Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics

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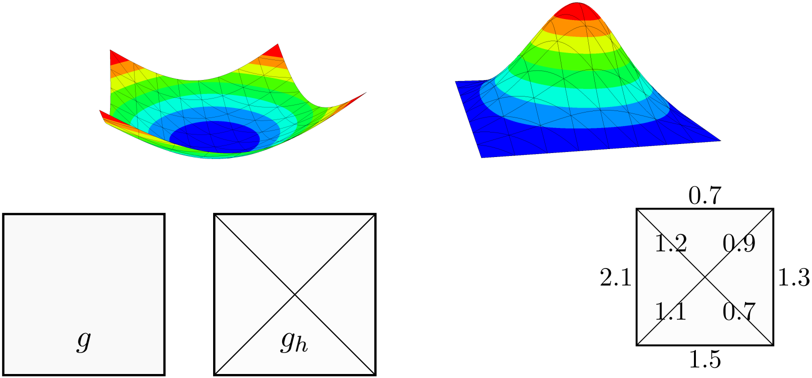
Oberwolfach, June 21st, 2022

Gauss curvature of approximated metric $\|K_h(g_h) - K(g)\|_? \leq ?$



Contents

Differential Geometry

Curvature

Riemannian manifold (M, g)

Riemannian manifold $(M \subset \mathbb{R}^2, g)$

Curvature

Riemannian manifold (M, g)

Levi-Civita connection ∇

- Riemann curvature tensor:

$$\mathfrak{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

$$R(X, Y, Z, W) = g(\mathfrak{R}(X, Y)Z, W)$$

Curvature

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$$R_{ijkl} = \partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma^p_{ik}\Gamma_{jpl} - \Gamma^p_{jk}\Gamma_{ipl}$$

- Christoffel symbols: $\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l$

$$\Gamma^k_{ij}(g) = g^{kl}\frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) = g^{kl}\Gamma_{ijl}$$

Curvature

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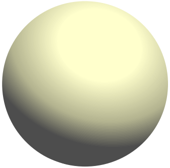
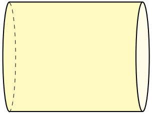
- Connection 1-form: $\varpi(X) = g(E_1, \nabla_X E_2) = -g(\nabla_X E_1, E_2)$

Gauss curvature:

$$K(g) = \frac{g(R(u,v)v,u)}{\|u\|_g\|v\|_g - g(u,v)^2} = \frac{R_{1221}}{\det g}$$

$$d^1\varpi = \sqrt{\det g}\,K(g)\,dx^1 \wedge dx^2$$





Curvature

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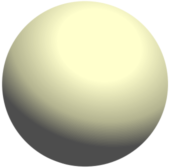
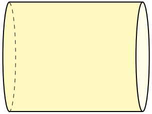
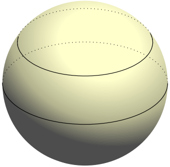




Geodesic curvature:

$$\kappa(g) = g(\nabla_{\hat{t}}\hat{t}, \hat{n}) = \frac{\sqrt{\det g}}{g_{tt}^{3/2}} (\partial_t t \cdot n + \Gamma^n_{tt})$$

$$\hat{n} = \hat{t} \times \hat{\nu}$$



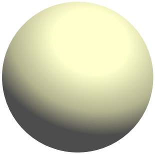
Gauss–Bonnet theorem

Gauss–Bonnet

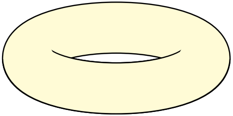
On manifold M:

$$\int_M K(g) + \int_{\partial M} \kappa(g) + \sum_V (\pi - \sphericalangle_V^M(g)) = 2\pi\chi_M$$

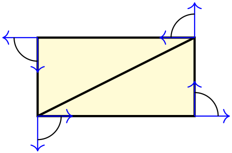
$$\chi_M(\mathscr{T}) = n_V - n_E + n_T$$



$\chi_M = 2$



$\chi_M = 0$



$\chi_M = 1$

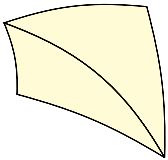
Gauss–Bonnet theorem

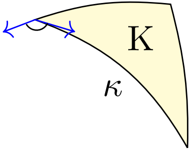
Gauss–Bonnet

On triangle T:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle^T_{V_i}(g)) = 2\pi$$

$$\chi_T = 3 - 3 + 1 = 1$$





Gauss–Bonnet

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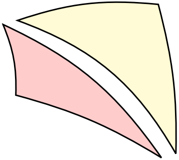

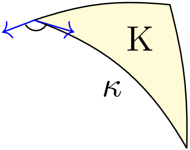

Curvature operator and analysis

Distributional Gauss curvature operator

Lifted distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k(\mathscr{T})$ find $K_h(g) \in \mathring{\mathcal{V}}_h^{k+1}$ s.t. for all $\varphi \in \mathring{\mathcal{V}}_h^{k+1}$

$$\int_{\mathscr{T}} K_h(g)\varphi = \sum_{T\in\mathscr{T}} \Big(K^T(\varphi,g) + \sum_{E\in\mathscr{E}_T} K_E^T(\varphi,g)\Big) + \sum_{V\in\mathscr{V}} K_V(\varphi,g)$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv:2111.02512* (2021).

Lifted distributional Gauss curvature

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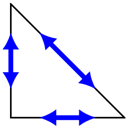
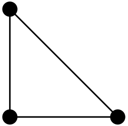


$$K^T(\varphi,g) = \int_T K(g)\,\varphi$$



$$K_E^T(\varphi,g) = \int_E \kappa(g)\varphi$$

$$K_V(\varphi,g) = \big(2\pi - \sum_{T:V\subset T} \sphericalangle_V^T(g)\big)\varphi(V)$$

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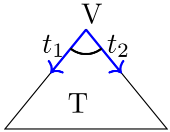

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$$K_V(\varphi, g) = \big(2\pi - \sum_{T:V\subset T} \sphericalangle_V^T(g)\big)\varphi(V)$$

$$\sphericalangle_V^T(g) = \arccos\left(\frac{t_1^\top g t_2}{\|t_1\|_g \|t_2\|_g}\right)$$

Distributional Gauss curvature operator

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Lifted distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k(\mathscr{T})$ find $K_h(g) \in \mathring{\mathcal{V}}_h^{k+1}$ s.t. for all $\varphi \in \mathring{\mathcal{V}}_h^{k+1}$

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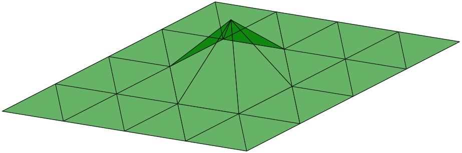
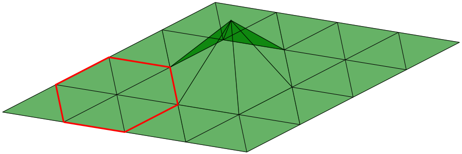
$$\int_{\mathscr{T}} K_h(g)\,\varphi\sqrt{\det g}\,da = \sum_{T\in\mathscr{T}} \Big(\int_T K(g)\,\varphi\,\sqrt{\det g}\,da + \int_{\partial T} \frac{\sqrt{\det g}}{g_{tt}}(\partial_t t\cdot n + \Gamma_{tt}^n)\,\varphi\,dl\Big) + \sum_{V\in\mathscr{V}} K_V(\varphi, g)$$

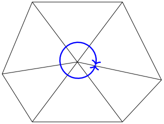
Regge elements



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

Regge elements





Regge: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

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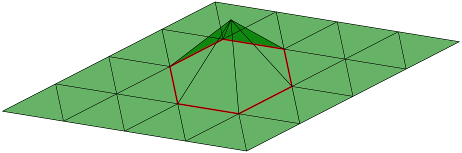

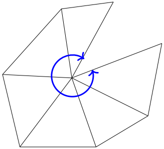

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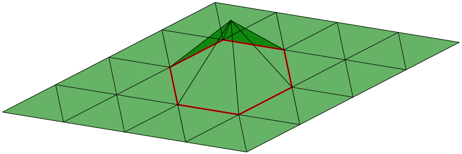
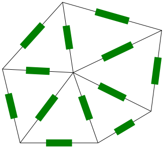
Regge elements



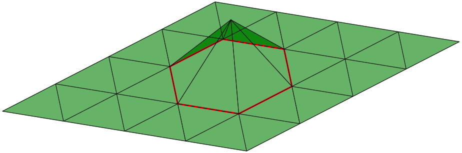


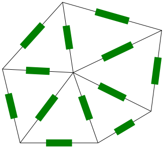
- metric tensor

REGGE: General relativity without coordinates, *Il Nuovo Cimento (1955-1965)*, 19 (1961).

SORKIN: Time-evolution problem in Regge calculus, *Phys. Rev. D 12* (1975).

Regge elements





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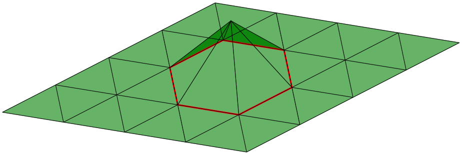
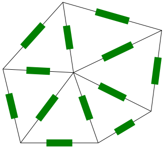
REGGE: General relativity without coordinates, *Il Nuovo Cimento (1955-1965)*, 19 (1961).

CHEEGER, MÜLLER, SCHRADER: On the curvature of piecewise flat spaces, *Communications in Mathematical Physics, 92(3) (1984).*

Regge elements



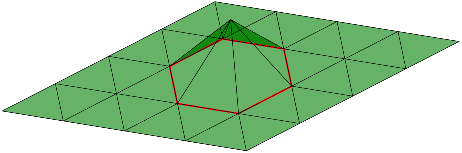


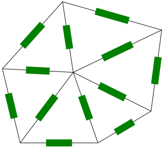
- metric tensor (tangential-tangential continuous)

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) = \{\varepsilon \in L^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid \mathrm{curl}^\top \mathrm{curl}(\varepsilon) \in H^{-1}(\Omega, \mathbb{R}^{(2d-3)\times(2d-3)})\}$$

CHRISTIANSEN: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*





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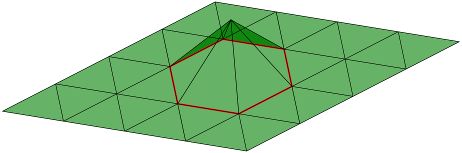
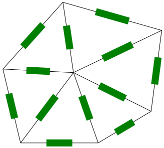
Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*





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Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011).*

N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*

First convergence result

$$\|K_h(g_h) - K(g)\|_? \leq h^?$$

Convergence

Let $g_h \in \mathrm{Reg}_h^k$ by the Regge interpolant of a smooth g. Then for sufficiently small h

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^k \|g\|_{H^{k+1}}.$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv:2111.02512* (2021).

First convergence result

$$\|K_h(g_h) - K(g)\|_? \leq h^?$$

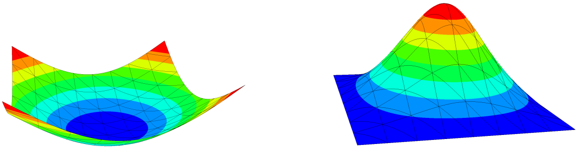
Convergence

Let $g_h \in \mathrm{Reg}_h^0$ by the Regge interpolant of a smooth g. Then for sufficiently small h

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^0 \|g\|_{H^1} \quad .$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv:2111.02512* (2021).

Numerical example

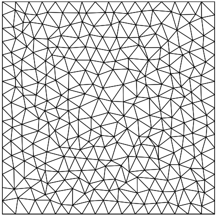


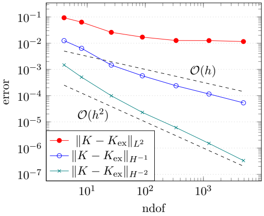
$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \qquad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

$$K(g) = \frac{81(1 - x^2)(1 - y^2)}{(9 + x^2(x^2 - 3)^2 + y^2(y^2 - 3)^2)^2}$$

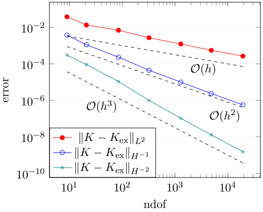
Numerical example (Gauss curvature)

NGSolve

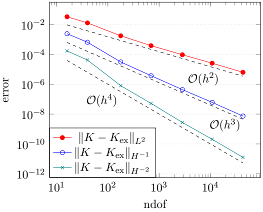




$k = 0$

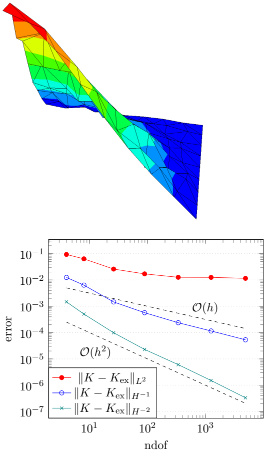


$k = 1$

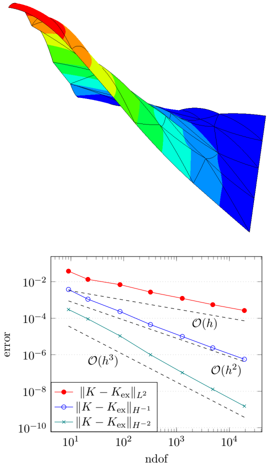


$k = 2$

Numerical example (Gauss curvature)



$k = 0$



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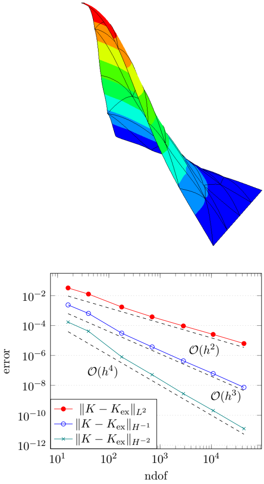
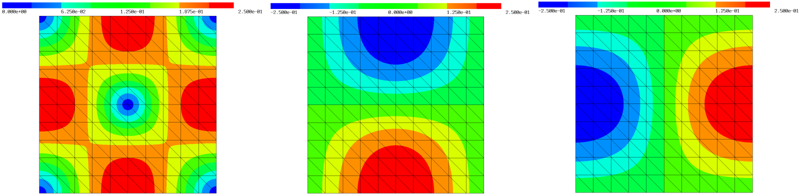
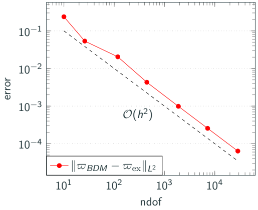
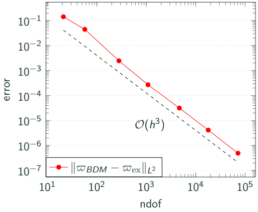


$k = 2$

Numerical example (connection 1-form)

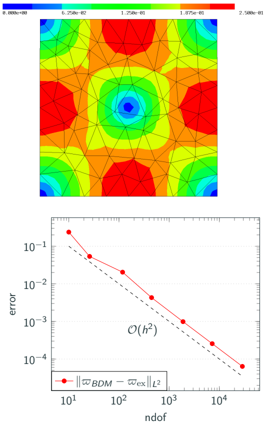




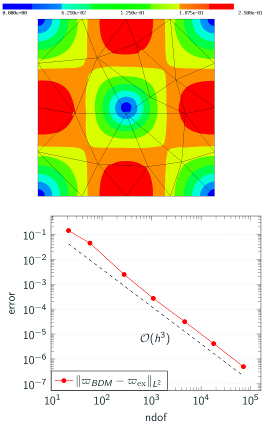


$k = 1$ $\qquad$ $k = 2$

Numerical example (connection 1-form)



$k = 1$



$k = 2$

Integral representation (Gauss curvature)

$$\int_{\mathscr{T}} K_h(g)u_h = \sum_{T\in\mathscr{T}}\Big(K^T(u_h,g)+\sum_{E\in\mathscr{E}_T} K_E^T(u_h,g)\Big)+\sum_{V\in\mathscr{V}} K_V(u_h,g)$$

- Consistency: For $g \in C^2(M,\mathcal{S})$, $u_h \in \mathring{\mathcal{V}}_h^{k+1}$ there holds $\int_{\mathscr{T}} K_h(g)\, u_h = \int_{\mathscr{T}} K(g) u_h$

$$\int_{\mathscr{T}} K_h(g_h)u_h = \frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)}\mathrm{div}_{G_h(t)} S_{G_h(t)}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

Integral representation (Gauss curvature)

$$\int_{\mathscr{T}} K_h(g)u_h = \sum_{T\in\mathscr{T}} \Big(K^T(u_h,g)+\sum_{E\in\mathscr{E}_T} K_E^T(u_h,g)\Big)+\sum_{V\in\mathscr{V}} K_V(u_h,g)$$

- Consistency: For $g \in C^2(M,\mathcal{S})$, $u_h \in \mathring{\mathcal{V}}_h^{k+1}$ there holds $\int_{\mathscr{T}} K_h(g)\, u_h = \int_{\mathscr{T}} K(g)u_h$

$$\int_{\mathscr{T}} K_h(g_h)u_h = \frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)}\mathrm{div}_{G_h(t)} S_{G_h(t)}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

Representation with covariant incompatibility operator

$$\int_{\mathscr{T}} K_h(g_h)u_h = -\frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h(t)}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

Integral representation (connection 1-form)

Find $\star\varpi_h(g_h) \in \mathring{\mathcal{N}}_{II}^k$ such that for all $v_h \in \mathring{\mathcal{N}}_{II}^k$

$$\int_{\mathscr{T}} \star\varpi_h(g_h) v_h = -\frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)} S_{G_h(t)} \sigma_h, v_h\rangle_{\mathring{\mathcal{N}}_{II}}\, dt$$

Integral representation (connection 1-form)

Find $\star\varpi_h(g_h) \in \mathring{\mathcal{N}}_{II}^k$ such that for all $v_h \in \mathring{\mathcal{N}}_{II}^k$

$$\int_{\mathscr{T}} \star\varpi_h(g_h) v_h = -\frac{1}{2}\int_0^1 \langle \operatorname{div}_{G_h(t)} S_{G_h(t)}\sigma_h, v_h\rangle_{\mathring{\mathcal{N}}_{II}}\, dt$$

$$\mathring{\mathcal{W}}_g = \{v \in C^\infty(\mathscr{T}, \mathbb{R}^2)\,|\,[\![g(v, n_g)]\!]_E = 0\}, \quad \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_\delta, \quad Q_g\mathring{\mathcal{W}} = \mathring{\mathcal{W}}_g$$

Find $\star\varpi_h(g_h) \in \mathring{\mathcal{N}}_{II}^k$ such that for all $v_h \in \mathring{\mathcal{N}}_{II}^k$

$$\int_{\mathscr{T}} \star\varpi_h(g_h) v_h = -\frac{1}{2}\int_0^1 \langle \mathrm{div}_{G_h(t)} S_{G_h(t)} \sigma_h, v_h \rangle_{\mathring{\mathcal{N}}_{II}} \, dt$$

$$\mathring{\mathcal{W}}_g = \{v \in C^\infty(\mathscr{T}, \mathbb{R}^2) \,|\, [\![g(v, n_g)]\!]_E = 0\}, \quad \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_\delta, \quad \frac{1}{\sqrt{\det g}} \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_g$$

Integral representation (connection 1-form)

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Find $\star\varpi_h(g_h) \in \mathring{\mathcal{N}}^k_{II}$ such that for all $v_h \in \mathring{\mathcal{N}}^k_{II}$

$$\int_{\mathscr{T}} \star\varpi_h(g_h) v_h = -\frac{1}{2}\int_0^1 \langle \operatorname{div}_{G_h(t)} S_{G_h(t)} \sigma_h, v_h \rangle_{\mathring{\mathcal{N}}_{II}}\, dt$$

$$\mathring{\mathcal{W}}_g = \{v \in C^\infty(\mathscr{T}, \mathbb{R}^2) \,|\, [\![g(v, n_g)]\!]_E = 0\}, \quad \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_\delta, \quad \frac{1}{\sqrt{\det g}}\mathring{\mathcal{W}} = \mathring{\mathcal{W}}_g$$

Find $\varpi_h(g_h) \in \mathring{\mathcal{W}}^k_h$ such that for all $v_h \in \mathring{\mathcal{W}}^k_h$

$$\int_{\mathscr{T}} \varpi_h(g_h) Q_g v_h = -\frac{1}{2}\int_0^1 \langle \operatorname{curl}_{G_h(t)}(\sigma_h), Q_g v_h \rangle_{\mathring{\mathcal{W}}_{G_h(t)}}\, dt$$

Integral representation (connection 1-form)

Find $\star\varpi_h(g_h) \in \mathring{\mathcal{N}}_{II}^k$ such that for all $v_h \in \mathring{\mathcal{N}}_{II}^k$

$$\int_{\mathscr{T}} \star\varpi_h(g_h) v_h = -\frac{1}{2}\int_0^1 \langle \operatorname{div}_{G_h(t)} S_{G_h(t)} \sigma_h, v_h \rangle_{\mathring{\mathcal{N}}_{II}} \, dt$$

$$\mathring{\mathcal{W}}_g = \{ v \in C^\infty(\mathscr{T}, \mathbb{R}^2) \mid [\![g(v, n_g)]\!]_E = 0 \}, \quad \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_\delta, \quad \frac{1}{\sqrt{\det g}} \mathring{\mathcal{W}} = \mathring{\mathcal{W}}_g$$

Find $\varpi_h(g_h) \in \mathring{\mathcal{W}}_h^k$ such that for all $v_h \in \mathring{\mathcal{W}}_h^k$

$$\int_{\mathscr{T}} \varpi_h(g_h) Q_g v_h = -\frac{1}{2}\int_0^1 \langle \operatorname{curl}_{G_h(t)}(\sigma_h), Q_g v_h \rangle_{\mathring{\mathcal{W}}_{G_h(t)}} \, dt$$

$$\begin{aligned}\int_{\mathscr{T}} \varpi_h(g_h) Q_g \operatorname{rot} u_h &= -\frac{1}{2}\int_0^1 \langle \operatorname{curl}_{G_h(t)}(\sigma_h), Q_g \operatorname{rot} u_h \rangle_{\mathring{\mathcal{W}}_{G_h(t)}} \, dt \\ &= -\frac{1}{2}\int_0^1 \langle \operatorname{inc}_{G_h(t)}(\sigma_h), u_h \rangle_{\mathring{\mathcal{V}}(\mathscr{T})} \, dt = \int_{\mathscr{T}} K_h(g_h) u_h\end{aligned}$$

Strategy of proof

- Goal

$$|(K_h(g_h) - K(g), u)_g| \leq C\Big(\|g - g_h\|_{L^\infty} + h\|g - g_h\|_{W_h^{1,\infty}} + h \inf_{v_h \in \mathring{\mathcal{V}}_h^{k+1}} \|K(g) - v_h\|_{L^2}\Big)\|u\|_{H^1}$$

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq C h^{k+1} \|g\|_{W^{k+1,\infty}}$$

- Goal

$$|(K_h(g_h) - K(g), u)_g| \leq C\Big(\|g - g_h\|_{L^\infty} + h\|g - g_h\|_{W_h^{1,\infty}} + h \inf_{v_h \in \mathring{\mathcal{V}}_h^{k+1}} \|K(g) - v_h\|_{L^2}\Big)\|u\|_{H^1}$$

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^{k+1}\|g\|_{W^{k+1,\infty}}$$

- $u \in H_0^1(\Omega)$, $u_h = P_{h,g}u \in \mathring{\mathcal{V}}_h^{k+1}$, $g_h = \mathcal{R}_h^k g$

$$(K_h(g_h) - K(g), u)_g = (K_h(g_h) - K(g), u - u_h + u_h)_g =$$

$$(K_h(g_h) - K(g), u - u_h)_g + (K_h(g_h) - K(g), u_h)_{g-g_h+g_h} =$$

$$(K_h(g_h), u_h)_{g_h} - (K(g), u_h)_g + (K_h(g_h) - K(g), u - u_h)_g + (K_h(g_h), u_h)_g - (K_h(g_h), u_h)_{g_h}$$

$G(t) = \delta + t(g - \delta)$, $\sigma = g - \delta$

$$(K(g), u_h)_g - (K_h(g_h), u_h)_{g_h} = \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

$$= \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} + \langle \mathrm{inc}_G(\sigma_h - \sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

$$G(t) = \delta + t(g-\delta),\ \sigma = g - \delta$$

$$(K(g), u_h)_g - (K_h(g_h), u_h)_{g_h} = \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

$$= \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} + \langle \mathrm{inc}_G(\sigma_h - \sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

$$\langle \mathrm{inc}_g(\sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} = \langle \mathrm{curl}_g(\sigma), Q_g \operatorname{rot} u_h\rangle_{\mathring{\mathcal{W}}_g}$$

Strategy of proof

$$G(t) = \delta + t(g - \delta),\ \sigma = g - \delta$$

$$(K(g), u_h)_g - (K_h(g_h), u_h)_{g_h} = \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

$$= \frac{1}{2}\int_0^1 \langle \mathrm{inc}_{G_h}(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_G(\sigma_h), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} + \langle \mathrm{inc}_G(\sigma_h - \sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})}\, dt$$

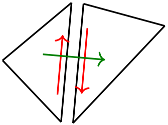
$$\langle \mathrm{inc}_g(\sigma), u_h\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} = \langle \mathrm{curl}_g(\sigma), Q_g \,\mathrm{rot}\, u_h\rangle_{\mathring{\mathcal{W}}_g}$$

$$|\langle \mathrm{curl}_{G_h}(\sigma_h), Q_{G_h} v_h\rangle_{\mathring{\mathcal{W}}_{G_h}} - \langle \mathrm{curl}_G(\sigma_h), Q_G v_h\rangle_{\mathring{\mathcal{W}}_G}| \leq C(\|G - G_h\|_{L^\infty} + h\|G - G_h\|_{W_h^{1,\infty}})\|v_h\|_{L^2}$$

$$|\langle \mathrm{curl}_G(\sigma - \sigma_h), Q_G v_h\rangle_{\mathring{\mathcal{W}}_G}| \leq C(\|\sigma - \sigma_h\|_{L^\infty} + h\|\sigma - \sigma_h\|_{W_h^{1,\infty}})\|v_h\|_{L^2}$$

Covariant distributional curl

$$(d^1\sigma_Z)(X,Y) = (\nabla_X\sigma)(Z,Y) - (\nabla_Y\sigma)(Z,X)$$
$$(\mathrm{curl}_g\,\sigma)(Z) = \star(d^1\sigma_Z), \quad \sigma\in\mathcal{T}^2_0(T), Z\in\mathfrak{X}(T)$$

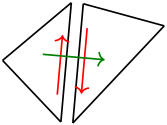


For g, $\sigma\in\mathrm{Reg}^k_h$ and $\varphi\in\mathring{\mathcal{W}}^k_h$ normal continuous the distributional covariant curl is

$$\langle\mathrm{curl}_g\,\sigma, Q_g\varphi\rangle_{\mathring{\mathcal{W}}_g} = \int_{\mathscr{T}}\frac{(\mathrm{curl}_g\,\sigma)(\varphi)}{\sqrt{\det g}} - \int_{\partial\mathscr{T}}\frac{g(\varphi,n_g)\sigma(n_g,t_g)}{\sqrt{\det g}}$$
$$= \sum_{T\in\mathscr{T}}\int_T\frac{\mathrm{curl}\,\sigma_i\varphi^i + \sigma_{ij}\varepsilon^{ik}\Gamma^j_{kl}\varphi^l}{\sqrt{\det g}}\,dx - \int_{\partial T}\frac{g_{tt}\sigma_{nt}-g_{nt}\sigma_{tt}}{\sqrt{\det g}\,g_{tt}}\varphi_n\,ds.$$

Covariant distributional curl

$$(d^1\sigma_Z)(X,Y) = (\nabla_X\sigma)(Z,Y) - (\nabla_Y\sigma)(Z,X)$$
$$(\mathrm{curl}_g\,\sigma)(Z) = \star(d^1\sigma_Z),\quad \sigma\in\mathcal{T}^2_0(T), Z\in\mathfrak{X}(T)$$



For g, $\sigma\in\mathrm{Reg}_h^k$ and $\varphi\in\mathring{\mathcal{W}}_h^k$ normal continuous the distributional covariant curl is

$$\langle\mathrm{curl}_g\,\sigma, Q_g\varphi\rangle_{\mathring{\mathcal{W}}_g} = \int_{\mathscr{T}}\frac{(\mathrm{curl}_g\,\sigma)(\varphi)}{\sqrt{\det g}} - \int_{\partial\mathscr{T}}\frac{g(\varphi,n_g)\sigma(n_g,t_g)}{\sqrt{\det g}}$$
$$= \sum_{T\in\mathscr{T}}\int_T\frac{\sigma_{mk}(\mathrm{rot}\,\varphi^{mk} - \varepsilon^{kj}(\Gamma^l_{lj}\varphi^m - \Gamma^m_{ji}\varphi^i))}{\sqrt{\det g}}\,dx + \int_{\partial T}\frac{\sigma_{tt}g_{it}\varphi^i}{\sqrt{\det g}\,g_{tt}}\,ds.$$

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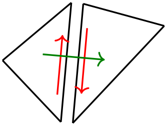
$$(d^1\sigma_Z)(X,Y) = (\nabla_X\sigma)(Z,Y) - (\nabla_Y\sigma)(Z,X)$$
$$(\mathrm{curl}_g\,\sigma)(Z) = \star(d^1\sigma_Z), \quad \sigma\in\mathcal{T}_0^2(T), Z\in\mathfrak{X}(T)$$



For g, $\sigma\in\mathrm{Reg}_h^k$ and $\varphi\in\mathring{\mathcal{W}}_h^k$ normal continuous the distributional covariant curl is

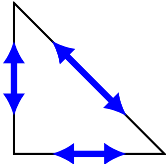
$$\langle\mathrm{curl}_g\,\sigma, Q_g\varphi\rangle_{\mathring{\mathcal{W}}_g} = \int_{\mathscr{T}}\frac{(\mathrm{curl}_g\,\sigma)(\varphi)}{\sqrt{\det g}} - \int_{\partial\mathscr{T}}\frac{g(\varphi,n_g)\sigma(n_g,t_g)}{\sqrt{\det g}}$$
$$= \sum_{T\in\mathscr{T}}\int_T\frac{\sigma_{mk}(\mathrm{rot}\,\varphi^{mk} - \varepsilon^{kj}(\Gamma_{lj}^l\varphi^m - \Gamma_{ji}^m\varphi^i))}{\sqrt{\det g}}\,dx + \int_{\partial T}\frac{\sigma_{tt}g_{it}\varphi^i}{\sqrt{\det g}\,g_{tt}}\,ds.$$

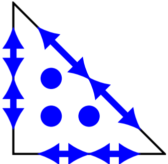
- Standard distributional curl
$$\langle\mathrm{curl}_\delta\,\sigma,\varphi\rangle_{\mathring{\mathcal{W}}} = \sum_{T\in\mathscr{T}}\int_T\mathrm{curl}\,\sigma\cdot\varphi\,da - \int_{\partial T}\sigma_{nt}\varphi_n\,dl$$
- Smooth g and σ leads to classical covariant curl

Orthogonality properties of Regge interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : q\, da = 0 \text{ for all } q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2})$$





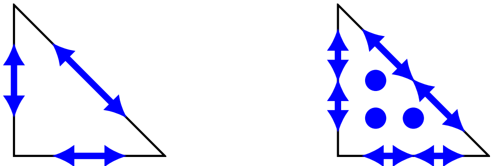
Orthogonality properties of Regge interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : q\, da = 0 \text{ for all } q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2})$$

$$\langle \Gamma_{ijk}(g - \mathcal{R}_h^k g), \Sigma_h^{ijk} \rangle = 0 \text{ for all } \Sigma_h \in \mathcal{P}^k(T, \mathbb{R}^{2\times 2\times 2})$$

$$\langle \Gamma_{ijk}(g), \Sigma^{ijk} \rangle = \sum_{T \in \mathscr{T}} \left(\int_T \Gamma_{ijk}(g) \Sigma^{ijk}\, da - \int_{\partial T} \Sigma^{nni} (g_{nt} t_i + \frac{1}{2} g_{nn} n_i)\, dl \right)$$



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Lemma

For $k \in \mathbb{N}_0$, $\sigma_h = \mathcal{R}_h^k\sigma$, $\sigma \in H^1(\Omega,\mathbb{S}) \cap C^0(\Omega,\mathbb{S})$, $v_h \in \mathring{\mathcal{W}}_h^k$, and $g \in W^{1,\infty}(\Omega,\mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega,\mathbb{S}^+)$ there holds

$$\langle \mathrm{curl}_g(\sigma-\sigma_h), Q_g v_h\rangle_{\mathring{\mathcal{W}}_g} \leq C\left(\|\sigma-\sigma_h\|_{L^2} + h|\sigma-\sigma_h|_{H_h^1}\right)\|v_h\|_{L^2(\Omega)}.$$

$$\Big|\int_E (\sigma-\sigma_h)_{tt} \qquad v_h\,dl\Big| = 0$$

Error analysis: Covariant distributional curl (I)

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Lemma

For $k \in \mathbb{N}_0$, $\sigma_h = \mathcal{R}_h^k\sigma$, $\sigma \in H^1(\Omega,\mathbb{S}) \cap C^0(\Omega,\mathbb{S})$, $v_h \in \mathring{\mathcal{W}}_h^k$, and $g \in W^{1,\infty}(\Omega,\mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega,\mathbb{S}^+)$ there holds

$$\langle \mathrm{curl}_g(\sigma - \sigma_h), Q_g v_h\rangle_{\mathring{\mathcal{W}}_g} \leq C\big(\|\sigma - \sigma_h\|_{L^2} + h|\sigma - \sigma_h|_{H_h^1}\big)\|v_h\|_{L^2(\Omega)}.$$

$$\Big|\int_E (\sigma - \sigma_h)_{tt}\, F(g)\; v_h\, dl\Big|$$
$$\leq C h^{-1}(\|\sigma - \sigma_h\|_{L^2(T)} + h\|\sigma - \sigma_h\|_{H_h^1(T)})\|v_h\|_{L^2(T)}$$

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Lemma

For $k \in \mathbb{N}_0$, $\sigma_h = \mathcal{R}_h^k \sigma$, $\sigma \in H^1(\Omega, \mathbb{S}) \cap C^0(\Omega, \mathbb{S})$, $v_h \in \mathring{\mathcal{W}}_h^k$, and $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$ there holds

$$\langle \mathrm{curl}_g(\sigma - \sigma_h), Q_g v_h \rangle_{\mathring{\mathcal{W}}_g} \leq C\left(\|\sigma - \sigma_h\|_{L^2} + h|\sigma - \sigma_h|_{H_h^1}\right)\|v_h\|_{L^2(\Omega)}.$$

$$\Big| \int_E (\sigma - \sigma_h)_{tt} \left(\Pi_0 + (\mathrm{id} - \Pi_0)\right)(F(g)) v_h \, dl \Big|$$
$$\leq C \quad \left(\|\sigma - \sigma_h\|_{L^2(T)} + h\|\sigma - \sigma_h\|_{H_h^1(T)}\right)\|v_h\|_{L^2(T)}$$

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Lemma

For $k \in \mathbb{N}_0$, $\sigma_h = \mathcal{R}_h^k \sigma$, $\sigma \in H^1(\Omega, \mathbb{S}) \cap C^0(\Omega, \mathbb{S})$, $v_h \in \mathring{\mathcal{W}}_h^k$, and $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$ there holds

$$\langle \mathrm{curl}_g(\sigma - \sigma_h), Q_g v_h \rangle_{\mathring{\mathcal{W}}_g} \leq C\big(\|\sigma - \sigma_h\|_{L^2} + h|\sigma - \sigma_h|_{H_h^1}\big)\|v_h\|_{L^2(\Omega)}.$$

$$\Big|\int_E (\sigma - \sigma_h)_{tt}\,(\Pi_0 + (\mathrm{id} - \Pi_0))(F(g))v_h\,dl\Big|$$

$$\leq C \qquad (\|\sigma - \sigma_h\|_{L^2(T)} + h\|\sigma - \sigma_h\|_{H_h^1(T)})\|v_h\|_{L^2(T)}$$

$$\Big|\int_T (\sigma - \sigma_h) : (f(g)\,\mathrm{rot}\, v_h)\,da\Big|$$

$$\leq Ch^{-1}(\|\sigma - \sigma_h\|_{L^2(T)} + h\|\sigma - \sigma_h\|_{H_h^1(T)})\|v_h\|_{L^2(T)}$$

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Lemma

Let $k \in \mathbb{N}_0$, $\sigma_h \in \mathrm{Reg}_h^k$, $g_h = \mathcal{R}_h^k g$, $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$, and $v_h \in \mathring{\mathcal{W}}_h^k$. Then for sufficiently small h

$$\langle \mathrm{curl}_g\, \sigma_h, Q_g v_h \rangle_{\mathring{\mathcal{W}}_g} - \langle \mathrm{curl}_{g_h}\, \sigma_h, Q_{g_h} v_h \rangle_{\mathring{\mathcal{W}}_{g_h}} = \langle \Gamma_{ijk}(g - g_h), \Sigma_h^{ijl} \rangle$$

$$+ \mathcal{O}\big(C(\|g - g_h\|_{L^\infty(\Omega)} + h\|g - g_h\|_{W_h^{1,\infty}(\Omega)})\|\sigma_h\|_{H_h^1(\Omega)}\|v_h\|_{L^2(\Omega)}\big).$$

- Keeping volume and boundary terms together

$$\langle \Gamma_{ijk}(g), \Sigma^{ijk} \rangle = \sum_{T \in \mathscr{T}} \Big(\int_T \Gamma_{ijk}(g) \Sigma^{ijk}\, da - \int_{\partial T} \Sigma^{nni}(g_{nt} t_i + \frac{1}{2} g_{nn} n_i)\, dl \Big)$$

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Lemma

Let $k \in \mathbb{N}_0$, $\sigma_h \in \mathrm{Reg}_h^k$, $g_h = \mathcal{R}_h^k g$, $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$, and $v_h \in \mathring{\mathcal{W}}_h^k$. Then for sufficiently small h

$$\langle \Gamma_{ijk}(g - g_h), \Sigma_h^{ijl} \rangle = \langle \Gamma_{ijk}(g - g_h), \Sigma_{h,0}^{ijl} \rangle$$
$$+ \mathcal{O}\big(C(\|g - g_h\|_{L^\infty(\Omega)} + h\|g - g_h\|_{W_h^{1,\infty}(\Omega)})\|\sigma_h\|_{H_h^1(\Omega)}\|v_h\|_{L^2(\Omega)}\big).$$

- Keeping volume and boundary terms together
- Use orthogonality property for $\Sigma_{h,0}^{ijl} \in \mathcal{P}^k(T, \mathbb{R}^{2\times2\times2})$

$$\langle \Gamma_{ijk}(g - \mathcal{R}_h^k g), \Sigma_h^{ijk} \rangle = 0 \text{ for all } \Sigma_h \in \mathcal{P}^k(T, \mathbb{R}^{2\times2\times2})$$

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$$\left|\langle \mathrm{curl}_g\, \sigma_h, Q_g v_h\rangle_{\mathring{\mathcal{W}}_g} - \langle \mathrm{curl}_{g_h}\, \sigma_h, Q_{g_h} v_h\rangle_{\mathring{\mathcal{W}}_{g_h}}\right| \leq$$

$$C(\|g - g_h\|_{L^\infty(\Omega)} + h\|g - g_h\|_{W_h^{1,\infty}(\Omega)})\|\sigma_h\|_{H_h^1(\Omega)}\|v_h\|_{L^2(\Omega)}.$$

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Why Christoffel symbol as remainder?

- $\Gamma_{klm}(g - g_h)$ is of sub-optimal order

$$\Big|\int_T \frac{(\mathrm{curl}\,\sigma_h)_i {v_h}^i + \sigma_{h,ij}\varepsilon^{ik}\Gamma^j_{kl}(g){v_h}^l}{\sqrt{\det g}} - \frac{(\mathrm{curl}\,\sigma_h)_i {v_h}^i + \sigma_{h,ij}\varepsilon^{ik}\Gamma^j_{kl}(g_h){v_h}^l}{\sqrt{\det g_h}}\,dx\Big|$$

$$\leq C\|g - g_h\|_{L^\infty}\|\sigma_h\|_{H^1_h}\|v_h\|_{L^2} + \Big|\int_T \frac{\sigma_{h,ij}\varepsilon^{ik}(\Gamma^j_{kl}(g) - \Gamma^j_{kl}(g_h)){v_h}^l}{\sqrt{\det g_h}}\,dx\Big|$$

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ARNOLD, WALKER: The Hellan–Herrmann–Johnson method with curved elements, *SIAM Journal on Numerical Analysis, 58(5)* (2020).

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For g, $\sigma \in \mathrm{Reg}_h^k$ and $u \in \mathring{\mathcal{V}}_h^{k+1}$ continuous the distributional covariant incompatibility operator

$$\langle \mathrm{inc}_g\,\sigma, u\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} = \langle \mathrm{curl}_g\,\sigma, Q_g \,\mathrm{rot}\, u\rangle_{\mathring{\mathcal{W}}_g} = \sum_{T\in\mathscr{T}} \int_{\mathscr{T}} \mathrm{inc}_g\,\sigma\, u$$
$$- \int_{\partial T} u\, g(\mathrm{curl}_g\,\sigma - \mathrm{grad}_g\,\sigma(n_g, t_g), t_g) - \sum_{V\in\mathscr{V}_T} [\![\sigma(n_g, t_g)]\!]_V^T u(V)$$

Covariant distributional inc

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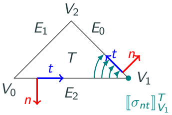
- Standard distributional inc

$$\langle \mathrm{inc}_\delta\,\sigma, u\rangle_{\mathring{\mathcal{V}}(\mathscr{T})} = \sum_{T\in\mathscr{T}} \int_{\mathscr{T}} \mathrm{inc}\,\sigma\, u - \int_{\partial T} u(\mathrm{curl}\,\sigma - \nabla\sigma_{nt})\cdot t$$

$$- \sum_{V\in\mathscr{V}_T} [\![\sigma_{nt}]\!]_V^T u(V)$$



- Smooth g and σ gives classical covariant inc

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Corollary

Let $k \in \mathbb{N}_0$, $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$, $\sigma \in H^1(\Omega, \mathbb{S}) \cap C^0(\Omega, \mathbb{S})$, $\sigma_h = \mathcal{R}_h^k \sigma$, and $u_h \in \mathring{\mathcal{V}}_h^{k+1}$. Then

$$|\langle \mathrm{inc}_g(\sigma - \sigma_h), u_h \rangle_{\mathring{\mathcal{V}}(\mathscr{T})}| \leq C\big(\|\sigma - \sigma_h\|_{L^2} + h\|\sigma - \sigma_h\|_{H_h^1}\big)\|\nabla u_h\|_{L^2}.$$

Corollary

Let $k \in \mathbb{N}_0$, $\sigma_h \in \mathrm{Reg}_h^k$, $g_h = \mathcal{R}_h^k g$, $g \in W^{1,\infty}(\Omega, \mathbb{S}^+)$ with $g^{-1} \in L^\infty(\Omega, \mathbb{S}^+)$, and , $u_h \in \mathring{\mathcal{V}}_h^{k+1}$. Then for sufficiently small h

$$\begin{aligned}|\langle \mathrm{inc}_g\, \sigma_h, u_h \rangle_{\mathring{\mathcal{V}}(\mathscr{T})} - \langle \mathrm{inc}_{g_h}\, \sigma_h, u_h \rangle_{\mathring{\mathcal{V}}(\mathscr{T})}| \\ \leq C(\|g - g_h\|_{L^\infty} + h\|g - g_h\|_{W_h^{1,\infty}})\|\sigma_h\|_{H_h^1}\|\nabla u_h\|_{L^2}.\end{aligned}$$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)

Let $k \in \mathbb{N}_0$, $g \in W^{k+1,\infty}(\Omega)$ with $K(g) \in H^k(\Omega)$, and $g_h = \mathcal{R}_h^k g$ the Regge interpolant. Then there holds for the lifted Gauss curvature $K_h(g_h) \in \mathring{\mathcal{V}}_h^{k+1}$ for sufficiently small h

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^{k+1}(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}).$$

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Corollary

There holds for $0 \leq l \leq k$

$$\|K_h(g_h) - K(g)\|_{L^2} \leq Ch^k(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}),$$
$$|K_h(g_h) - K(g)|_{H_h^l} \leq Ch^{k-l}(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}).$$

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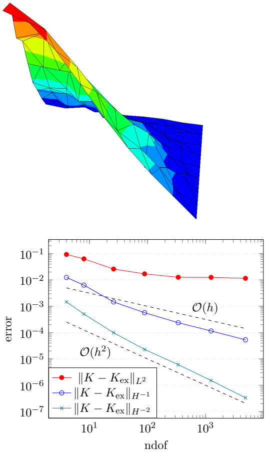
Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)

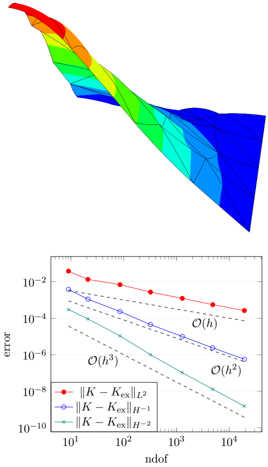
Let $k \in \mathbb{N}_0$, $g \in H^{k+1}(\Omega)$ with $\varpi(g) \in H^k(\Omega)$, and $g_h = \mathcal{R}_h^k g$ the Regge interpolant. Then there holds for the lifted connection 1-form $\varpi_h(g_h) \in \mathring{\mathcal{W}}_h^k$ for sufficiently small h

$$\|\varpi_h(g_h) - \varpi(g)\|_{L^2} \leq Ch^{k+1}(\|g\|_{H^{k+1}} + |\varpi(g)|_{H^k}).$$

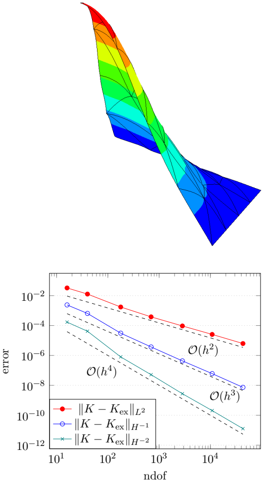
Numerical example (Gauss curvature)



$k = 0$



$k = 1$



$k = 2$

Numerical example (connection 1-form)

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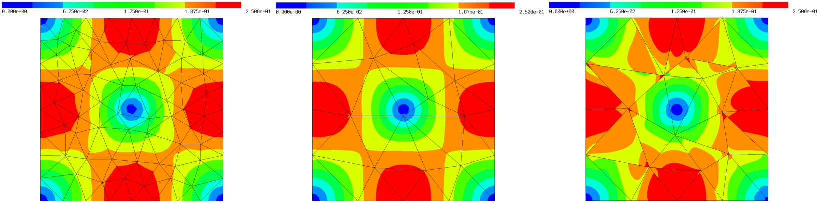
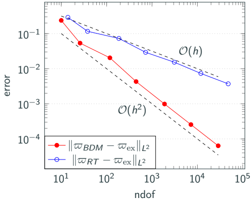
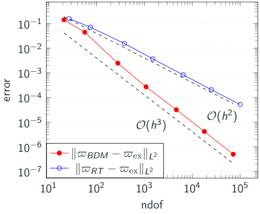




$k = 1$



$k = 2$

Extension to 3D

Extension to 3D

- Riemann curvature tensor R_{ijkl} has 6 independent entries
- Curvature operator $Q : M \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$\langle Q(u \wedge v), w \wedge z\rangle = \langle R(u,v)z, w\rangle \text{ for all } u, v, w, z \in \mathfrak{X}(M)$$

$$Q^{ij} = -\frac{1}{4\det g}\varepsilon^{ikl}\varepsilon^{jmn}R_{klmn}, \quad Q^{xx} = -\frac{R_{yzyz}}{\det g}, \; Q^{yz} = \frac{R_{xzxy}}{\det g}$$

$$\mathrm{Ric}_{ij} = g^{kl}R_{kilj} = -(Q \times \mathrm{cof}(g))_{ij}$$

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$$\mathrm{Ric}_{ij} = g^{kl}R_{kilj} = -(Q \times \mathrm{cof}(g))_{ij}$$

- No Gauss–Bonnet theorem in 3D

Lifted distributional curvature

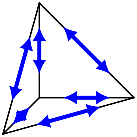
For $g \in \mathrm{Reg}_h^k(\mathscr{T})$ find $Q_h(g) \in \mathrm{Reg}_h^k(\mathscr{T})$ s.t. $\forall\, v \in \mathrm{Reg}_h^k(\mathscr{T})$

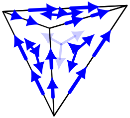
$$\int_{\mathscr{T}} Q_h(g) : v = \sum_{T\in\mathscr{T}} \big(K^T(v,g) + \sum_{F\in\mathcal{F}_T} K_F^T(v,g)\big) + \sum_{E\in\mathscr{E}} K_E(v,g)$$

$$K^T(v,g) = \int_T Q(g) : v$$

$$K_F^T(v,g) = \int_F {\color{red}?} : v$$

$$K_E(v,g) = \big(2\pi - \sum_{T:E\subset T} \sphericalangle_E^T(g)\big) v_{t_E t_E}$$





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$$\int_{\mathscr{T}} Q_h(g) : v\sqrt{\det g}\, dx = \sum_{T\in\mathscr{T}} \Big(\int_T Q(g) : v\sqrt{\det g}\, dx + \int_{\partial T} \frac{\sqrt{\det g}}{\mathrm{cof}(g)_{nn}}((n\otimes n)\times \Gamma^n_{\bullet\bullet}) : v\, da\Big) + \sum_{E\in\mathscr{E}} K_E(v,g)$$

$$\mathrm{cof}(A)^{ij} = \det(A)A^{ji}, \quad (A\times B)^{ij} = \varepsilon^{ikl}\varepsilon^{jmn}A_{km}B_{ln}$$

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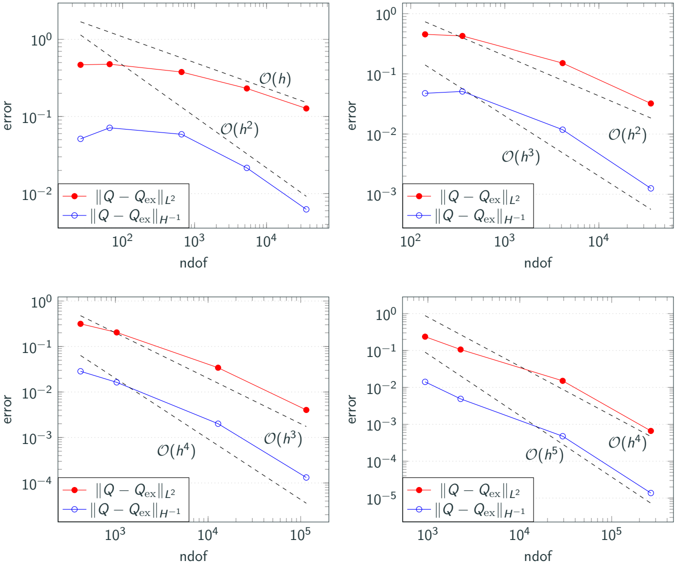
$$\mathrm{2D} : \int_{\partial T} \frac{\sqrt{\det g}}{g_{tt}} \Gamma^n_{tt}\, v\, dl$$



error
ndof
10^0
10^{-1}
10^{-2}
10^2
10^3
10^4
$\mathcal{O}(h)$
$\mathcal{O}(h^2)$
$\|Q - Q_{ex}\|_{L^2}$
$\|Q - Q_{ex}\|_{H^{-1}}$
10^{-3}
$\mathcal{O}(h^2)$
$\mathcal{O}(h^3)$
10^{-4}
10^5
$\mathcal{O}(h^3)$
$\mathcal{O}(h^4)$
10^{-5}
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$\mathcal{O}(h^5)$

Summary

- Improved error analysis (Gauss curvature, connection 1-form)
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https://www.asc.tuwien.ac.at/~schoeberl/wiki/index.php/Michael_Neunteufel

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Thank You for Your attention!

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