Mixed Finite Element Methods for Nonlinear Elasticity and Shells

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FWF Der Wissenschaftsfonds.

Nonstandard Finite Element Methods, Oberwolfach, 13th January 2021

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Nonlinear elasticity

Formulation

Find $\boldsymbol{\sigma} \in H(\text{div div}, \Omega)$ and $u \in H(\text{curl}, \Omega)$ s.t.

$$\begin{aligned} \int_\Omega \mathbb{C}^{-1}\boldsymbol{\sigma} : \delta\boldsymbol{\sigma}\, dx + \langle \operatorname{div}(\delta\boldsymbol{\sigma}), u\rangle &= 0 && \forall \delta\boldsymbol{\sigma} \\ \langle \operatorname{div}(\boldsymbol{\sigma}), u\rangle &= -\int_\Omega f \cdot \delta u\, dx && \forall \delta u \end{aligned}$$

$$\begin{aligned} \langle \operatorname{div}(\boldsymbol{\sigma}), u\rangle &:= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}(\boldsymbol{\sigma}) \cdot u\, dx - \sum_{E\in\mathcal{E}_h} \int_E [\![\boldsymbol{\sigma}_{nt}]\!] u_t\, ds \\ &= \sum_{T\in\mathcal{T}_h} -\int_T \boldsymbol{\sigma} : \nabla u\, dx + \sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\sigma}_{nn} [\![u_n]\!]\, ds \end{aligned}$$

A. Pechstein, J. Schöberl: Tangential-displacement and normal-normal-stress continuous mixed finite elements for elasticity *(2011).*

Formulation

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- Works for linear material law $\mathbb{C}$
- Problem for nonlinear (not invertible) material law

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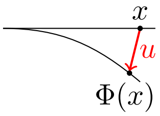
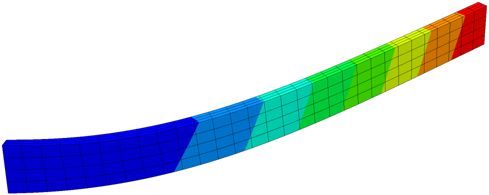
- Works for linear material law $\mathbb{C}$
- Problem for nonlinear (not invertible) material law
- Hu–Washizu principle

Notation: nonlinear elasticity

Displacement u

Deformation gradient $\quad \boldsymbol{F} := \boldsymbol{I} + \nabla u$

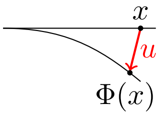


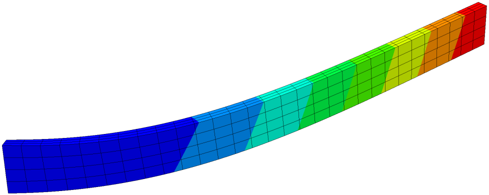


Notation: nonlinear elasticity

Displacement u

- Deformation gradient $\boldsymbol{F} := \boldsymbol{I} + \nabla u$
- Cauchy-Green strain tensor $\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$
- Green strain tensor $\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$





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Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}$
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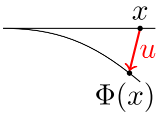
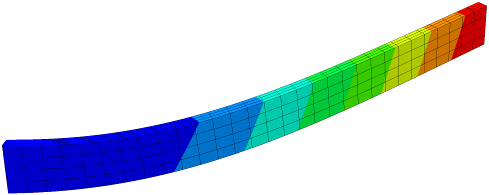
Energy density	$\mathcal{W}(\boldsymbol{C}) : \mathbb{R}^{3\times 3} \to \mathbb{R}$
Stress tensor	$\boldsymbol{\Sigma} := 2\frac{\partial \mathcal{W}}{\partial \boldsymbol{C}}$



$$\int_\Omega \mathcal{W}(\boldsymbol{C}(u)) - f \cdot u \, dx \to \text{min!}$$



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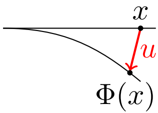
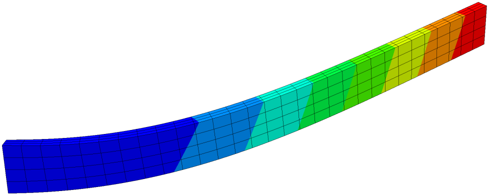


$$-\mathrm{div}(\boldsymbol{F}\boldsymbol{\Sigma}) = f \qquad \text{in } \Omega \qquad + \text{bc}$$



Notation: nonlinear elasticity

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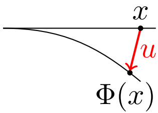
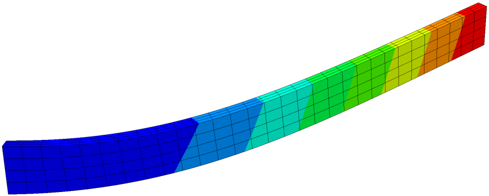


$$-\mathrm{div}(\boldsymbol{P}) = f \qquad \text{in } \Omega \qquad + \text{bc}$$



Hu–Washizu principle

$$\min_{u \in V_h} \int_\Omega \mathcal{W}(\boldsymbol{F}(u)) - f \cdot u \, dx$$

Hu–Washizu principle

$$\min_{\substack{u \in V_h \\ \boldsymbol{F} = \boldsymbol{F}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{F}) - f \cdot u \, dx$$

Hu–Washizu principle

$$\min_{\substack{u \in V_h \\ \boldsymbol{F} = \boldsymbol{F}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{F}) - f \cdot u \, dx$$

$$\mathcal{L}(u, \boldsymbol{F}, \boldsymbol{P}) = \int_\Omega \mathcal{W}(\boldsymbol{F}) - f \cdot u \, dx - \langle \boldsymbol{F} - (\nabla u + \boldsymbol{I}), \boldsymbol{P} \rangle$$

Hu–Washizu principle

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$$\min_{\substack{u\in V_h\\ \boldsymbol{F}=\boldsymbol{F}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{F}) - f\cdot u\,dx$$

$$\mathcal{L}(u,\boldsymbol{F},\boldsymbol{P}) = \int_\Omega \mathcal{W}(\boldsymbol{F}) - f\cdot u\,dx - \langle \boldsymbol{F} - (\nabla u + \boldsymbol{I}), \boldsymbol{P}\rangle$$

- Lifting distribution ∇u to $\boldsymbol{F}\in[L^2]^{3\times 3}$
- 1^{st} Piola–Kirchhoff stress tensor $\boldsymbol{P}\,(=\boldsymbol{F}\boldsymbol{\Sigma})$ as Lagrange multiplier
- $\boldsymbol{P} = \boldsymbol{P}_{\text{sym}} + \boldsymbol{P}_{\text{skew}}, \quad \boldsymbol{P}_{\text{sym}}\in H(\text{div div}),\ \boldsymbol{P}_{\text{skew}}\in[L^2]^{3\times 3}_{\text{skew}}$
- $\boldsymbol{F} = \boldsymbol{F}_{\text{sym}} + \boldsymbol{F}_{\text{skew}},\ \boldsymbol{F}_{\text{sym}}\in H(\text{curl curl})^{\mathrm{dc}},\ \boldsymbol{F}_{\text{skew}}\in[L^2]^{3\times 3}_{\text{skew}}$

Idea: Instead of

$$\min_{\substack{u \in V_h \\ \boldsymbol{F} = \boldsymbol{F}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{F}) - f \cdot u \, dx$$

Second method

Idea: Use $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$

$$\min_{\substack{u \in V_h \\ \boldsymbol{C} = \boldsymbol{C}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{C}) - f \cdot u \, dx$$

Second method

Idea: Use $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$

$$\min_{\substack{u \in V_h \\ \boldsymbol{C} = \boldsymbol{C}(u)}} \int_\Omega \mathcal{W}(\boldsymbol{C}) - f \cdot u \, dx$$

$$\mathcal{L}(u, \boldsymbol{C}, \boldsymbol{\Sigma}, \quad) = \int_\Omega \mathcal{W}(\boldsymbol{C}) - f \cdot u \, dx + \frac{1}{2} \langle \boldsymbol{C} - (\nabla u + \boldsymbol{I})^\top \underbrace{(\nabla u + \boldsymbol{I})}_{=:\boldsymbol{F}(\boldsymbol{u})}, \boldsymbol{\Sigma} \rangle$$

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- $u \in H(\mathrm{curl}) \rightarrow \nabla u$ is a distribution
- How to define $\nabla u^\top \nabla u$?

Idea: Use $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$

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$$+ \sum_{T \in \mathcal{T}} \int_T \frac{1}{2} (\boldsymbol{C} - \boldsymbol{C}(u)) : \boldsymbol{\Sigma} \, dx - \int_{\partial T} (\boldsymbol{F}(u) \boldsymbol{\Sigma})_{nn} (u - \hat{u})_n ds$$

- $u \in H(\mathrm{curl}) \rightarrow \nabla u$ is a distribution
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(Nearly) Incompressible case

Incompressibility $1 \stackrel{!}{=} J := \det(\boldsymbol{F}) = \det(\boldsymbol{I} + \nabla u)$, $\lambda \gg 1$

$$\min_{\substack{u \in V_h \\ J = J(u)}} \int_\Omega \widetilde{\mathcal{W}}(u) + \frac{\lambda}{2}(1 - J)^2 - f \cdot u \, dx$$

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$$\mathcal{L}(u, \hat{u}, J, \theta) = \int_\Omega \widetilde{\mathcal{W}}(u) + \frac{\lambda}{2}(1 - J)^2 - f \cdot u \, dx + \langle J - \det(\boldsymbol{I} + \nabla u), \theta \rangle$$

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- $\theta \in L^2(\Omega)$ is the conjugate stress to J
- $\frac{\partial J(u)}{\partial u}[\delta u] = J(u)\boldsymbol{F}^{-\top}(u) : \nabla \delta u = \operatorname{cof}(\boldsymbol{F}(u)) : \nabla \delta u$

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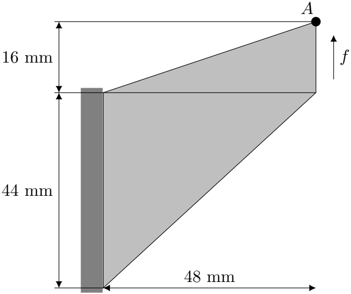
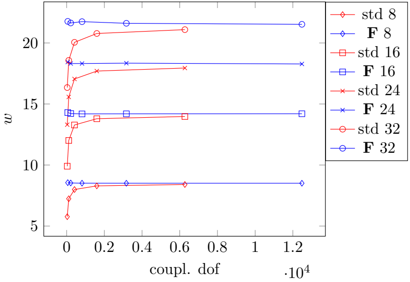
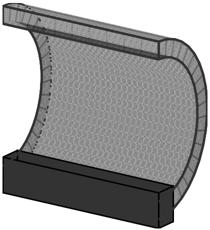
Cook's Membrane

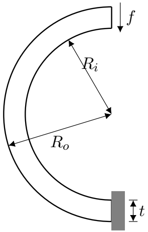
NGSolve

www.gitlab.com/mneunteufel/nonlinear_elasticity





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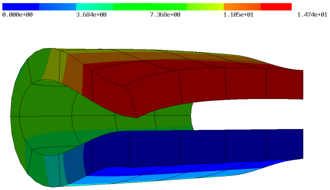
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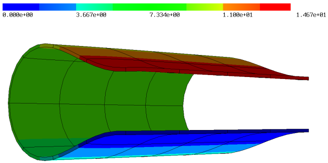
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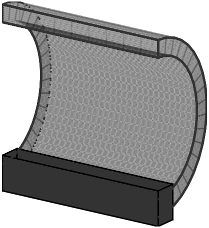
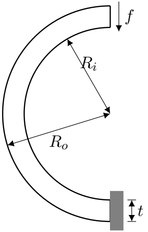
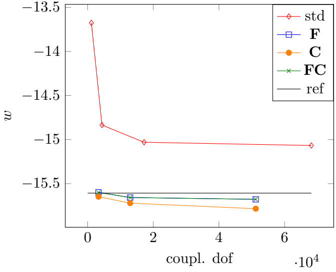
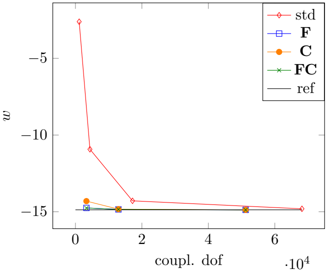
Cylindrical Shell









Nonlinear shells

Shell energy

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

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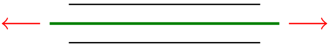

- Membrane energy

Shell energy

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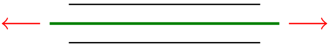
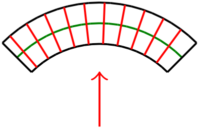
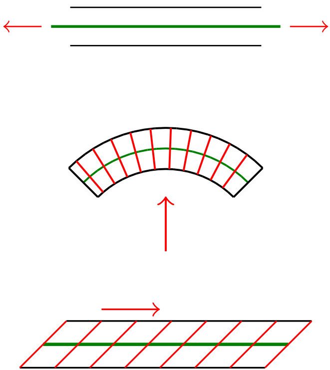




- Membrane energy
- Bending energy

$$
\begin{aligned}
\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} \\
&+ \frac{t^3}{24}\|\mathrm{sym}(\boldsymbol{F}^T\nabla\tilde{\nu}\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} \\
&+ \frac{tG\kappa}{2}\|\boldsymbol{F}^T\tilde{\nu}\circ\phi\|^2
\end{aligned}
$$



- Membrane energy
- Bending energy
- Shearing energy

Moment tensor

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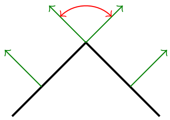

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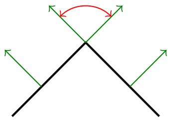
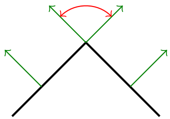


- Measure change of angles

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu}, \boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R), \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

Moment tensor

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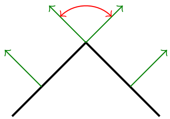
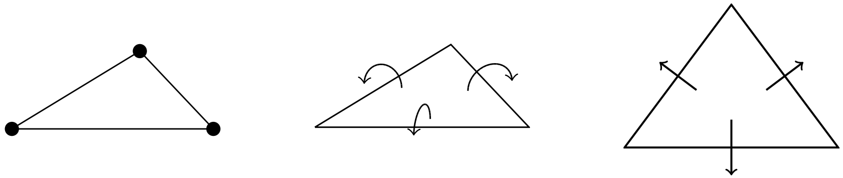


- Measure change of angles

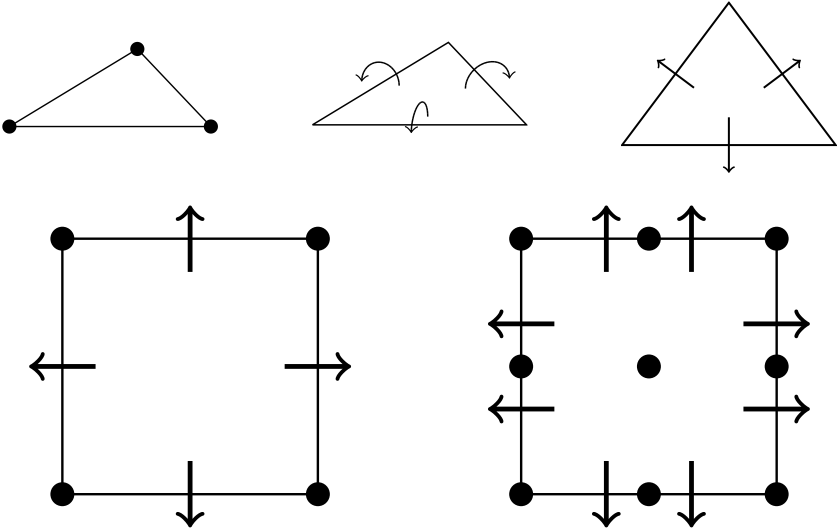
$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle\boldsymbol{F}^\top\nabla\nu - \nabla\hat{\nu},\boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R),\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\rightarrow$ second order problem
- Hybridization possible

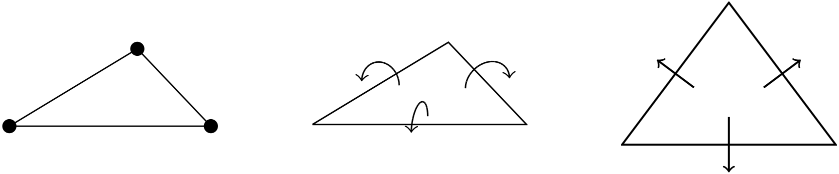
Shell element



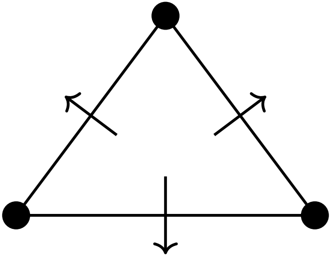
Shell element



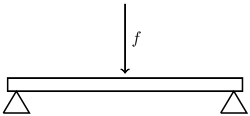
Shell element



Morley triangle:



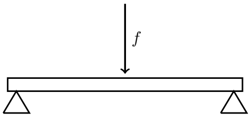
$$\operatorname{div}(\operatorname{div}(\nabla^2 w)) = f$$



$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, &&\Rightarrow w \in H^1(\Omega) \\ \operatorname{div}(\operatorname{div}(\boldsymbol{\sigma})) &= f, &&\Rightarrow \boldsymbol{\sigma} \in H(\mathrm{div}\ \mathrm{div}, \Omega) \end{aligned}$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17–29.*

$$\mathrm{div}(\mathrm{div}(\nabla^2 w)) = f$$

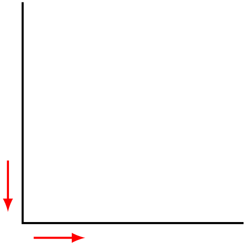


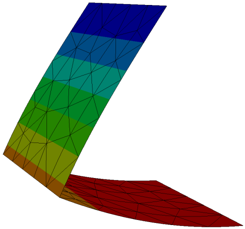
$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 w, \quad && \Rightarrow w \in H^1(\Omega) \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, \quad && \Rightarrow \boldsymbol{\sigma} \in H(\operatorname{div}\operatorname{div}, \Omega) \end{aligned}$$

Linearization

If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ method is the linearization of the bending energy of our method.

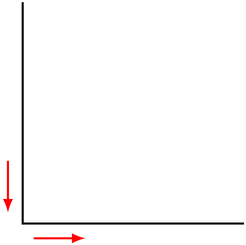
Structures with kinks

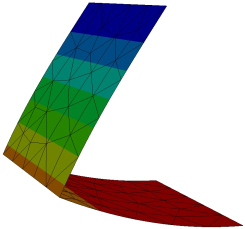




Structures with kinks

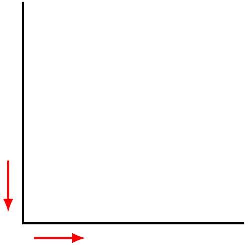
- Normal-normal continuous moment $\boldsymbol{\sigma}$

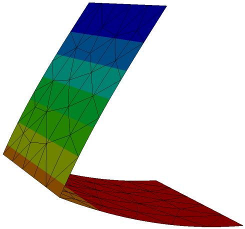




Structures with kinks

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks



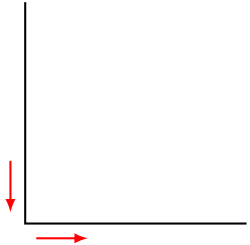


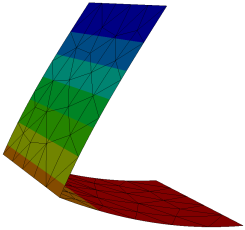
Structures with kinks

- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

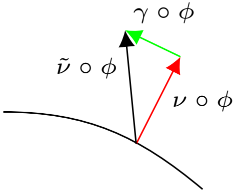
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s} \overset{!}{=} 0$$

$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$



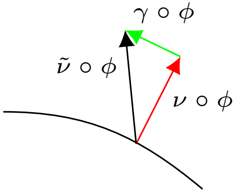


- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu} \circ \phi = \nu \circ \phi + \gamma \circ \phi$



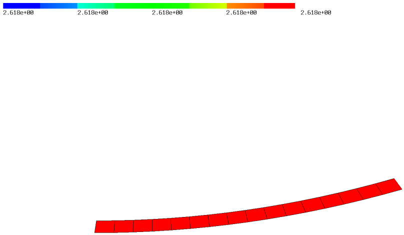
Echter, R. and Oesterle, B. and Bischoff, M.: A hierarchic family of isogeometric shell finite elements, *CMAME* (2013) 254, pp. 170–180.

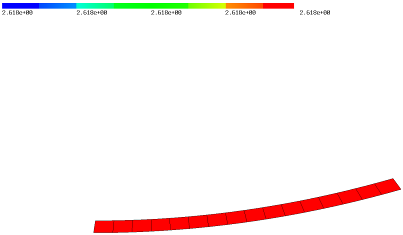
- Use hierarchical shell model
- Additional shearing dofs γ in $H(\mathrm{curl})$
- $\tilde{\nu}\circ\phi = \nu\circ\phi + \gamma\circ\phi$
- Linearization recovers TDNNS method for Reissner–Mindlin plates



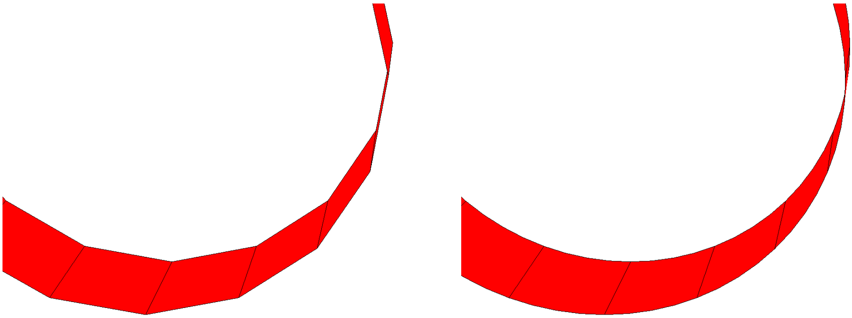
A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math.* (2017) 137, pp. 713–740.

Cantilever subjected to end moment

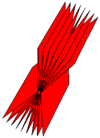


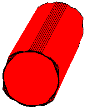


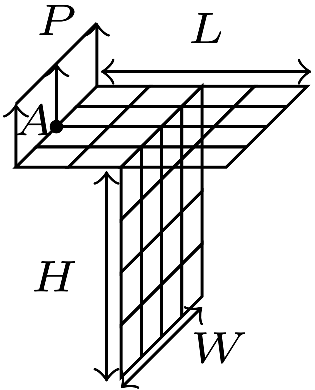
Cantilever subjected to end moment



Cantilever subjected to end moment



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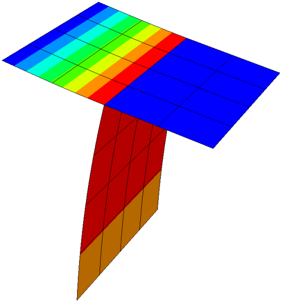

Membrane locking

Membrane locking

Membrane locking

Membrane locking

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} - f\cdot u$$

Membrane locking

TECHNISCHE UNIVERSITÄT WIEN

$$\mathcal{W}(u) = t\, E_{\mathrm{mem}}(u) + t^3\, E_{\mathrm{bend}}(u) - f \cdot u$$

Membrane locking

$$\mathcal{W}(u) = \frac{1}{t^2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u$$

- Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

Membrane locking

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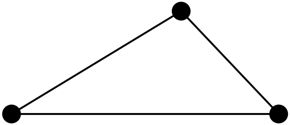

$V_h = \Pi(\mathcal{T}_h) \cap C(\Omega) \subset H^1(\Omega)$

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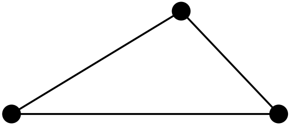
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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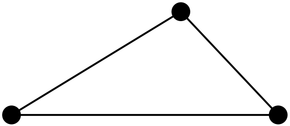

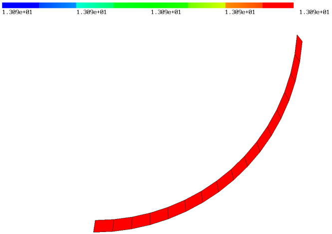

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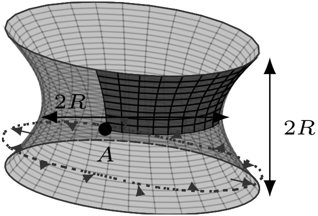

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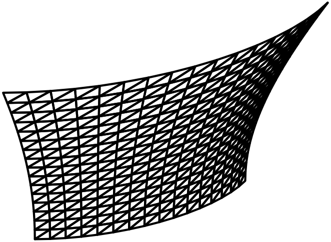

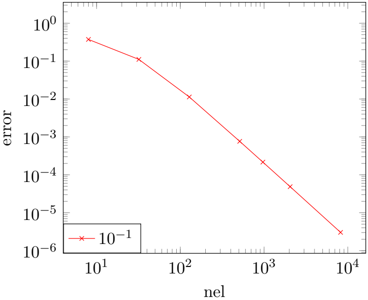

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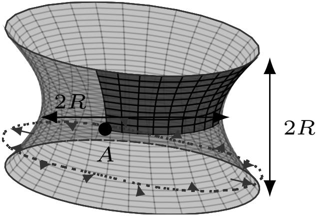
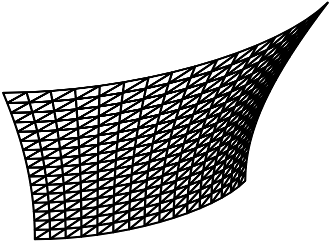
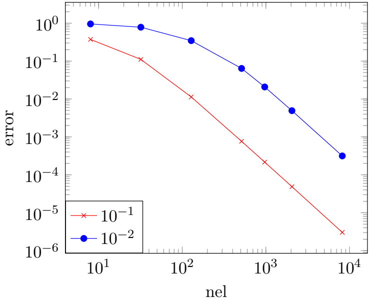
Hyperboloid with free ends

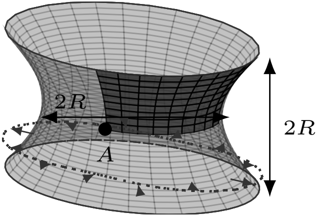


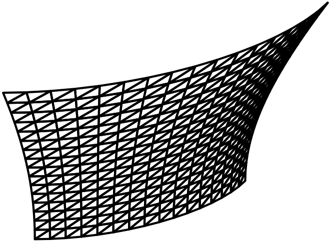


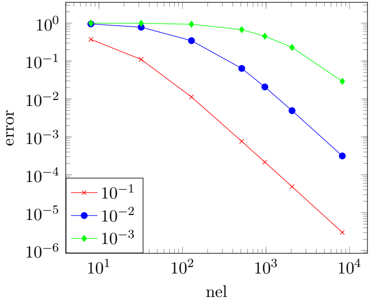


Pre-asymptotic regime

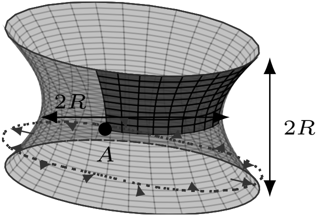
Hyperboloid with free ends

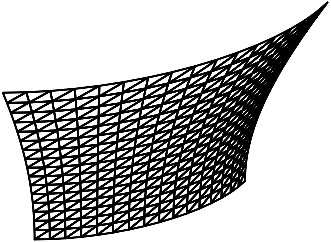


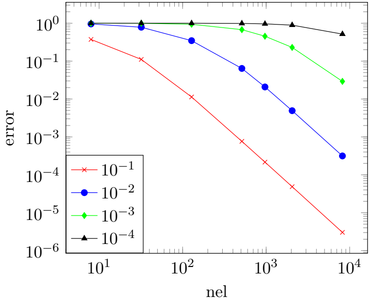




Pre-asymptotic regime

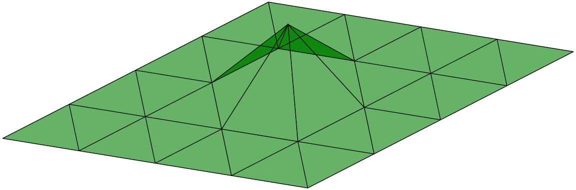




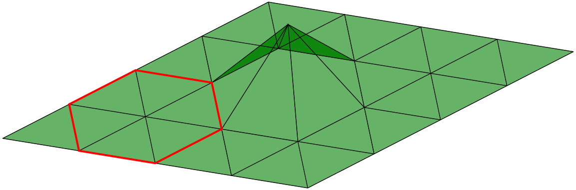


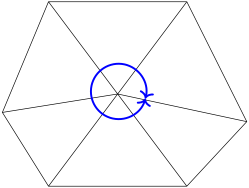
Pre-asymptotic regime

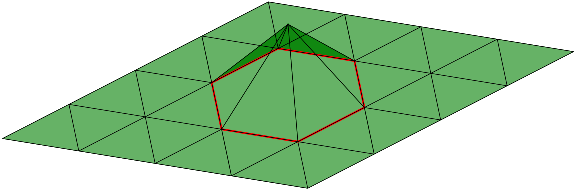
Regge calculus

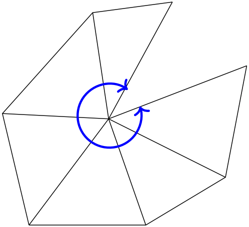


Regge calculus

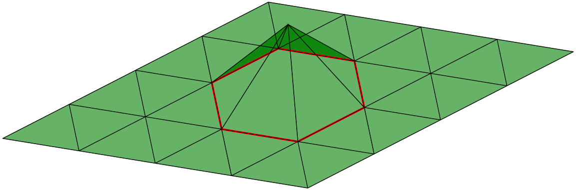


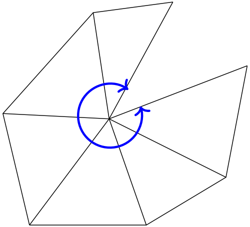




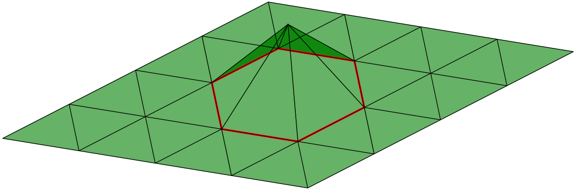


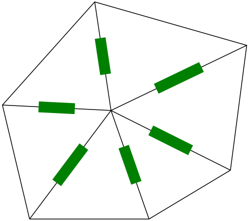
Regge calculus





T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*.

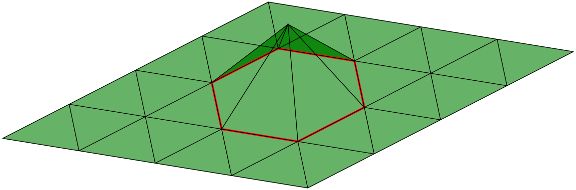


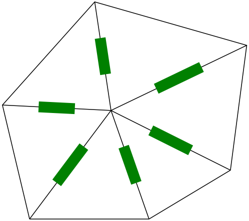


- Metric tensor

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*.

Regge calculus





- Metric tensor
- tangential-tangential continuous

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571.*

$$\mathrm{Reg}_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\mathcal{T}_h)]^{d\times d}_{sym} \mid t^T \boldsymbol{\sigma}\, t \text{ is continuous over elements}\}$$

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

Regge elements

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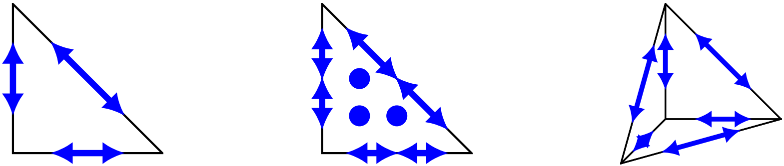

L. Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

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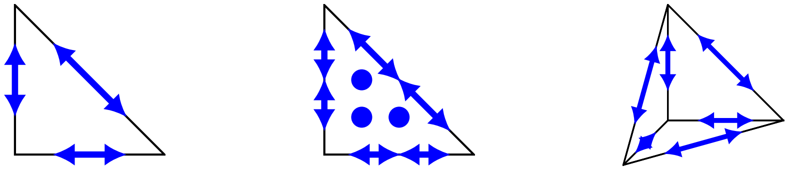
$$H(\text{curl curl}) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{d\times d}_{sym} \mid \text{curl }(\text{curl }\boldsymbol{\sigma})^T \in [H^{-1}(\Omega)]^{2d-3\times 2d-3}\}$$



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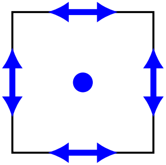
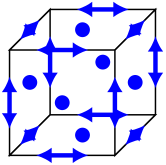
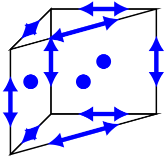




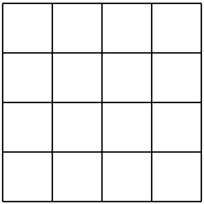


Membrane locking

$$\frac{1}{t^2}\|\boldsymbol{E}_{\tau\tau}(u_h)\|^2_{\boldsymbol{M}}$$

Membrane locking

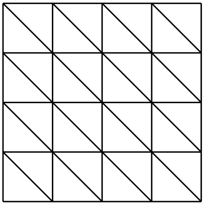
$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}_{\tau\tau}(u_h)\|^2_{\boldsymbol{M}}$$



- Reduced integration for quadrilateral meshes

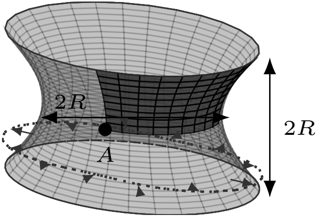
Membrane locking

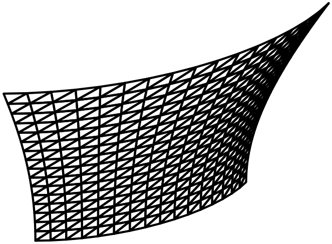
$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^k \boldsymbol{E}_{\tau\tau}(u_h)\|_{\boldsymbol{M}}^2$$

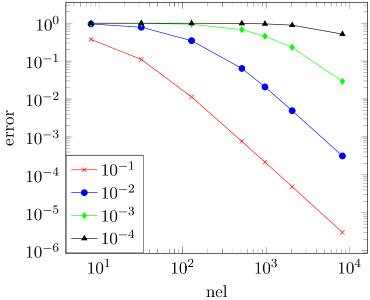


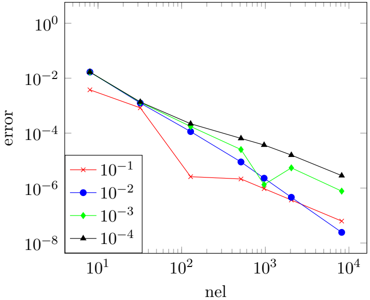
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

Hyperboloid with free ends

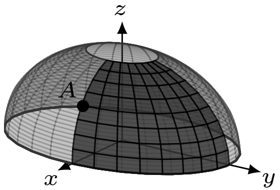


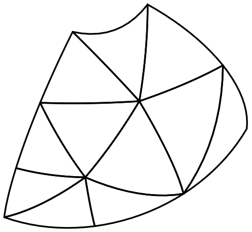


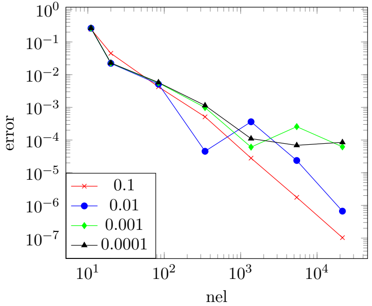


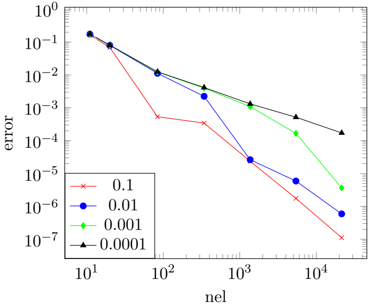


Open hemisphere with clamped ends









Summary

- Lifting of distributions for nonlinear elasticity
- Hellan–Herrmann–Johnson method for nonlinear Koiter/Naghdi shells
- Regge interpolation avoids membrane locking

N., Pechstein, Schöberl: Three-field mixed finite element methods for nonlinear elasticity *arxiv.org/abs/2009.03928*

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *C&S 225 (2019)*

N., Schöberl: Avoiding membrane locking with Regge interpolation, *CMAME 373 (2021)*

N., Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien*