Elasticity & thin-walled structures

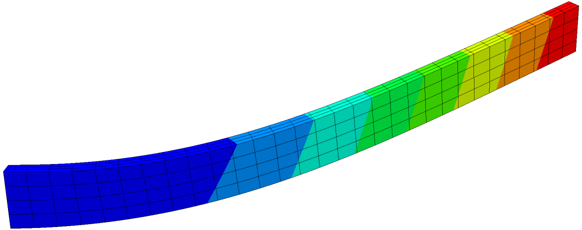
Deformation $\Phi : \Omega \to \mathbb{R}^3$



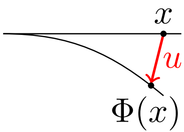
Deformation $\Phi : \Omega \to \mathbb{R}^3$

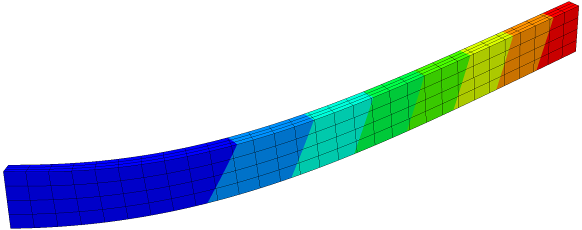
Displacement $u := \Phi - id$





Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla \Phi$





Deformation	$\Phi : \Omega \to \mathbb{R}^3$
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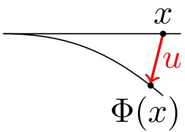
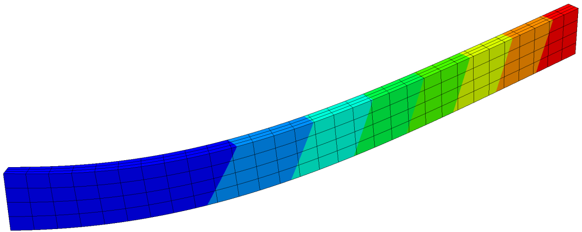
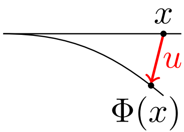
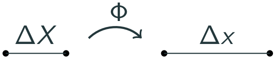




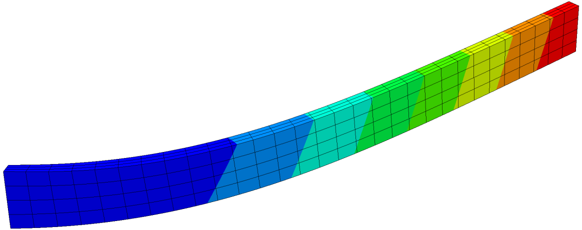
Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{I} + \nabla u + \nabla u^T + \nabla u^T \nabla u$



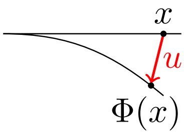


$$\Delta x^T \Delta x = \Delta X^T \boldsymbol{C} \Delta X$$



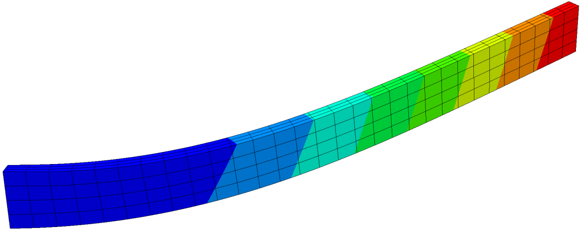
Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{I} + \nabla u + \nabla u^T + \nabla u^T \nabla u$
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$
Linearized strain	$\varepsilon := \mathrm{sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$



$$\Delta X \xrightarrow{\Phi} \Delta x$$

$$\Delta x^T \Delta x = \Delta X^T \boldsymbol{C} \Delta X$$



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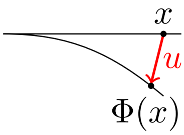
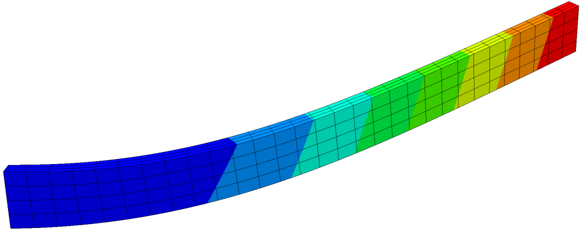
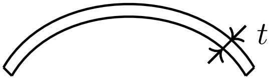


Elasticity

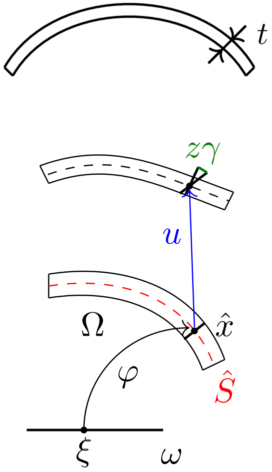
$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|_{\boldsymbol{M}}^2 - \langle f, u \rangle \to \min_{u \in V}!$$



Thin-walled structures



- Reduce 3D elasticity to 2D shell model



- Reduce 3D elasticity to 2D shell model
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z\underbrace{(\nu + \gamma) \circ \phi}_{=\tilde{\nu}\circ\phi}(\hat{x})$
- Reissner-Mindlin/Naghdi shell

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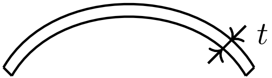

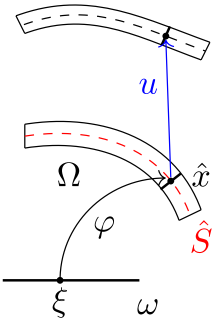

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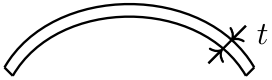
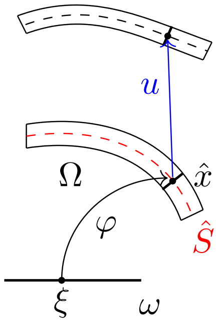
- Kirchhoff–Love/Koiter shell

Thin-walled structures





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- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \underbrace{\phi(\hat{x})}_{=\hat{x}+u(\hat{x})} + z\,\nu \circ \phi(\hat{x})$
- Kirchhoff–Love/Koiter shell
- Insert Φ in 3D elasticity and integrate over thickness, neglect higher order terms $\mathcal{O}(t^4)$ (asymptotical analysis)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu \circ \phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$

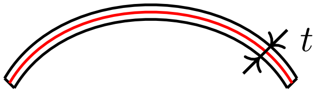
$u \ldots$ displacement of mid-surface

$t \ldots$ thickness

$\mathbb{M} \ldots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu} \otimes \hat{\nu}$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top \nabla u + \nabla u^\top \boldsymbol{P} + \boldsymbol{P}\nabla u)$$



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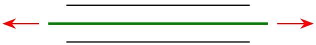


membrane energy

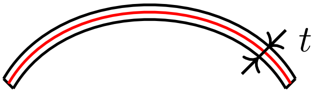
u . . . displacement of mid-surface

t . . . thickness

$\mathbb{M}$ . . . material tensor

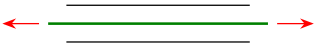
$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$



u . . . displacement of mid-surface

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$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

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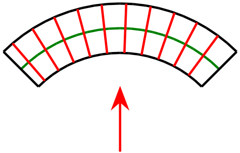
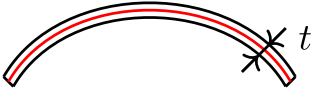




Naghdi shell

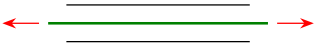
$$\mathcal{W}(u,\gamma) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla(\tilde{\nu}\circ\phi)) - \nabla\hat{\nu}\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2$$

γ . . . shearing

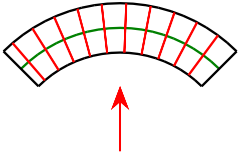
$\tilde{\nu} = \dfrac{\nu+\gamma}{\|\nu+\gamma\|}$ . . . director

G . . . shearing modulus

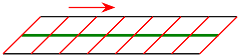
$\kappa = 5/6$ . . . shear correction factor



membrane energy



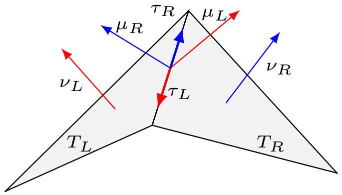
bending energy



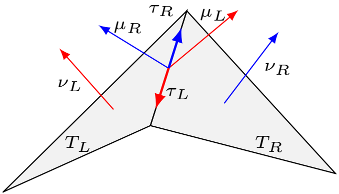
shearing energy

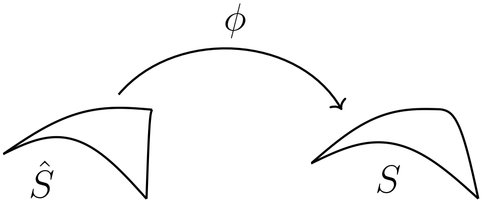
Differential geometry

- Normal vector ν
  Tangent vector τ
  Element normal vector $\mu = \nu \times \tau$



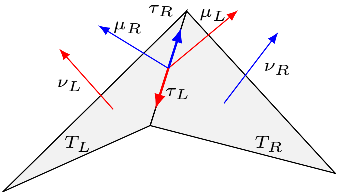
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

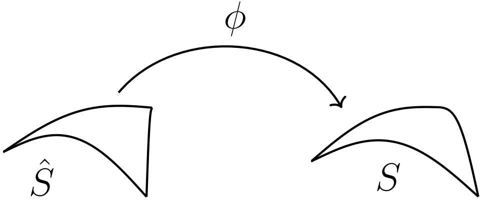




- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$





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  Tangent vector $\hat{\tau}$
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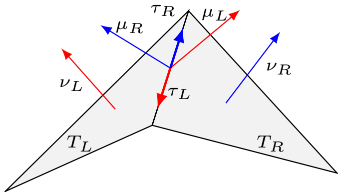
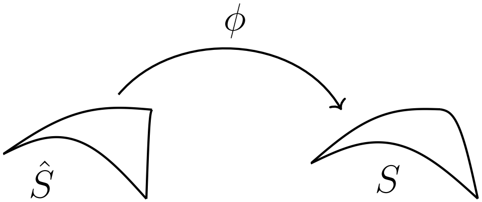
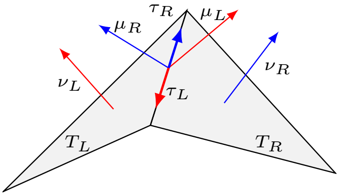


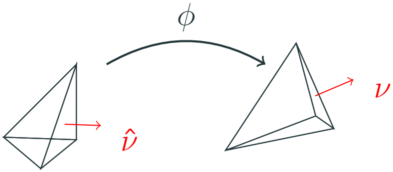
- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



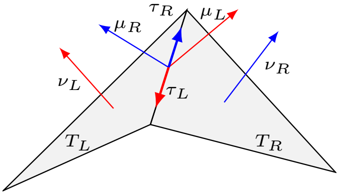
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$



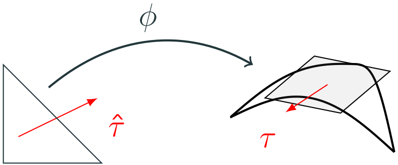
- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$



- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

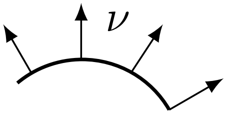


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$
  $\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi$



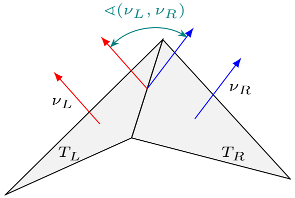
Curvature approximation

- Change of normal vector measures curvature $\nabla\nu$

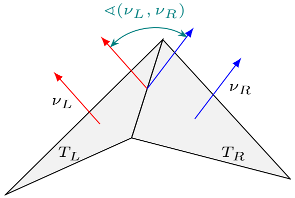


Curvature approximation

- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



Grinspun, Gingold, Reisman, Zorin: Computing discrete shape operators on general meshes, *Computer Graphics Forum* (2006).



- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?

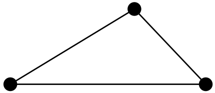
  - Sobolev perspective: $\nu \notin H^1$, but $\nu \in L^2$
  - $\nabla\nu \notin L^2$, it is a distribution (or measure)
  - Define distributional Weingarten tensor ($\boldsymbol{\Psi}_{\mu\mu} = (\boldsymbol{\Psi}\mu)\cdot\mu$)

$$\langle \nabla\nu, \boldsymbol{\Psi}\rangle_{\mathscr{T}} = \sum_{T\in\mathscr{T}} \int_T \nabla\nu : \boldsymbol{\Psi}\, da + \sum_{E\in\mathscr{E}^{\mathrm{int}}} \int_E \sphericalangle(\nu_L, \nu_R)\, \boldsymbol{\Psi}_{\mu\mu}\, dl$$

  - Test function $\boldsymbol{\Psi}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023).

Lagrange finite elements:



$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in [L^2(\Omega)]^d\}$

$\mathcal{L}_h^k(\mathscr{T}_h) = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

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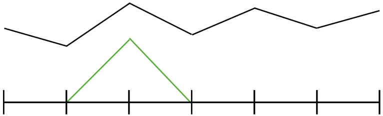

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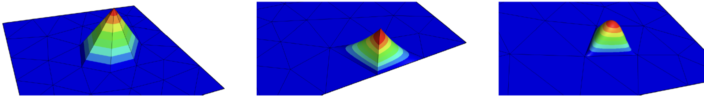

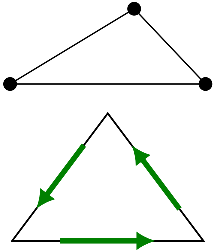

Lagrange finite elements:

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Nédélec finite elements:

$$H(\mathrm{curl}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \mid \mathrm{curl}\sigma \in [L^2(\Omega)]^{2d-3}\}$$

$$\mathcal{N}_{II}^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid [\![\sigma_\tau]\!]_F = 0\} \subset H(\mathrm{curl}, \Omega)$$

Lagrange finite elements:

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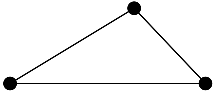

Nédélec finite elements:

$$H(\mathrm{curl}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{curl}\sigma \in [L^2(\Omega)]^{2d-3}\}$$

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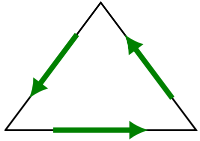
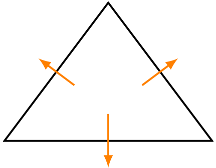


Raviart-Thomas/Brezzi-Douglas-Marini elements:

$$H(\mathrm{div}, \Omega) = \{\sigma \in [L^2(\Omega)]^d \,|\, \mathrm{div}\sigma \in L^2(\Omega)\}$$

$$BDM^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div}, \Omega)$$



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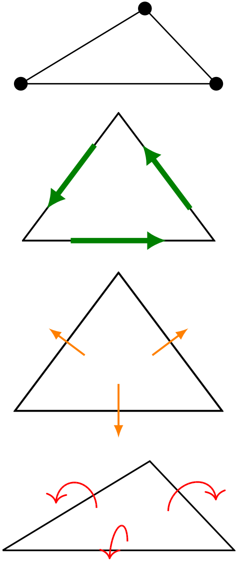
$$BDM^k = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \,|\, [\![\sigma_n]\!]_F = 0\} \subset H(\mathrm{div}, \Omega)$$

Hellan-Herrmann-Johnson finite elements:

$$H(\mathrm{divdiv}, \Omega) = \{\sigma \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}} \,|\, \mathrm{div}\mathbf{div}\sigma \in H^{-1}(\Omega)\}$$

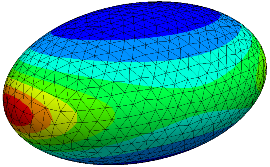
$$M_h^k(\mathscr{T}_h) = \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \,|\, [\![n^T \sigma n]\!]_F = 0\}$$

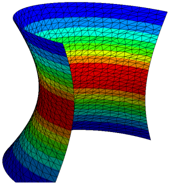


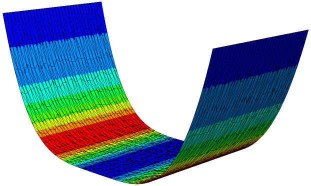
Lifting of distributional Weingarten tensor

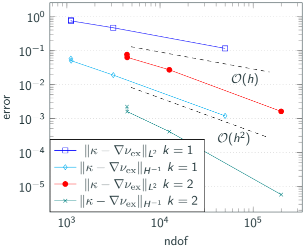
Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

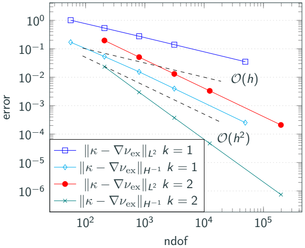
$$\int_{\mathscr{T}} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, da = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} .$$

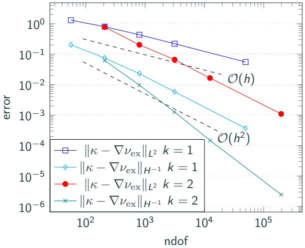












Example curvature approximation

Portland State University

Lifting of distributional Weingarten tensor

Find $\boldsymbol{\kappa} \in \Sigma_h^{k-1}$ for $\mathscr{T}$ with curving order k such that for all $\boldsymbol{\sigma} \in \Sigma_h^{k-1}$

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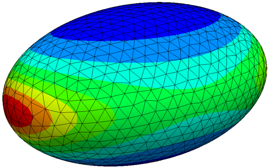
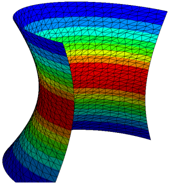
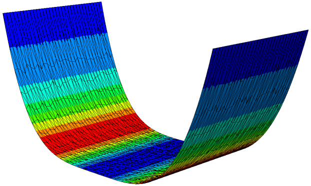






- Numerical analysis: WIP

GOPALAKRISHNAN, N.: Analysis of generalized shape operator on surfaces *(in preparation)*

Application (cell membrane)

Canham–Helfrich–Evans energy:

$$\mathcal{W}(\mathcal{S}) = 2\kappa_b \int_{\mathcal{S}} (H - H_0)^2 \, ds$$

κ_b bending elastic constant

H mean curvature

$2H_0$ spontaneous curvature

Constraints:

$$|\Omega| = V_0, \quad |\mathcal{S}| = A_0, \qquad V_0 \leq \frac{A_0^{\frac{3}{2}}}{6\sqrt{\pi}}$$

Functional:

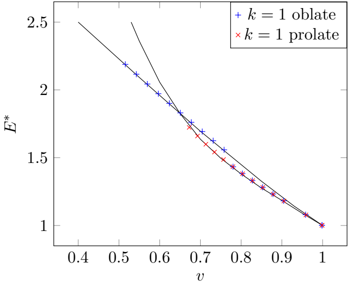
$$\mathcal{J}(\mathcal{S}) = \mathcal{W}(\mathcal{S}) + c_A(|\mathcal{S}| - A_0)^2 + c_V(|\Omega| - V_0)^2$$

$$H = 0.5\,\mathrm{tr}(\nabla \boldsymbol{\nu})$$

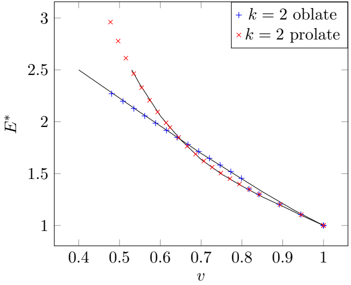
N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023)

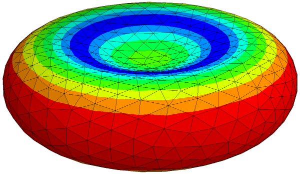
Gangl, Sturm, N., Schöberl, Fully and Semi-Automated Shape Differentiation in NGSolve, *Structural and Multidisciplinary Optimization* (2021)

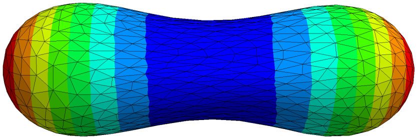
Application (cell membrane)



NGSolve

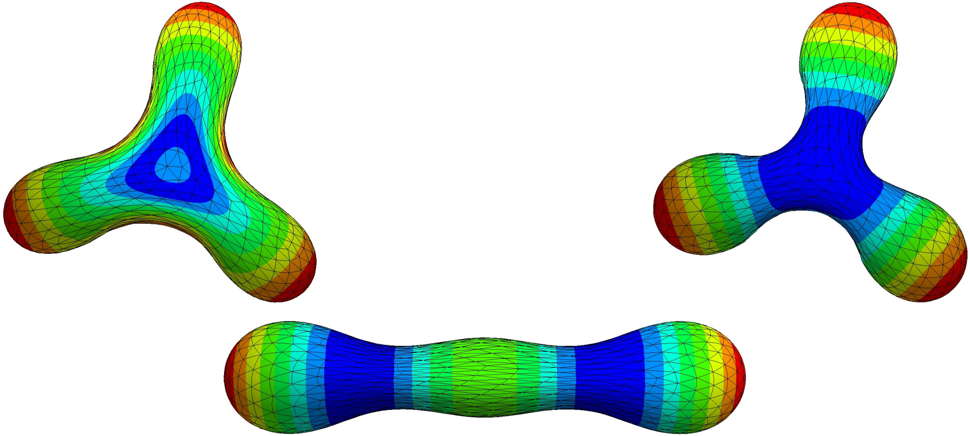






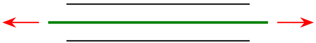
Seifert, Berndl, Lipowsky, Shape transformations of vesicles: Phase diagram for spontaneous-curvature and bilayer-coupling models, *Phys. Rev. A (1991)*

More complicated shapes with non-zero spontaneous curvature H_0:



Nonlinear shells

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\mathbb{M}}$$



membrane energy

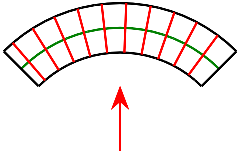
$u \ldots$ displacement of mid-surface

$t \ldots$ thickness

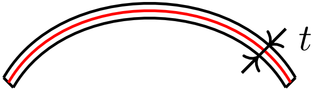
$\mathbb{M} \ldots$ material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$



bending energy



Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\mathbb{M}}^2$$

u . . . displacement of mid-surface

t . . . thickness

$\mathbb{M}$ . . . material tensor

$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$

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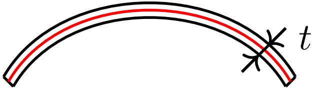
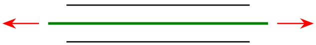
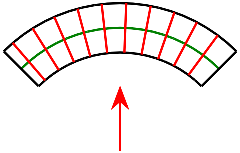




membrane energy



bending energy

- $\boldsymbol{F}^T\nabla(\nu\circ\phi) = f(\nabla^2 u)$
- Generalized curvature $\nabla\hat{\nu}$

$$\text{Lifting:} \quad \int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma}\, dx = \sum_{T\in\mathscr{T}_h} \int_T \nabla\nu : \boldsymbol{\sigma}\, dx + \sum_{E\in\mathscr{E}_h} \int_E \sphericalangle(\nu_L,\nu_R)\boldsymbol{\sigma}_{\mu\mu}\, ds$$

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via Hu–Washizu three-field formulation

$$\mathcal{L}(u,\boldsymbol{\kappa}^{\mathrm{diff}},\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|_{\mathbb{M}}^2 + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|_{\mathbb{M}}^2 - \langle f,u\rangle + \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma}\in M_h^{k-1}$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ Hellinger–Reissner two-field formulation in $(u,\boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)

Portland State University

Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\
&+ \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\
&+ \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019)*.

Portland State University

Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

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N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

Portland State University

Shell problem

Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$
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\mathcal{L}(u, \boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\
&+ \sum_{T\in\mathscr{T}_h} \int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad\qquad\qquad\qquad) \, dx \\
&+ \sum_{E\in\mathscr{E}_h} \int_E (\sphericalangle(\nu_L, \nu_R) \qquad\qquad\qquad\qquad)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, ds
\end{aligned}
$$

Use hybridization to eliminate $\boldsymbol{\sigma}$ $\rightarrow$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

Portland State University

Shell problem

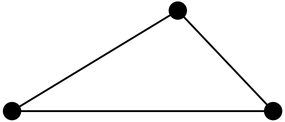
Find $u \in [V_h^k]^3$, $\boldsymbol{\sigma} \in M_h^{\mathrm{dc},k-1}$, and $\alpha \in \Gamma_h^{k-1}$ for

$$\begin{aligned}\mathcal{L}(u,\boldsymbol{\sigma},\alpha) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} - \langle f, u\rangle \\ + \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\ + \sum_{E\in\mathscr{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} + [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\alpha_{\hat{\mu}}\,ds\end{aligned}$$

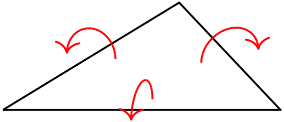
$\{\{\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\}\} = \frac{1}{2}((\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} + (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R})$, $[\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_L} - (\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}})|_{T_R}$

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. (2019).*

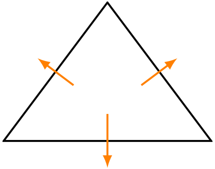
Shell element (Koiter)



displacement u

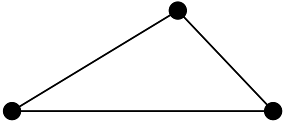


bending moment tensor $\boldsymbol{\sigma}$

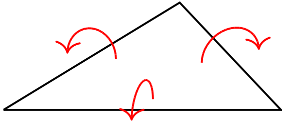


hybridization α

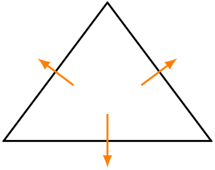
Shell element (Koiter)



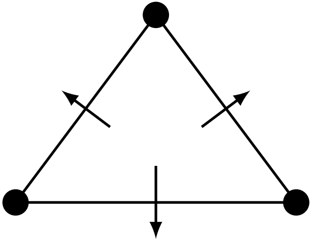
displacement u



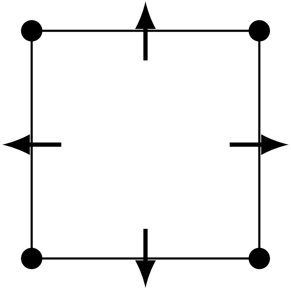
bending moment tensor $\boldsymbol{\sigma}$



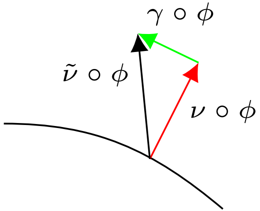
hybridization α



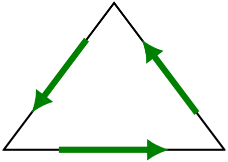
Morley element



- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \frac{\nu\circ\phi+\gamma\circ\phi}{\|\nu\circ\phi+\gamma\circ\phi\|}$
- Free of shear locking



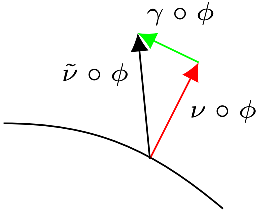
$$H(\mathrm{curl}) := \{u \in [L^2(\Omega)]^d \mid \mathrm{curl}\, u \in [L^2(\Omega)]^{2d-3}\}$$
$$\mathcal{N}_{II}^k := \{u \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid u_t \text{ is continuous over elements}\}$$



Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg.* (2013).

Extension to nonlinear Naghdi shells

- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi+\gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking



$$H(\mathrm{curl}) := \{u\in[L^2(\Omega)]^d \mid \mathrm{curl}\,u\in[L^2(\Omega)]^{2d-3}\}$$
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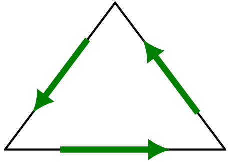

Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *Comput. Methods Appl. Mech. Engrg.* (2013).

- Use hierarchical shell model
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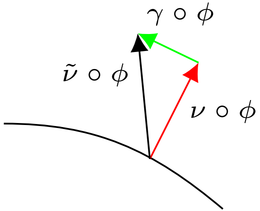
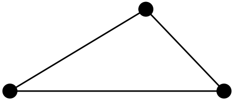
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi = \frac{1}{J}\,\mathrm{cof}(\boldsymbol{F})\hat{\nu} + (\boldsymbol{F}^{\dagger})^{\top}\hat{\gamma}$
- Free of shear locking

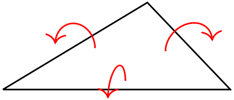


$$\mathcal{L}(u,\boldsymbol{\sigma},\hat{\gamma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}}$$
$$+ \sum_{T\in\mathscr{T}_h}\int_T \left(\boldsymbol{H}_{\tilde{\nu}} + (1-\tilde{\nu}\cdot\hat{\nu})\,\nabla\hat{\nu} - \nabla\hat{\gamma}\right):\boldsymbol{\sigma}\,dx$$
$$+ \sum_{E\in\mathscr{E}_h}\int_E \left(\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R) + [\![\hat{\gamma}_{\hat{\mu}}]\!]\right)\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds$$

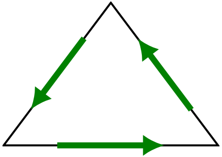
Shell element (Naghdi)



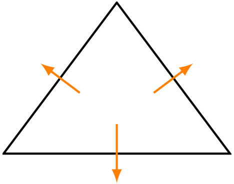
displacement u



bending moment tensor $\boldsymbol{\sigma}$

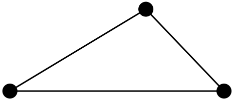


shear vector γ

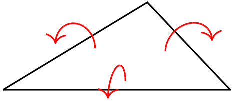


hybridization α

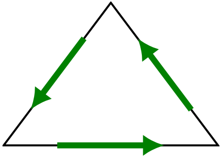
Shell element (Naghdi)



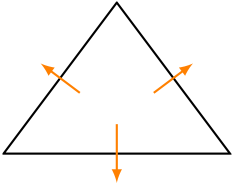
displacement u



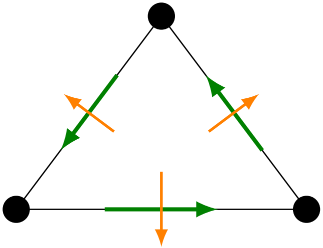
bending moment tensor $\boldsymbol{\sigma}$

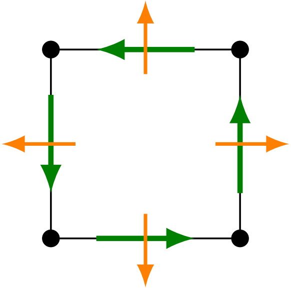


shear vector γ



hybridization α





$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{shell}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|_{\mathbb{M}}^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\mathbb{M}^{-1}}^2 + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\, dx - \int_{\partial T} (\nabla u^\top \hat{\nu})_{\hat{\mu}} \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

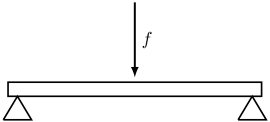
$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{plate}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|_{\mathbb{M}^{-1}}^2 + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\, dx - \int_{\partial T} \frac{\partial w}{\partial \hat{\mu}} \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{shell}}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \boldsymbol{H}_{\hat{\nu}} : \boldsymbol{\sigma}\, dx - \int_{\partial T}(\nabla u^\top \hat{\nu})_{\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

$$\mathcal{L}_{\mathrm{lin}}^{\mathrm{plate}}(w,\boldsymbol{\sigma}) = -\frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} + \sum_{T\in\mathscr{T}_h}\left(\int_T \nabla^2 w : \boldsymbol{\sigma}\, dx - \int_{\partial T}\frac{\partial w}{\partial\hat{\mu}}\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)$$

Biharmonic plate equation:

$$\mathbf{divdiv}(\mathbb{M}\nabla^2 w) = f \Leftrightarrow \begin{cases}\boldsymbol{\sigma} = \mathbb{M}\nabla^2 w,\\ \mathbf{divdiv}\boldsymbol{\sigma} = f,\end{cases}$$



M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. (1989)*

$$
\begin{aligned}
\mathcal{L}^{\text{shell}}_{\text{lin}}(u, \boldsymbol{\sigma}, \hat{\gamma}) &= \frac{t}{2}\|\mathrm{sym}(\nabla^{\mathrm{cov}} u)\|^2_{\mathbb{M}} + \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} \\
&\quad + \sum_{T\in\mathscr{T}_h}\left(\int_T (\boldsymbol{H}_{\hat{\nu}} - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}((\nabla u^\top \hat{\nu})_{\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right) \\
\mathcal{L}^{\text{plate}}_{\text{lin}}(w, \boldsymbol{\sigma}, \hat{\gamma}) &= \frac{t\kappa G}{2}\|\hat{\gamma}\|^2 - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\mathbb{M}^{-1}} \\
&\quad + \sum_{T\in\mathscr{T}_h}\left(\int_T (\nabla^2 w - \nabla\hat{\gamma}) : \boldsymbol{\sigma}\, dx - \int_{\partial T}(\frac{\partial w}{\partial\hat{\mu}} - \hat{\gamma}_{\hat{\mu}})\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds\right)
\end{aligned}
$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017)

Membrane locking

Membrane locking

$$\mathcal{W}(u) = t\, E_{\mathrm{mem}}(u) + t^3\, E_{\mathrm{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

Portland State University

$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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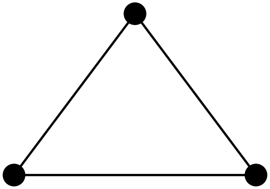
Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$



$$V_h = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

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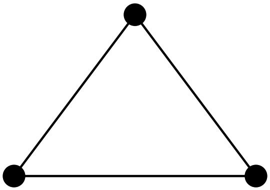
Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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Membrane locking

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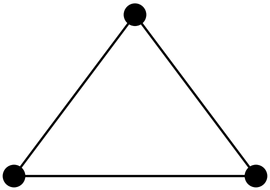
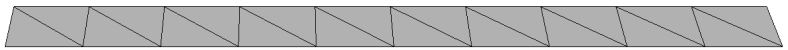
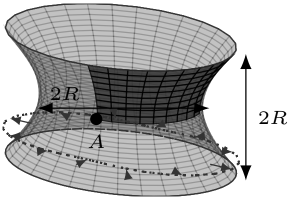
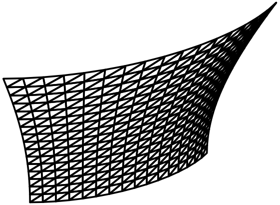


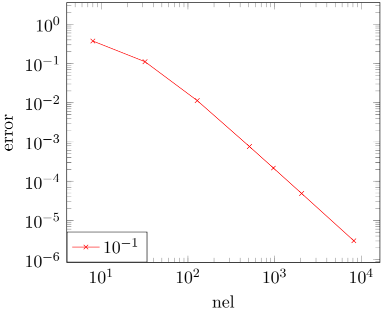
$$V_h = \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$



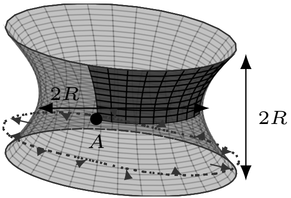
Hyperboloid with free ends

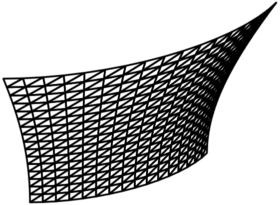


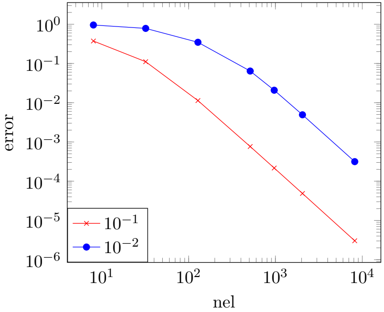




Hyperboloid with free ends

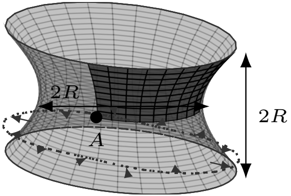


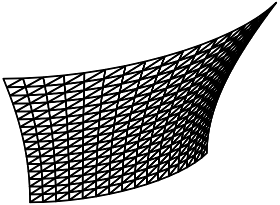


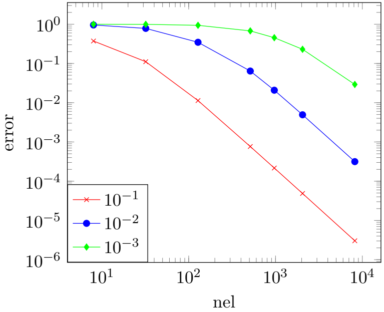


- Pre-asymptotic regime

Hyperboloid with free ends

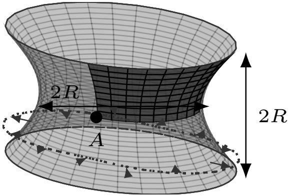


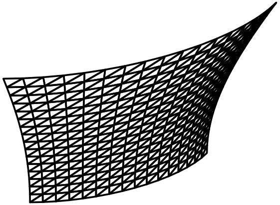


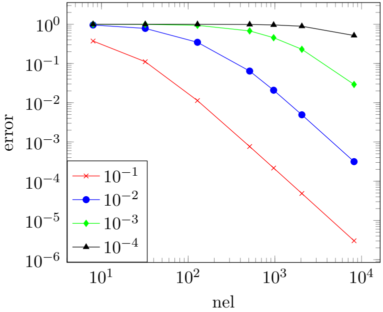


- Pre-asymptotic regime

Hyperboloid with free ends





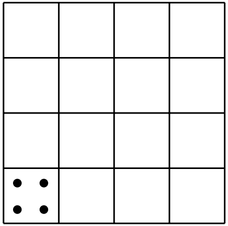


- Pre-asymptotic regime

Membrane locking

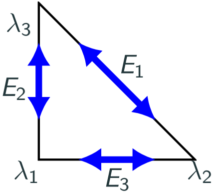
$$\frac{1}{t^2}\| \qquad \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

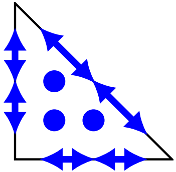
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\mathbb{M}}$$



- Reduced integration for quadrilateral meshes

$$H(\operatorname{curl}\operatorname{curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \operatorname{curl}\operatorname{curl}\sigma \in H^{-1}(\Omega)\}$$
$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





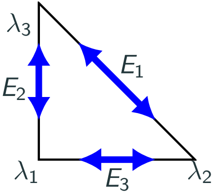
$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i\delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

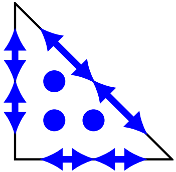
- CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik (2011).*
- LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*

Regge elements

$$H(\mathrm{curl\,curl}) := \{\sigma \in [L^2(\Omega)]^{2\times 2}_{sym} \mid \mathrm{curl\,curl}\,\sigma \in H^{-1}(\Omega)\}$$

$$\mathrm{Reg}_h^k := \{\varepsilon \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{\mathrm{sym}} \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





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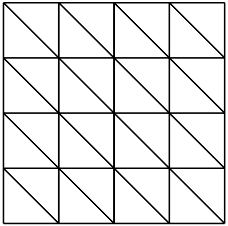
$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

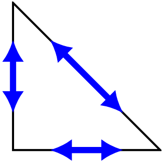
$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Membrane locking

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}(u_h)\|_{\mathbb{M}}^2$$

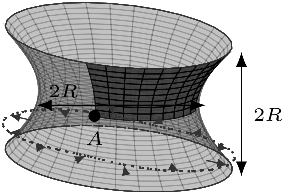


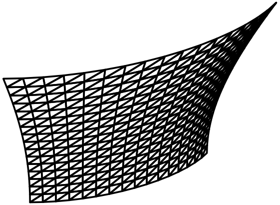


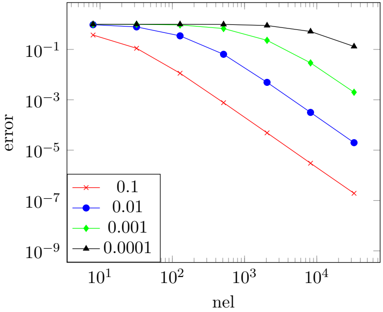
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

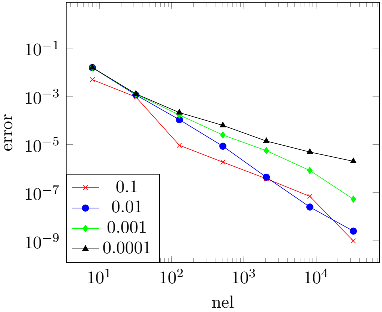
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg. (2021).*

Hyperboloid with free ends

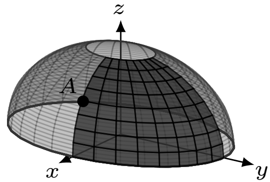


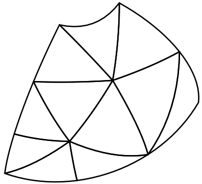


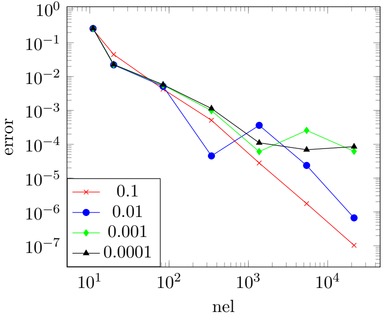


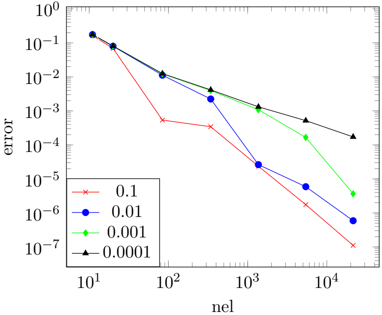


Open hemisphere with clamped ends





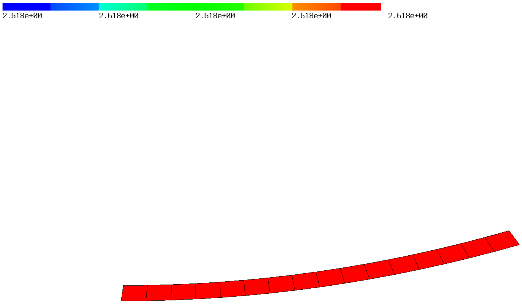


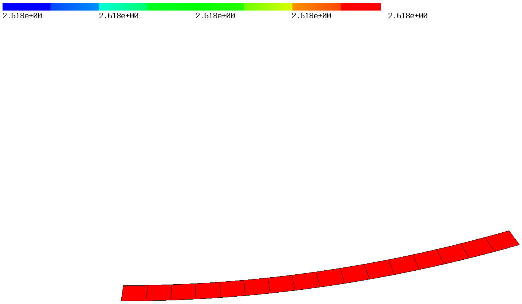


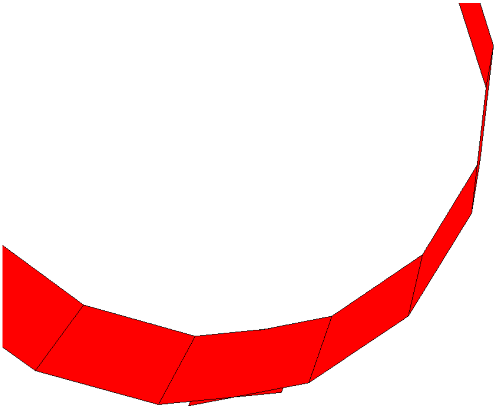
Numerical examples

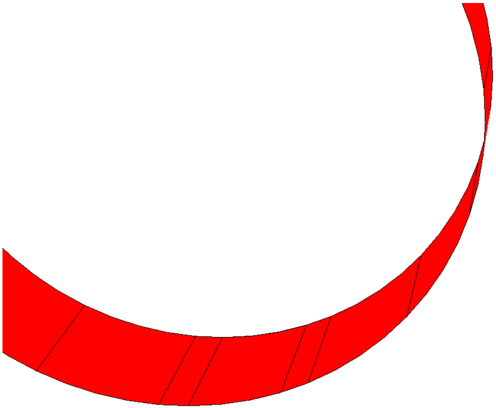
Cantilever subjected to end moment

NGSolve

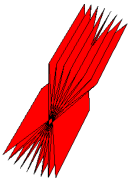


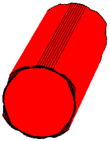




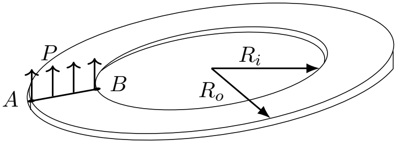


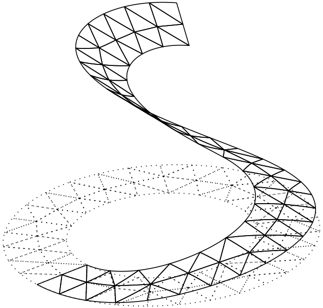
Cantilever subjected to end moment

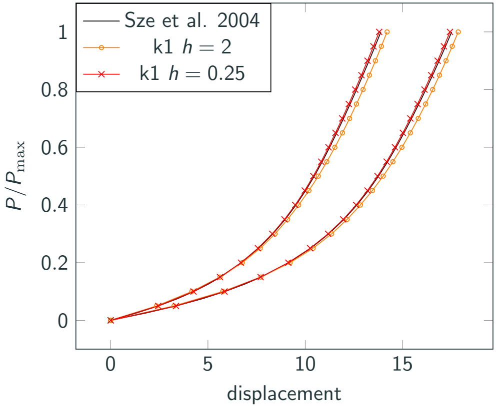




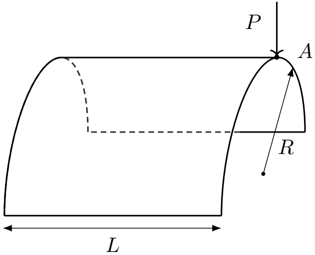
Slit annular plate

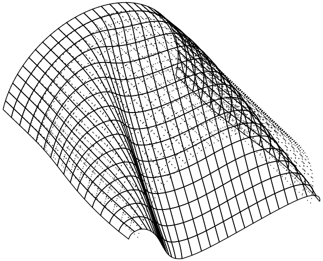


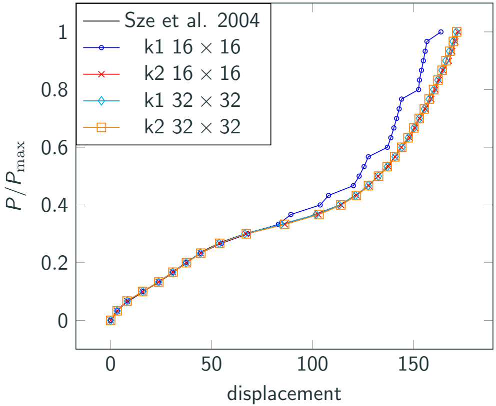




Pinched cylinder







Structures with kinks and branched shells

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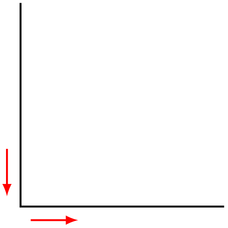
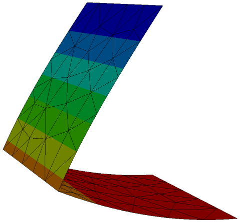
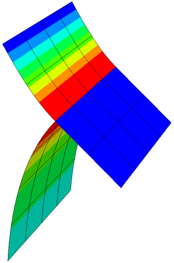
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

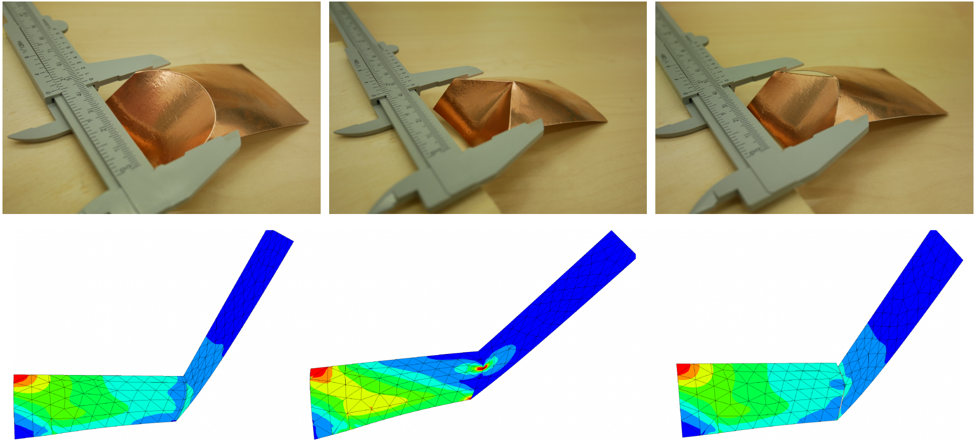


$$\int_E (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds \overset{!}{=} 0 \quad \Rightarrow \quad \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$





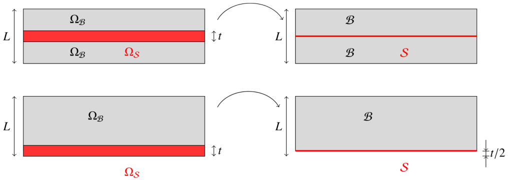
Isometry constraint $\boldsymbol{C} = \boldsymbol{I} \Rightarrow \boldsymbol{E} = 0$, i.e., zero membrane energy

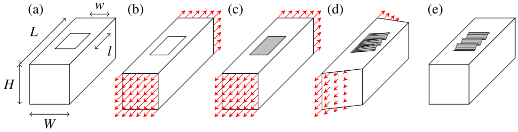


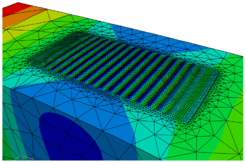
Bartels, Bonito, Hornung, N., Babuška's paradox in a nonlinear bending model, *arXiv:2503.17190*

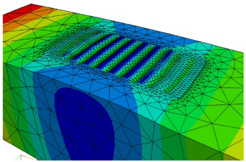
Portland State UNIVERSITY

- Composite materials, blood vessels, etc.
- Lagrange elements for elasticity and shell displacement $\rightarrow$ easy to couple









Pechstein, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)

Summary

- Locking-free shell elements
- Combine mixed FEM & differential geometry
- Generalized curvature
- Potential to couple shell models with different equations

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- Combine mixed FEM & differential geometry
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Thank You for Your attention!

Literature

- GOPALAKRISHNAN, N.: Analysis of generalized shape operator on surfaces *(in preparation)*
- BARTELS, BONITO, HORNUNG, N., Babuška's paradox in a nonlinear bending model, *arXiv:2503.17190*
- PECHSTEIN, N., Direct coupling of continuum and shell elements in large deformation problems, *Comput. Methods Appl. Mech. Engrg.* (2025)
- N., SCHÖBERL: The Hellan–Herrmann–Johnson and TDNNS methods for linear and nonlinear shells, *Comput. Struct.* (2024)
- N., SCHÖBERL, STURM, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *J. Comput. Phys.* (2023)
- N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*
- N., SCHÖBERL: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg.* (2021).
- N., SCHÖBERL: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct.* (2019)