The Hellan-Herrmann-Johnson method for nonlinear shells

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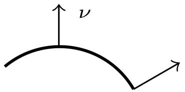
Curvatures

Weingarten operator

- $\nu : S \to \mathbb{S}^2$ normal vector field

$$\nabla_\tau \nu : TS \to TS$$



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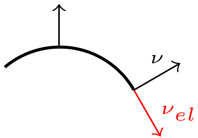

- First approach: Find $\kappa \in H(\mathrm{divdiv})$ s.t. for all $\sigma \in H(\mathrm{divdiv})$

$$\begin{aligned}\int_S \kappa : \sigma \, dx &= \int_S \nabla_\tau \nu : \sigma \, dx \\ &= -\int_S \nu \cdot \mathrm{div}_\tau(\sigma) \, dx + \int_{\partial S} \nu \cdot (\sigma \nu_{el}) \, ds\end{aligned}$$

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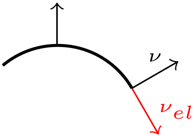

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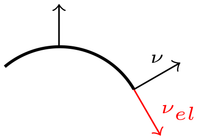

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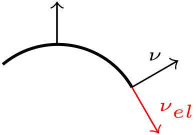

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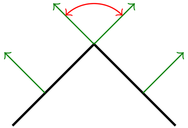
- Affine geometry
  - $\nabla_\tau \nu = 0$ on T
  - jump over edges



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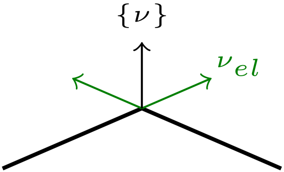
Distributional derivative

Let $\mathcal{T}_h$ be an affine triangulation of a smooth manifold S. Then there holds

$$\langle \nabla_\tau \nu, \sigma \rangle = \sum_{T\in\mathcal{T}_h} \int_T \nabla_\tau \nu : \sigma \, dx + \int_{\partial T} \{\nu\} \cdot \nu_{el} \sigma_{\nu_{el}\nu_{el}} \, ds + \mathcal{O}(h).$$



Curvatures on discrete geometries

Distributional derivative

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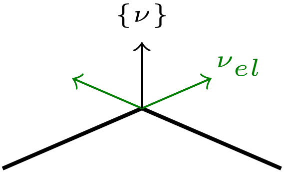
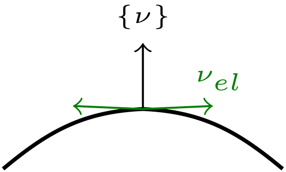
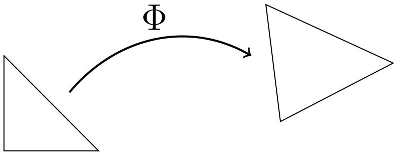




Curvatures on deformed surfaces

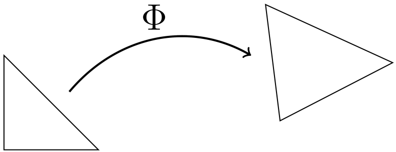
Transformation of H(divdiv) functions



Transformation of H(divdiv) functions

- Piola transformation

$$u \circ \Phi = P[\hat{u}] = \frac{1}{J} F \hat{u} \qquad F = \nabla_{\hat{x}} \Phi, \; J = \det(F)$$



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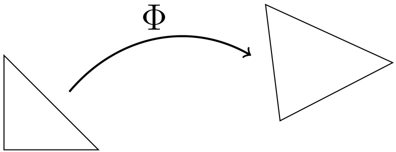
$$u \circ \Phi = P[\hat{u}] = \frac{1}{J} F \hat{u} \qquad F = \nabla_{\hat{x}} \Phi, \; J = \det(F)$$

- Preserve normal-normal continuity

$$\sigma \circ \Phi = \frac{1}{J^2} F \hat{\sigma} F^T$$



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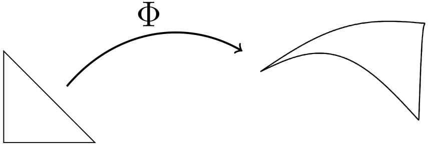

Transformation of H(divdiv) functions

- Piola transformation

$$u \circ \Phi = P[\hat{u}] = \frac{1}{J} F \hat{u} \qquad F = \nabla_{\hat{x}} \Phi, \; J = \sqrt{\det(F^T F)}$$

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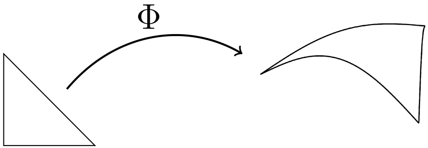

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$$u \circ \Phi = P[\hat{u}] = \frac{1}{J} F \hat{u} \qquad F = \nabla_{\hat{x}} \Phi, \; J = \| \operatorname{cof}(F) \|$$

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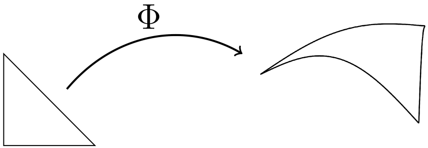

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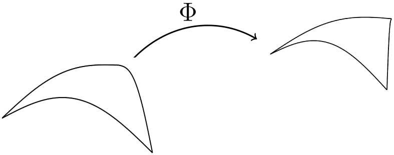

Deformed surfaces

- Solving on simple reference domain
- Avoid meshing complicated geometries

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$$\int_S \kappa : \sigma \, dx = \int_{\hat{S}} \frac{1}{J^3} F \hat{\kappa} F^T : F \hat{\sigma} F^T \, d\hat{x}$$

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$$\int_T \nu \operatorname{div}(\sigma) \, dx = \int_{\hat{T}} J \frac{\operatorname{cof}(F)\hat{\nu}}{\| \operatorname{cof}(F)\hat{\nu} \|} \operatorname{div}(\sigma) \circ \Phi \, d\hat{x}$$

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$$\int_{\partial T} \{\nu\} \cdot \nu_{el} \sigma_{\nu_{el} \nu_{el}} \, ds = \int_{\partial \hat{T}} \frac{J_{bnd}}{J^2} \{\bar{\nu}\} \cdot \bar{\nu}_{el} (F \hat{\sigma} F^T)_{\bar{\nu}_{el} \bar{\nu}_{el}} \, d\hat{s}$$

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$$\{\bar{\nu}\} = \frac{\mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R}{\| \mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R \|}$$

Deformed surfaces

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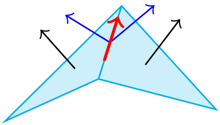

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- Solving on simple reference domain
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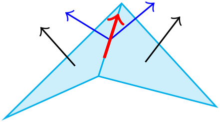

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Deformed surfaces

Transformation

Let $\sigma \circ \Phi = \frac{1}{J^2} F \hat{\sigma} F^T$ and $\Phi = \mathrm{id} + u$. Then

$$\int_{\hat{T}} J \bar{\nu} \operatorname{div}(\sigma) \circ \Phi \, d\hat{x} = - \int_{\hat{T}} \frac{1}{J} \sum_i \bar{\nu}_i (\nabla(\hat{\nu} \otimes \hat{\nu})_i - H_i) : \hat{\sigma} \, d\hat{x},$$

where $H_i := \nabla^2 u_i$.

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Curvature on deformed surface

$$\int_{\hat{S}} \frac{1}{J^3} F \hat{\kappa} F^T F \hat{\sigma} F^T \, d\hat{x} = \sum_{\hat{T}} \int_{\hat{T}} \frac{1}{J} \sum_i \bar{\nu}_i (\nabla(\hat{\nu} \otimes \hat{\nu})_i - H_i) : \hat{\sigma} \, d\hat{x}$$
$$+ \int_{\partial \hat{T}} \frac{J_{bnd}}{J^2} \{\bar{\nu}\} \bar{\nu}_{el} (F \hat{\sigma} F^T)_{\bar{\nu}_{el} \bar{\nu}_{el}} \, d\hat{s}$$

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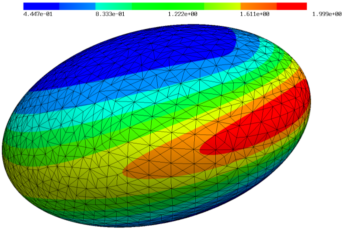
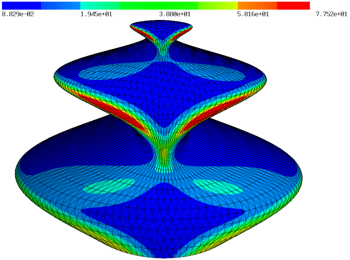
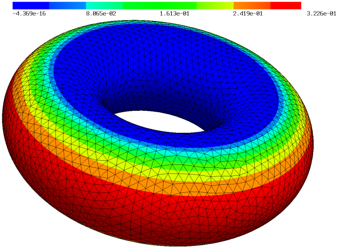
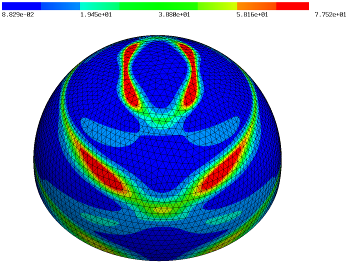
Example curvatures









Nonlinear shells

Shell energy

$$\mathcal{W}(u) = \|E_{\tau\tau}(u)\|_M^2 + t^2\|\hat{\kappa}(u) - \kappa_R\|_M^2$$

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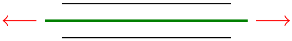

- Membrane energy

Shell energy

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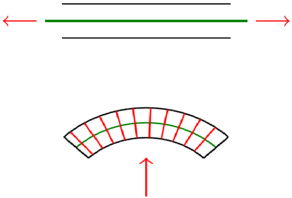

- Membrane energy
- Bending energy

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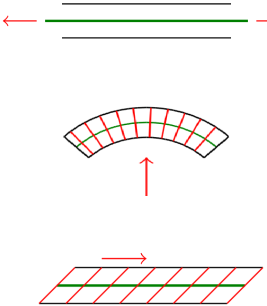

- Membrane energy
- Bending energy
- Shearing energy

Momentum

- Assumption: $\kappa_R = 0$, neglect F, J

$$L(u, \hat{\kappa} \quad) = \|E_{\tau\tau}(u)\|_M^2 + t^2 \|\hat{\kappa}\|_M^2$$

Momentum

- Assumption: $\kappa_R = 0$, neglect F, J
- Use Lagrange multiplier σ

$$L(u,\hat{\kappa},\sigma) = \|E_{\tau\tau}(u)\|_M^2 + t^2\|\hat{\kappa}\|_M^2 + \langle\hat{\kappa},\sigma\rangle_{L^2} - G(u,\sigma)$$

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$$\frac{\partial}{\partial \hat{\kappa}} L(u, \hat{\kappa}, \sigma) = 2t^2 M\hat{\kappa} + \sigma \stackrel{!}{=} 0$$

Momentum

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$$\Rightarrow \hat{\kappa} = -\frac{1}{2t^2} M^{-1}\sigma$$

Momentum

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- Lagrange parameter has physical meaning of momentum

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Shell problem

Find $u \in [H^1(\hat{S})]^d$ and $\sigma \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{W}(u,\sigma) = \|E_{\tau\tau}(u)\|_M^2 - \frac{1}{2t^2}\|\sigma\|_{M^{-1}}^2 + G(u,\sigma)$$

with

$$G(u,\sigma) = \sum_{\hat{T}} \int_{\hat{T}} \sigma : H_{\bar{\nu}}\, d\hat{x} - \int_{\partial\hat{T}} \{\bar{\nu}\} \cdot \bar{\nu}_{el} \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}}\, d\hat{s}.$$

New formulation

Shell problem

Find $u \in [H^1(\hat{S})]^d$ and $\sigma \in H(\mathrm{divdiv}, \hat{S})$ for the saddle point problem

$$\mathcal{W}(u,\sigma) = \|E_{\tau\tau}(u)\|^2_M - \frac{1}{2t^2}\|\sigma\|^2_{M^{-1}} + G(u,\sigma)$$

with

$$G(u,\sigma) = \sum_{\hat{T}} \int_{\hat{T}} \sigma : H_{\bar{\nu}}\, d\hat{x} - \int_{\partial \hat{T}} \{\bar{\nu}\} \cdot \bar{\nu}_{el} \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}}\, d\hat{s}.$$

- Saddle point problem, indefinite matrix

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- Saddle point problem, indefinite matrix
- Hybridization techniques possible

Hybridization

Shell problem

Find $u \in [H^1(\hat{S})]^d$, $\sigma \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\beta \in H(\mathrm{div}, \hat{S})^{bnd}$ for the saddle point problem

$$\mathcal{W}(u, \sigma, \beta) = \|E_{\tau\tau}(u)\|_M^2 - \frac{1}{2t^2}\|\sigma\|_{M^{-1}}^2 + G(u, \sigma, \beta)$$

with

$$G(u, \sigma, \beta) = \sum_{\hat{T}} \int_{\hat{T}} \sigma : H_{\bar{\nu}} \, d\hat{x} - \int_{\partial \hat{T}} (\{\bar{\nu}\}\bar{\nu}_{el} + \beta \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s}.$$

Hybridization

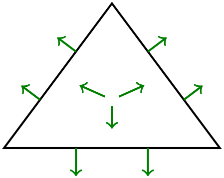
Shell problem

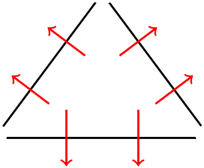
Find $u \in [H^1(\hat{S})]^d$, $\sigma \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\beta \in H(\mathrm{div}, \hat{S})^{bnd}$ for the saddle point problem

$$\mathcal{W}(u,\sigma,\beta) = \|E_{\tau\tau}(u)\|_M^2 - \frac{1}{2t^2}\|\sigma\|_{M^{-1}}^2 + G(u,\sigma,\beta)$$

with

$$G(u,\sigma,\beta) = \sum_{\hat{T}} \int_{\hat{T}} \sigma : H_{\bar{\nu}}\, d\hat{x} - \int_{\partial\hat{T}} (\{\bar{\nu}\}\bar{\nu}_{el} + \beta\hat{\nu}_{el})\sigma_{\hat{\nu}_{el}\hat{\nu}_{el}}\, d\hat{s}.$$



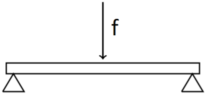


Hellan-Herrmann-Johnson method

Hellan-Herrmann-Johnson

Find $u \in H^1(\Omega)$ and $\sigma \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

$$\mathcal{W}(u,\sigma) = -\frac{1}{2}\|\sigma\|^2 + \sum_{T\in\mathcal{T}} \int_T \nabla u \,\mathrm{div}(\sigma)\, dx - \int_{\partial T} (\nabla u)_\tau \sigma_{\nu_{el}\tau}\, ds$$
$$- \int_\Omega f \cdot u \, dx$$



Hellan-Herrmann-Johnson

Find $u \in H^1(\Omega)$ and $\sigma \in H(\mathrm{divdiv}, \Omega)$ for the saddle point problem

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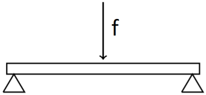
$$- \int_\Omega f \cdot u \, dx$$

Linearization

If the undeformed configuration is a flat plane and f works orthogonal on it, the HHJ-method is the linearization of the bending energy of our method.



$$\int_{\partial \hat{T}} (\{\bar{\nu}\} \cdot \bar{\nu}_{el} + \beta \cdot \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el} \hat{\nu}_{el}} \, d\hat{s}$$

$$\int_{\partial \hat{T}} (\{\bar{\nu}\} \cdot \bar{\nu}_{el} + \beta \cdot \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s} \qquad \{\bar{\nu}\} = \frac{\mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R}{\| \mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R \|}$$

Algorithm

$$\int_{\partial \hat{T}} (\{\bar{\nu}\} \cdot \bar{\nu}_{el} + \beta \cdot \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s} \qquad \{\bar{\nu}\} = \frac{\mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R}{\| \mathrm{cof}(F_L)\hat{\nu}_L + \mathrm{cof}(F_R)\hat{\nu}_R \|}$$

- $\{\bar{\nu}\}$ needs information of two elements the same time

Algorithm

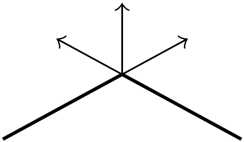
$$\int_{\partial \hat{T}} (\{\bar{\nu}\} \cdot \bar{\nu}_{el} + \beta \cdot \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s} \qquad \{\bar{\nu}\} = \frac{\operatorname{cof}(F_L)\hat{\nu}_L + \operatorname{cof}(F_R)\hat{\nu}_R}{\| \operatorname{cof}(F_L)\hat{\nu}_L + \operatorname{cof}(F_R)\hat{\nu}_R \|}$$

- $\{\bar{\nu}\}$ needs information of two elements the same time
- DG, fixpoint iteration

Algorithm

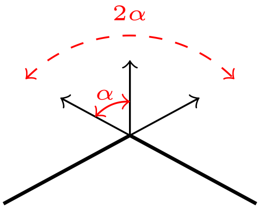
$$\int_{\partial\hat{T}} (\{\bar{\nu}\} \cdot \bar{\nu}_{el} + \beta \cdot \hat{\nu}_{el}) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s} \qquad \{\bar{\nu}\} = \frac{\operatorname{cof}(F_L)\hat{\nu}_L + \operatorname{cof}(F_R)\hat{\nu}_R}{\|\operatorname{cof}(F_L)\hat{\nu}_L + \operatorname{cof}(F_R)\hat{\nu}_R\|}$$

- $\{\bar{\nu}\}$ needs information of two elements the same time
- DG, fixpoint iteration
- $\{\bar{\nu}\} \cdot \bar{\nu}_{el}$ measures angle



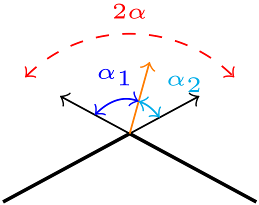
Algorithm

$$\left(\quad \{\bar{\nu}\} \;\; \bar{\nu}_{el} + \beta \hat{\nu}_{el} \right)$$



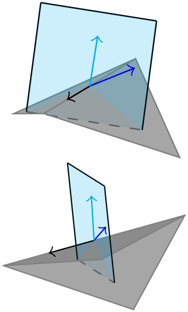
Algorithm

$\Big(\quad \{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \hat{\nu}_{el} \Big)$



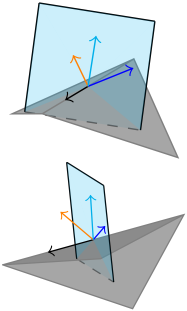
- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step

$$\left(\quad \{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \hat{\nu}_{el} \right)$$



- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step

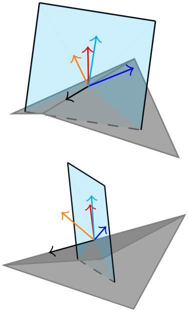
$$(\quad \{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \hat{\nu}_{el})$$



- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step

Algorithm

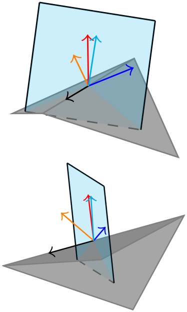
$$\left(P \{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \hat{\nu}_{el} \right)$$



- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step

Algorithm

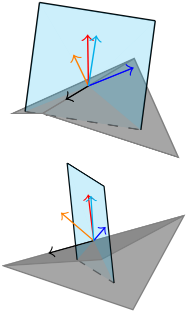
$$\left({\color{red}P}\{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \hat{\nu}_{el} \right)$$



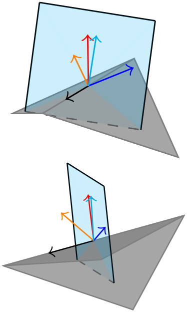
- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step

Algorithm

$$\left(\color{red}{P}\color{black}\{\bar{\nu}\}^n\bar{\nu}_{el} + \beta\hat{\nu}_{el}\right)$$



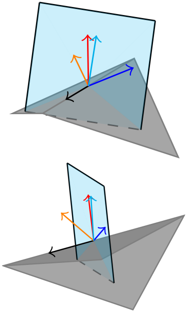
- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step
- Only element wise information needed, but fixpoint iteration



$$\left(P\{\bar{\nu}\}^n\bar{\nu}_{el} + \beta\hat{\nu}_{el}\right)$$

- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step
- Only element wise information needed, but fixpoint iteration
- $\beta\hat{\nu}_{el}$ has physical meaning of correction rotation

$$\frac{1}{\sqrt{1-(\beta\hat{\nu}_{el})^2}}\left(P\{\bar{\nu}\}^n\bar{\nu}_{el}+\beta\hat{\nu}_{el}\right)$$



- Use averaged normal vector $\{\bar{\nu}\}^n$ from last step
- Only element wise information needed, but fixpoint iteration
- $\beta\hat{\nu}_{el}$ has physical meaning of correction rotation
- Add correction terms to avoid iterations

Final algorithm for one loadstep

For given u^n compute

$$\{\bar{\nu}\}^n = Av(u^n).$$

Then find $u \in [H^1(\hat{S})]^d$, $\sigma \in H(\mathrm{divdiv}, \hat{S})^{dc}$ and $\beta \in H(\mathrm{div}, \hat{S})^{bnd}$ for saddle point problem

$$\mathcal{W}_{\{\bar{\nu}\}^n}(u, \sigma, \beta) = \|E_{\tau\tau}(u)\|_M^2 - \frac{1}{2t^2}\|\sigma\|_{M^{-1}}^2 + G_{\{\bar{\nu}\}^n}(u, \sigma, \beta)$$

with

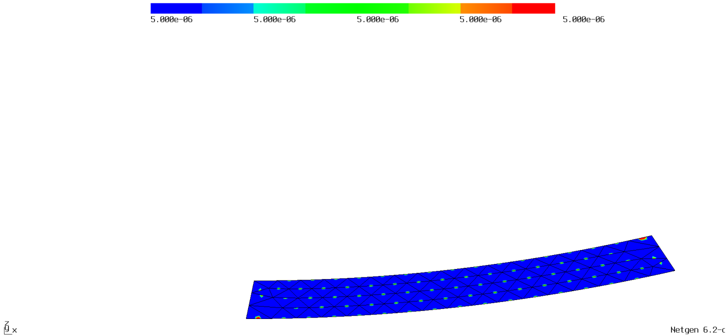
$$\begin{aligned} G_{\{\bar{\nu}\}^n}(u, \sigma, \beta) = &\sum_{\hat{T}} \int_{\hat{T}} \sigma : H_{\bar{\nu}} \, d\hat{x} \\ &- \int_{\partial \hat{T}} \frac{1}{\sqrt{1 - (\beta \nu_{el})^2}} \left(P\{\bar{\nu}\}^n \bar{\nu}_{el} + \beta \nu_{el}\right) \sigma_{\hat{\nu}_{el}\hat{\nu}_{el}} \, d\hat{s}. \end{aligned}$$

Numerical examples

1d example

1d example

2d example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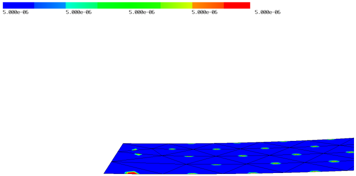
Netgen 6.2-c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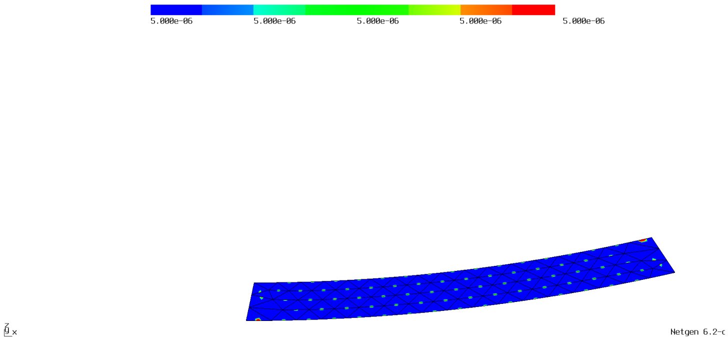
2d example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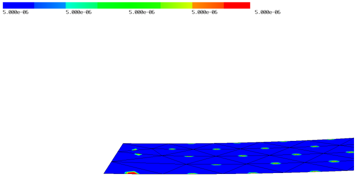
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