The Hellan–Herrmann–Johnson Method for Nonlinear Shells

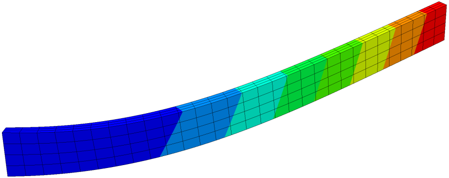
Michael Neunteufel, Joachim Schöberl

GAMM, Vienna, February 18-22, 2019

Contents

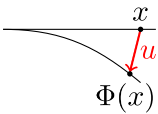
Notation

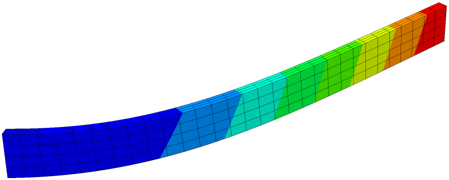
Deformation $\Phi : \Omega \to \mathbb{R}^3$



Deformation $\Phi : \Omega \to \mathbb{R}^3$

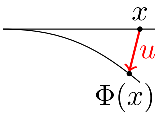
Displacement $u := \Phi - id$

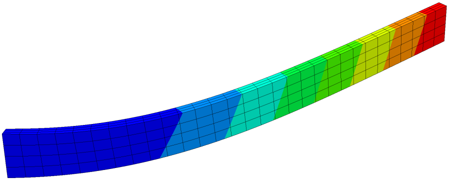




Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \nabla\Phi$





Deformation	$\Phi : \Omega \to \mathbb{R}^3$
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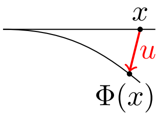

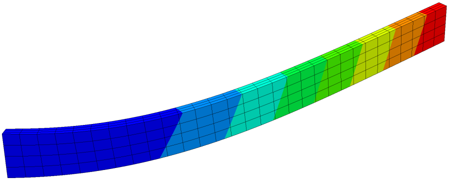

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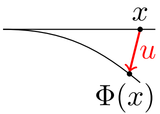
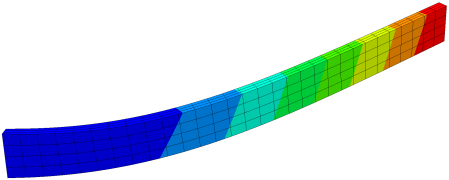
Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$\boldsymbol{F} := \boldsymbol{I} + \nabla u$
Cauchy-Green strain tensor	$\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$





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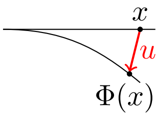
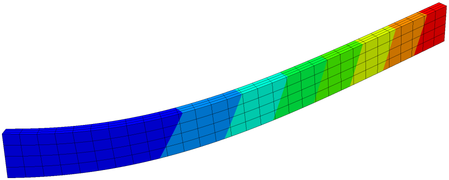
Green strain tensor	$\boldsymbol{E} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$





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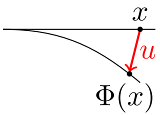
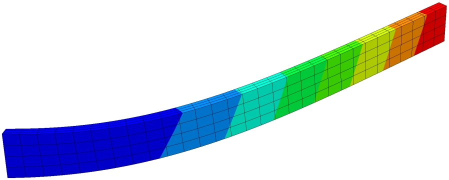


Elasticity

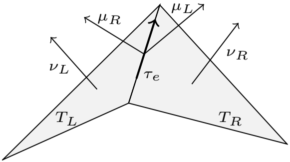
$$\mathcal{W}(u) = \frac{1}{2}\|\boldsymbol{E}\|_{\boldsymbol{M}}^2 - \langle f, u \rangle$$



- Normal vector ν

  Tangent vector τ_e

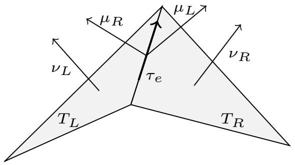
  Element normal vector $\mu = \pm\nu \times \tau_e$

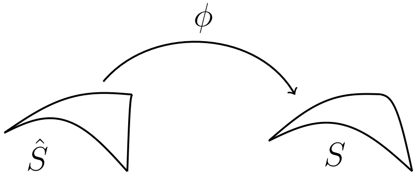


- Normal vector $\hat{\nu}$

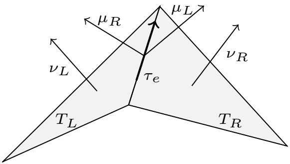
  Tangent vector $\hat{\tau}_e$

  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$

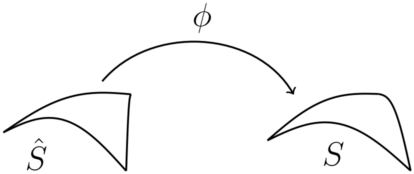




- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}_e$
  Element normal vector $\hat{\mu} = \pm\hat{\nu} \times \hat{\tau}_e$



- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$



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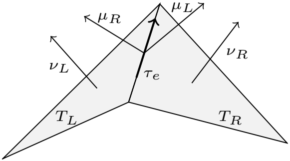
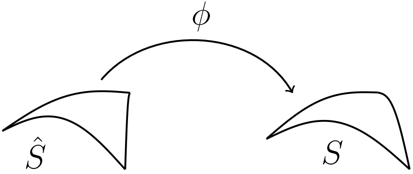
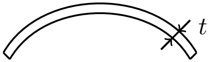
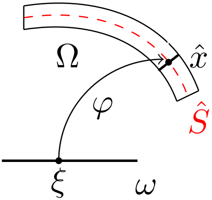


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau_e \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}_e$
  $\mu \circ \phi = \pm\nu \times \tau_e$

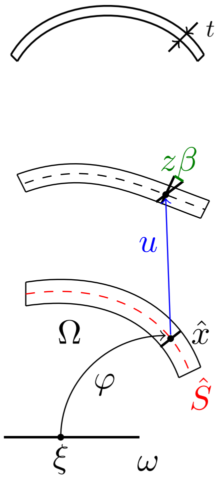




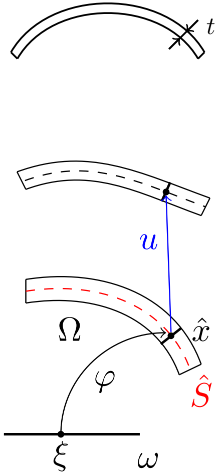
- Model of reduced dimensions



- Model of reduced dimensions
- $\Omega = \left\{ \varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right] \right\}$



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + z\hat{\nu}(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\ (\nu + \beta) \circ \phi(\hat{x})$



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- $\Phi(\hat{x} + z\hat{\nu}(\xi)) = \phi(\hat{x}) + z\, \nu \circ \phi(\hat{x})$

Shell energy (Kirchhoff–Love)

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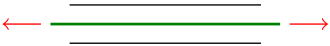
$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

Shell energy (Kirchhoff–Love)



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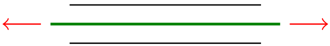
- Membrane energy

Shell energy (Kirchhoff–Love)



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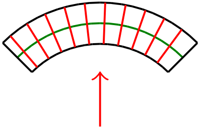
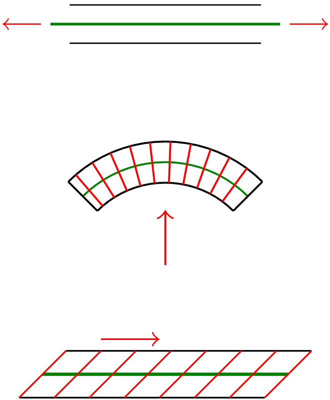


- Membrane energy
- Bending energy

Shell energy (Kirchhoff–Love)

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|_{\boldsymbol{M}}^2 + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|_{\boldsymbol{M}}^2$$

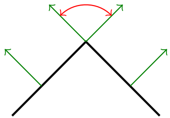


- Membrane energy
- Bending energy
- Shearing energy

Method and Shell Element

Moment tensor

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



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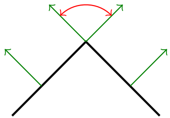
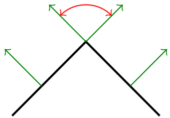


- Measure change of angles

Moment tensor

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}_{\tau\tau}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\sum_{\hat{E}\in\hat{\mathcal{E}}_h}\|\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R)\|^2_{\boldsymbol{M},\hat{E}}$$



- Measure change of angles

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \langle \boldsymbol{F}^T\nabla\nu - \nabla\hat{\nu}, \boldsymbol{\sigma}\rangle + \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\langle\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R), \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\rangle_{\hat{E}}$$

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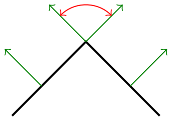

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- $\boldsymbol{\sigma}$ has physical meaning of moment

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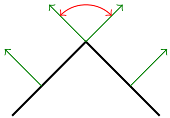

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- $\boldsymbol{\sigma}$ has physical meaning of moment
- Fourth order problem $\rightarrow$ second order problem

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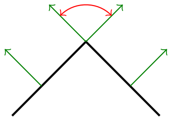

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- $\boldsymbol{\sigma}$ has physical meaning of moment
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Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u,\boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

$$G(u,\boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h}\int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h}\int_{\hat{E}} (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

Shell problem

Find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

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with

$$G(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu \qquad\qquad) \, d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) \qquad\qquad) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}} \, d\hat{s}.$$

$\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i$

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Shell problem (Hybridization)

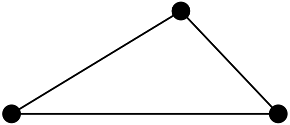
Find $u \in [H^1(\hat{S})]^3$, $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\alpha \in \Gamma(\hat{S})$ for

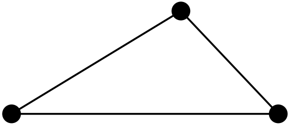
$$\mathcal{L}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G(u, \boldsymbol{\sigma}, \alpha) - \langle f, u\rangle,$$

with

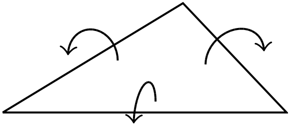
$$G(u, \boldsymbol{\sigma}, \alpha) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$
$$- \sum_{\hat{E}\in\hat{\mathcal{E}}_h} \int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\frac{1}{2}(\boldsymbol{\sigma}_{\hat{\mu}_L\hat{\mu}_L} + \boldsymbol{\sigma}_{\hat{\mu}_R\hat{\mu}_R})\, d\hat{s}$$
$$+ \int_{\hat{E}} \alpha_{\hat{\mu}} [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!]\, d\hat{s}.$$

$$V_h^k := \Pi^k(\hat{\mathcal{T}}_h) \cap C^0(\hat{S}_h)$$





$$V_h^k := \Pi^k(\hat{\mathcal{T}}_h) \cap C^0(\hat{S}_h)$$



$$\Sigma_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\hat{\mathcal{T}}_h)]^{2\times 2}_{sym} |\, [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = 0\}$$

A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Finite element spaces

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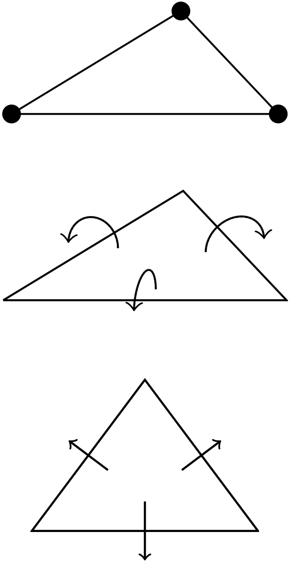
$$V_h^k := \Pi^k(\hat{\mathcal{T}}_h) \cap C^0(\hat{S}_h)$$

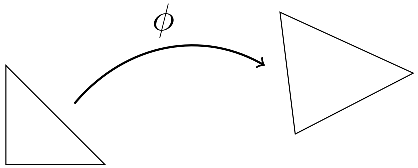
$$\Sigma_h^k := \{\boldsymbol{\sigma} \in [\Pi^k(\hat{\mathcal{T}}_h)]^{2\times 2}_{sym} \,|\, [\![\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}]\!] = 0\}$$

$$\Gamma_h^k := \{\alpha \in [\Pi^k(\hat{\mathcal{T}}_h)]^2 \,|\, [\![\alpha_{\hat{\mu}}]\!] = 0\}$$



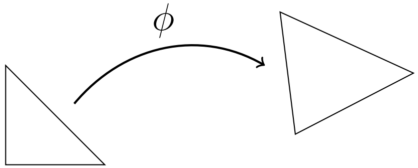
A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

Mapping to the surface



- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J}\boldsymbol{F}\hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}}\phi,\ J = \det(\boldsymbol{F})$$



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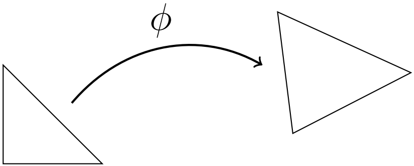
- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$



Mapping to the surface

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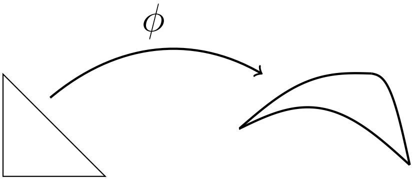


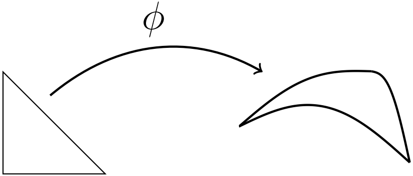
Mapping to the surface

- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \sqrt{\det(\boldsymbol{F}^T \boldsymbol{F})}$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$

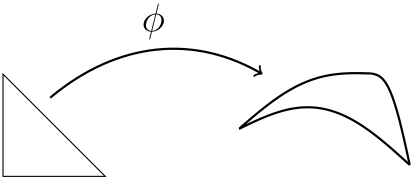


- Piola transformation

$$u \circ \phi = P[\hat{u}] = \frac{1}{J} \boldsymbol{F} \hat{u} \qquad \boldsymbol{F} = \nabla_{\hat{x}} \phi, \; J = \| \operatorname{cof}(\boldsymbol{F}) \|$$

- Preserve normal-normal continuity

$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2} \boldsymbol{F} \hat{\boldsymbol{\sigma}} \boldsymbol{F}^T$$



Mapping to the surface

- Piola transformation

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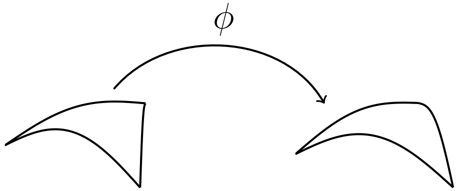
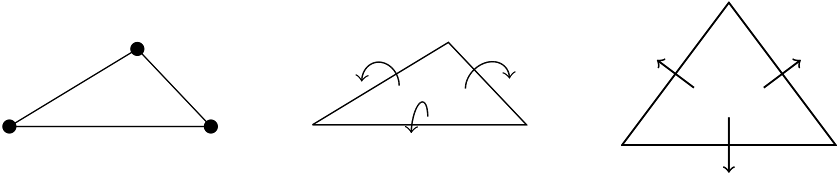
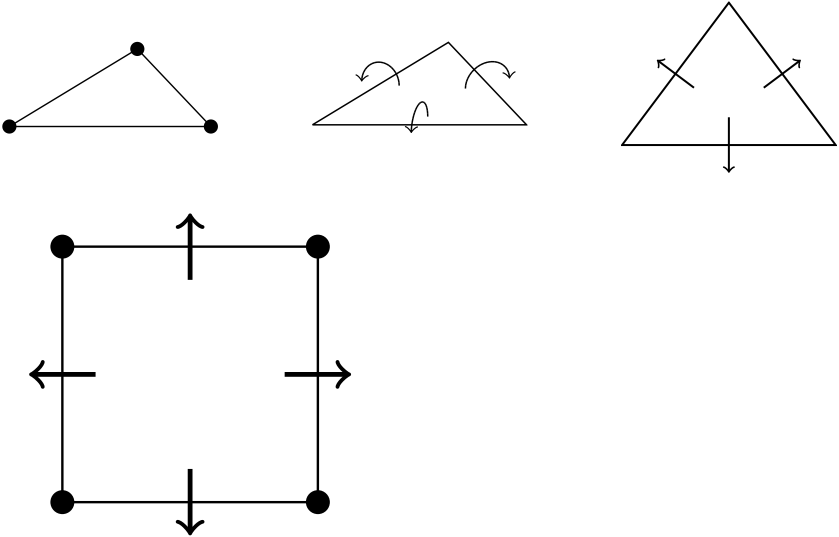
$$\boldsymbol{\sigma} \circ \phi = \frac{1}{J^2}\boldsymbol{F}\hat{\boldsymbol{\sigma}}\boldsymbol{F}^T$$

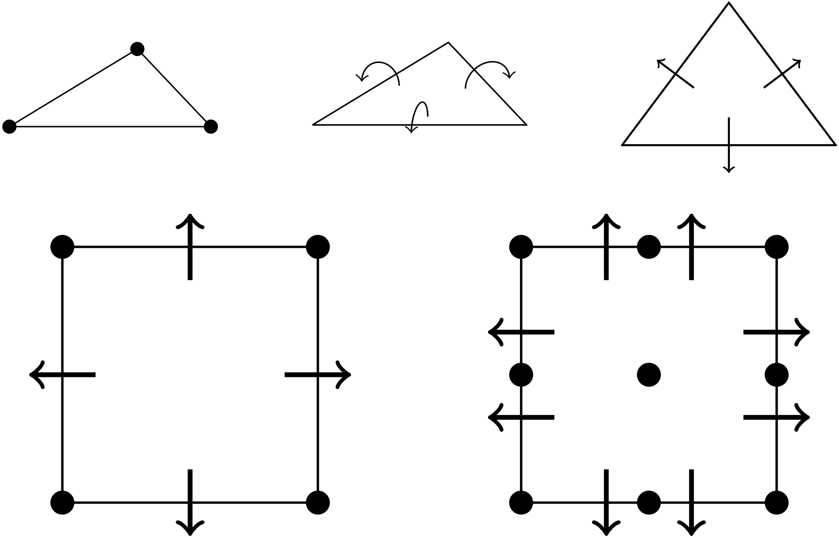


Shell element





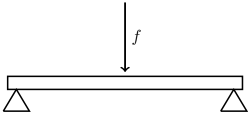
Shell element



Relation to HHJ

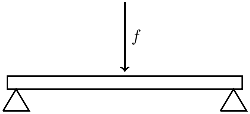
- Discretization method for 4th order elliptic problems

$$\operatorname{div}(\operatorname{div}(\nabla^2 u)) = f$$



- Discretization method for 4th order elliptic problems

$$\mathrm{div}(\mathrm{div}(\nabla^2 u)) = f \quad \Rightarrow u \in H^2(\Omega)$$



Hellan–Herrmann–Johnson method

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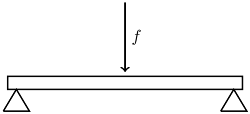
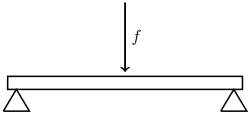


$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 u, \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, \end{aligned}$$

Hellan–Herrmann–Johnson method

- Discretization method for 4th order elliptic problems

$$\mathrm{div}(\mathrm{div}(\nabla^2 u)) = f \quad \Rightarrow u \in H^2(\Omega)$$



$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 u, &\quad \Rightarrow u \in H^1(\Omega) \\ \mathrm{div}(\mathrm{div}(\boldsymbol{\sigma})) &= f, &\quad \Rightarrow \boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega) \end{aligned}$$

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Hellan–Herrmann–Johnson

Find $u \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\mathcal{L}(u, \boldsymbol{\sigma}) = -\frac{1}{2}\|\boldsymbol{\sigma}\|^2 + \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \mathrm{div}(\boldsymbol{\sigma})\, dx - \int_{\partial T} (\nabla u)_\tau \boldsymbol{\sigma}_{\mu\tau}\, ds - \langle f, u \rangle.$$

M. Comodi: The Hellan-Herrmann-Johnson method: some new error estimates and postprocessing, *Math. Comp. 52 (1989) pp. 17-29.*

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Linearization

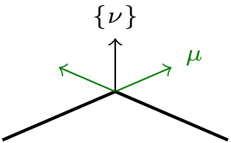
If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ method is the linearization of the bending energy of our method.

Kinks

Computational aspects

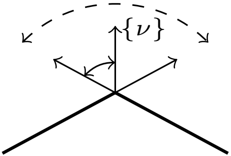
$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

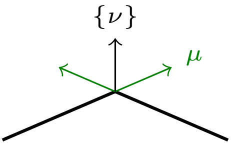


$$\int_{\partial\hat{T}} (\sphericalangle(\{\nu\}\ ,\mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

$$\{\nu\}\ := \frac{1}{\|\nu_L\ + \nu_R\ \|}(\nu_L\ + \nu_R\ )$$

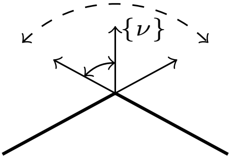


$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R)) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

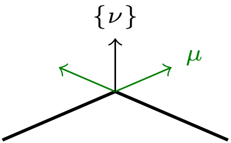


$$\int_{\partial \hat{T}} (\sphericalangle(\{\nu\}\ , \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu})) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

$$\{\nu\} \ := \frac{\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R}{\|\operatorname{cof}(\boldsymbol{F}_L)\hat{\nu}_L + \operatorname{cof}(\boldsymbol{F}_R)\hat{\nu}_R\|}$$

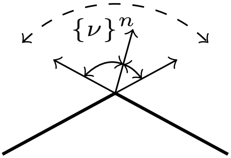


$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R)) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

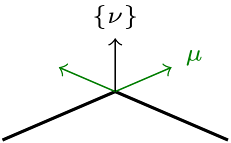


$$\int_{\partial \hat{T}} (\sphericalangle(\{\nu\}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu})) \boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

$$\{\nu\}^n := \frac{1}{\|{\nu_L}^n + {\nu_R}^n\|} ({\nu_L}^n + {\nu_R}^n)$$

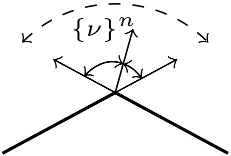


$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$



$$\int_{\partial \hat{T}} (\sphericalangle(\overline{\{\nu\}}^n, \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}$$

$$\overline{\{\nu\}}^n := \boldsymbol{P}^{\perp}_{\tau_e}(\{\nu\}^n)$$



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Final algorithm

For given u^n compute

$$\{\nu\}^n = Av(u^n).$$

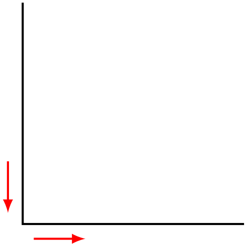
Then find $u \in [H^1(\hat{S})]^3$ and $\boldsymbol{\sigma} \in H(\mathrm{divdiv}, \hat{S})$ for

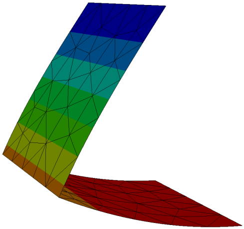
$$\mathcal{L}_{\{\nu\}^n}(u, \boldsymbol{\sigma}) = \frac{t}{2}\|E_{\tau\tau}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + G_{\{\nu\}^n}(u, \boldsymbol{\sigma}) - \langle f, u\rangle,$$

with

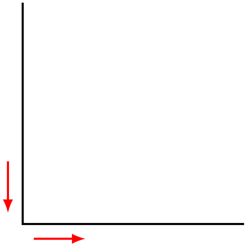
$$G_{\{\nu\}^n}(u, \boldsymbol{\sigma}) = \sum_{\hat{T}\in\hat{\mathcal{T}}_h} \int_{\hat{T}} \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1 - \hat{\nu}\cdot\nu)\nabla\hat{\nu})\, d\hat{x}$$
$$- \int_{\partial\hat{T}} (\sphericalangle(\boldsymbol{P}^{\perp}_{\tau_e}(\{\nu\}^n), \mu) - \sphericalangle(\{\hat{\nu}\}, \hat{\mu}))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s}.$$

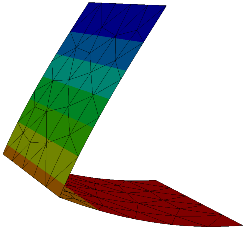
Structures with kinks



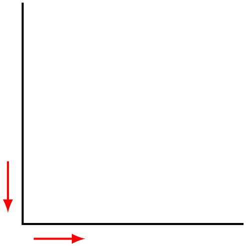


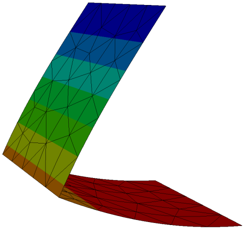
- Normal-normal continuous moment $\boldsymbol{\sigma}$





- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks





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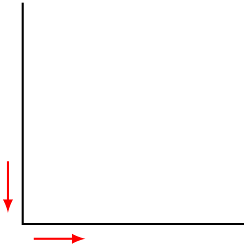
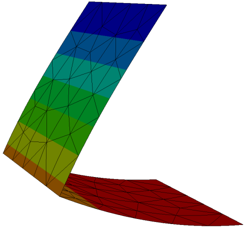
- Normal-normal continuous moment $\boldsymbol{\sigma}$
- Preserve kinks
- Variation of $\mathcal{L}(u, \boldsymbol{\sigma})$ in direction $\delta\boldsymbol{\sigma}$

$$\int_{\hat{E}} (\sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R))\delta\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, d\hat{s} \stackrel{!}{=} 0$$

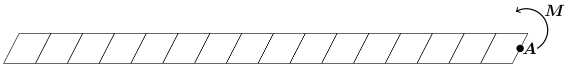
$$\Rightarrow \sphericalangle(\nu_L, \nu_R) - \sphericalangle(\hat{\nu}_L, \hat{\nu}_R) = 0$$

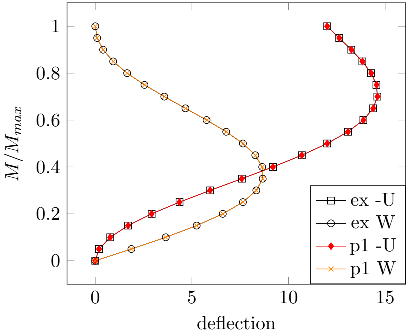




Numerical Examples

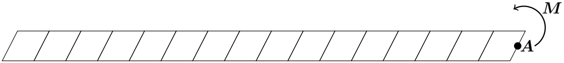
Cantilever subjected to end moment

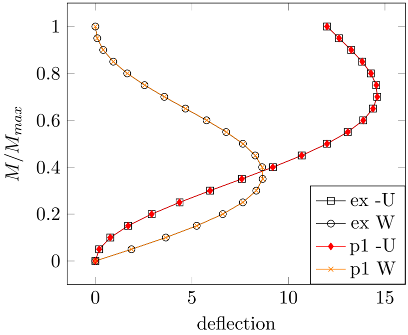




- $E = 1.2 \times 10^6$
  $\nu = 0$
  $L = 12$
  $W = 1$
  $t = 0.1$
  $M = 50\frac{\pi}{3}$

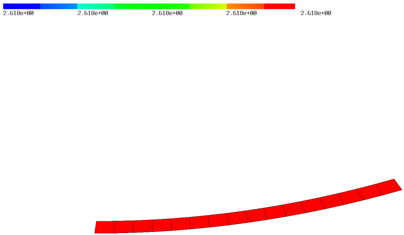
Cantilever subjected to end moment

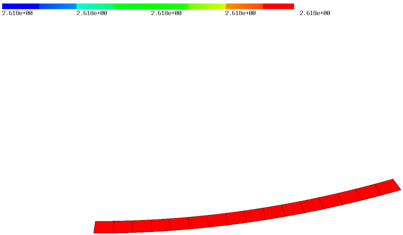




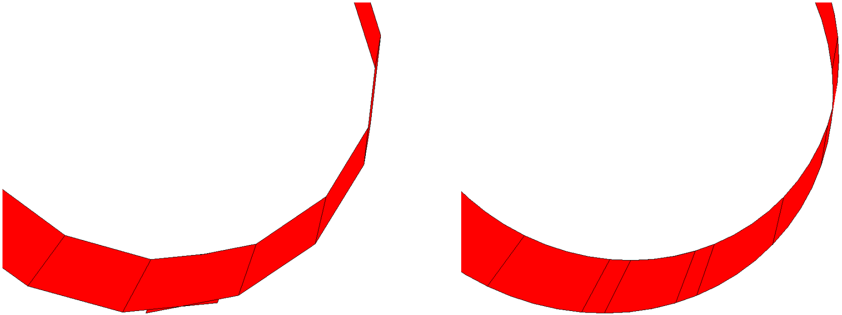
NGSolve

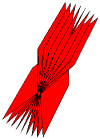
Cantilever subjected to end moment

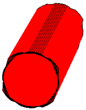




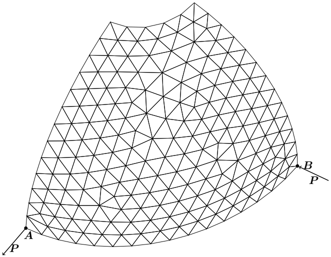
Cantilever subjected to end moment





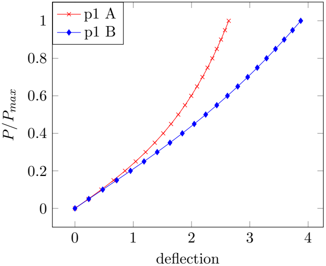


Hemispherical Shell

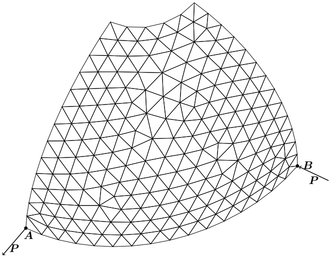


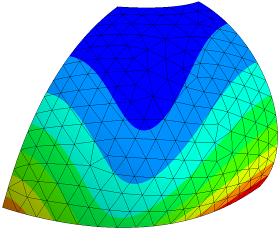
- $t = 0.04$
  $P = 50$
  $E = 6.825 \times 10^7$
  $\nu = 0.3$
  $R = 10$

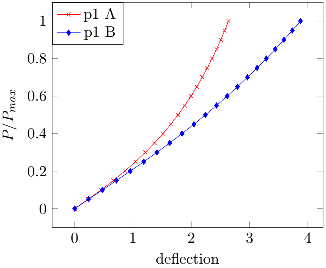
h	2	1	0.5	0.25
p1	4.1218	3.8811	3.8560	3.8735
p3	3.8319	3.8781	3.8796	3.8796



Hemispherical Shell

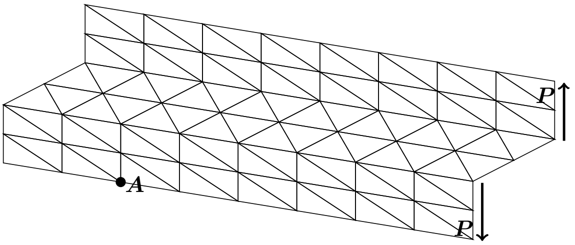






h	2	1	0.5	0.25
p1	4.1218	3.8811	3.8560	3.8735
p3	3.8319	3.8781	3.8796	3.8796

Z-Section Cantilever



- $P = 6 \times 10^5$
  $E = 2.1 \times 10^{11}$
  $\nu = 0.3$
  $t = 0.1$
  $L = 10$
  $W = 2$
  $H = 1$

- Membrane stress $\boldsymbol{\Sigma}_{xx}$ at point $\boldsymbol{A}$

	p1	p3
8x6	-0.7620×10^8	-1.0929×10^8
32x15	-1.0777×10^8	-1.0933×10^8
64x30	-1.0989×10^8	-1.0933×10^8
ref		-1.08×10^8

Summary

- Kirchhoff–Love shell element

Summary

- Kirchhoff–Love shell element
- Moment tensor

Summary

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Thank you for your attention!