Distributional computation of intrinsic curvature with Regge finite elements

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AC2T research GmbH

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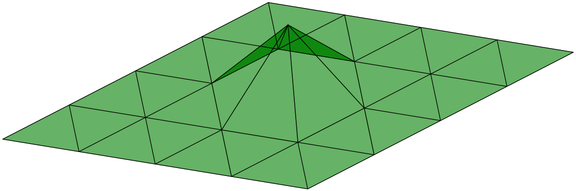
Motivation

Curvature approximation with piece-wise linear spaces



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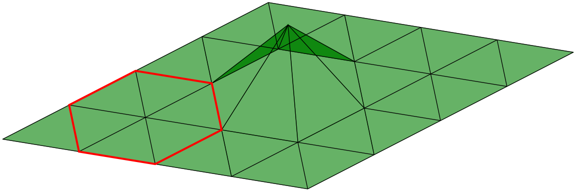

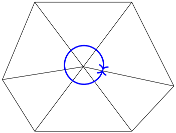

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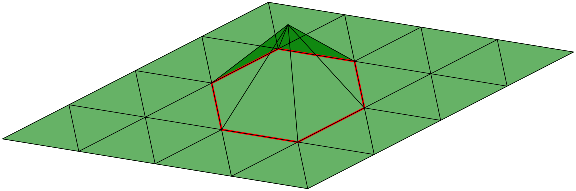
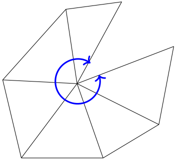
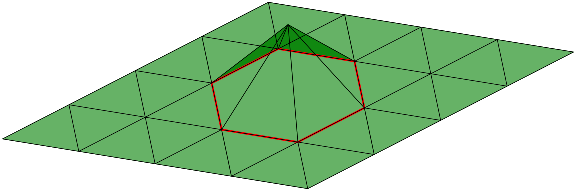
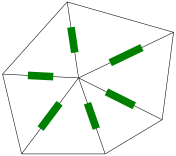




Curvature approximation with piece-wise linear spaces





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Riemannian manifold $(\Omega \subset \mathbb{R}^2, g)$

Curvature (2D)

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Curvature (2D)

Riemannian manifold (M, g)

Levi-Civita connection ∇

Riemann curvature tensor

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{uv-vu}$$

Curvature (2D)

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Riemannian manifold (M, g)

Levi-Civita connection ∇

Christoffel symbols: $\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l$

Riemann curvature tensor

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{uv-vu}$$

$$R_{ijkl} = \left(\frac{\partial}{\partial x_j} \Gamma^p_{ik} - \frac{\partial}{\partial x_i} \Gamma^p_{jk} + \Gamma^q_{ik} \Gamma^l_{jq} - \Gamma^q_{jk} \Gamma^l_{iq} \right) g_{lp}.$$

Curvature (2D)

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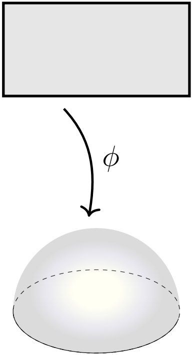
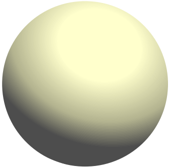
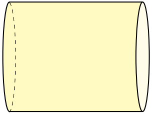


$$g = (D\Phi)' D\Phi, \quad \Gamma^k_{ij}(g) = g^{kl}\frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

Gauss curvature:

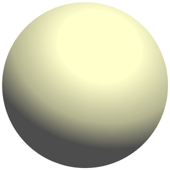
$$K(g) = \frac{g(R(u,v)v,u)}{\|u\|_g \|v\|_g - g(u,v)^2} = \frac{R_{1221}}{\det g}$$





Curvature

Gauss curvature:



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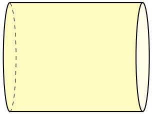
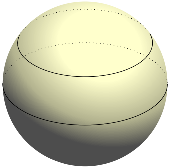


Geodesic curvature:

$$\kappa(g) = g(\nabla_{\hat{t}}\hat{t}, \hat{n}) = \frac{\sqrt{\det g}}{g_{tt}^{3/2}} \left(\partial_t t \cdot n + \Gamma^n_{tt}\right)$$

$$\hat{n} = \hat{t} \times \hat{\nu}$$



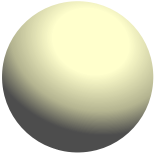
Gauss–Bonnet theorem

Gauss–Bonnet

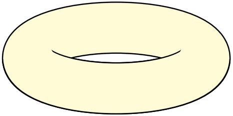
On manifold M:

$$\int_M K(g) + \int_{\partial M} \kappa(g) + \sum_V (\pi - \sphericalangle_V^M(g)) = 2\pi\chi_M$$

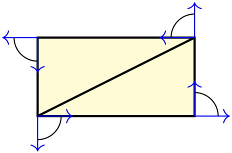
$$\chi_M(\mathcal{T}) = n_V - n_E + n_T$$



$\chi_M = 2$



$\chi_M = 0$



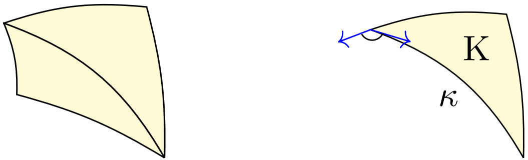
$\chi_M = 1$

Gauss–Bonnet

On triangle T:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle_{V_i}^{T}(g)) = 2\pi$$

$$\chi_T = 3 - 3 + 1 = 1$$



Gauss–Bonnet

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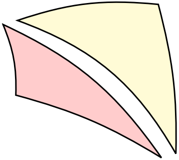
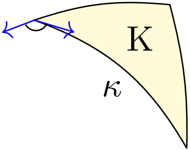




Curvature and Inc operator

Distributional curvature operator

Lifted distributional curvature

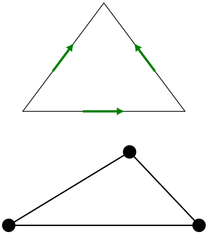
For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $K_h(g) \in V_h^{k+1}$ s.t. for all $\varphi \in V_h^{k+1}$

$$\int_{\mathcal{T}} K_h(g)\varphi = \sum_{T\in\mathcal{T}} \Big(K^T(\varphi,g) + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(\varphi,g) + \sum_{V\in\mathcal{V}_T^{\mathrm{int}}} K_V^T(\varphi,g)\Big)$$

$$K^T(\varphi,g) = \int_T K(g)\,\varphi$$

$$K_E^T(\varphi,g) = \int_E \kappa(g)\varphi$$

$$K_V^T(\varphi,g) = \Big(\sphericalangle_V^T(\delta) - \sphericalangle_V^T(g)\Big)\,\varphi(V)$$



Distributional curvature operator

Lifted distributional curvature

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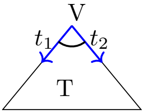

$$K^T(\varphi, g) = \int_T K(g)\,\varphi$$

$$K_E^T(\varphi, g) = \int_E \kappa(g)\varphi$$

$$K_V^T(\varphi, g) = \Big(\sphericalangle_V^T(\delta) - \sphericalangle_V^T(g)\Big)\,\varphi(V)$$

$$\sphericalangle_V^T(g) = \arccos\left(\frac{t_1' g t_2}{\|t_1\|_g \|t_2\|_g}\right)$$

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Lifted distributional curvature

For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $K_h(g) \in V_h^{k+1}$ s.t. for all $\varphi \in V_h^{k+1}$

$$\int_{\mathcal{T}} K_h(g)\varphi = \sum_{T\in\mathcal{T}} \Big(K^T(\varphi, g) + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(\varphi, g) + \sum_{V\in\mathcal{V}_T^{\mathrm{int}}} K_V^T(\varphi, g) \Big)$$

$$\int_{\mathcal{T}} K_h(g)\,\varphi\sqrt{\det g}\,da = \sum_{T\in\mathcal{T}} \Big(\int_T \frac{R_{1221}\,\varphi}{\sqrt{\det g}}\,da$$

$$+ \int_{\partial T\setminus\partial\Omega} \frac{\sqrt{\det g}}{g_{tt}} (\partial_t t \cdot n + \Gamma_{tt}^n)\,\varphi\,dl + \sum_{V\in\mathcal{V}_T^{\mathrm{int}}} K_V^T(\varphi, g) \Big)$$

Distributional curvature operator

Discrete Gauss–Bonnet

For $g \in \mathrm{Reg}_h^k(\mathcal{T})$

$$\int_{\mathcal{T}} K_h(g) + \sum_{E\in\mathcal{E}^{\mathrm{bnd}}} \int_E \kappa(g) + \sum_{V\in\mathcal{V}^{\mathrm{bnd}}} \Big(\pi - \sum_{T\in\mathcal{T}_V} \sphericalangle_V^T(g)\Big) = 2\pi\chi_M$$

Consistency

For any $g \in C^2(M,\mathcal{S})$

$$\int_{\mathcal{T}} K_h(g)\, v = \int_{\mathcal{T}} K(g)\, v, \qquad v \in V_h^k.$$

Distributional incompatibility operator

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1, \ \operatorname{curl} \sigma = \begin{bmatrix} \partial_1 \sigma_{12} - \partial_2 \sigma_{11} \\ \partial_1 \sigma_{22} - \partial_2 \sigma_{21} \end{bmatrix}, \ \operatorname{inc}(g) = \operatorname{curl} \operatorname{curl}(g)$$

$$H^2(\mathcal{T}, \mathbb{S}) = \{ g : \Omega \to \mathbb{S} \ \big| \ g_{ij}|_T \in H^2(T) \}$$

Distributional incompatibility operator

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1, \ \operatorname{curl} \sigma = \begin{bmatrix} \partial_1 \sigma_{12} - \partial_2 \sigma_{11} \\ \partial_1 \sigma_{22} - \partial_2 \sigma_{21} \end{bmatrix}, \ \operatorname{inc}(g) = \operatorname{curl} \operatorname{curl}(g)$$

$$H^2(\mathcal{T}, \mathbb{S}) = \{g : \Omega \to \mathbb{S} \mid g_{ij}|_T \in H^2(T)\}$$
$$\operatorname{Reg}(\mathcal{T}) = \{g \in H^2(\mathcal{T}, \mathbb{S}) \mid g \text{ is tangential-tangential continuous}\}$$
$$\operatorname{Reg}_h^k(\mathcal{T}) = \{g \in \operatorname{Reg}(\mathcal{T}) \mid g_{ij}|_T \in \mathcal{P}^k(T)\}$$

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$$\operatorname{Reg}(\mathcal{T}) = \{ g \in H^2(\mathcal{T}, \mathbb{S}) \mid [\![t' g t]\!]_E = 0 \text{ for all } E \in \mathcal{E}^{\mathrm{int}} \}$$

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$$H(\operatorname{curl} \operatorname{curl}, \Omega) = \{g \in L^2(\Omega, \mathbb{S}) \ \mid \ \operatorname{inc}(g) \in H^{-1}(\Omega)\}$$

Distributional incompatibility operator

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1,\ \operatorname{curl}\sigma = \begin{bmatrix} \partial_1\sigma_{12} - \partial_2\sigma_{11} \\ \partial_1\sigma_{22} - \partial_2\sigma_{21} \end{bmatrix},\ \operatorname{inc}(g) = \operatorname{curl}\operatorname{curl}(g)$$

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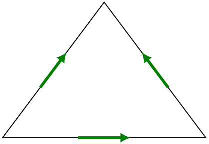
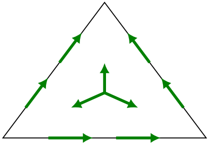
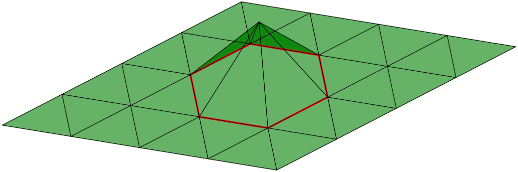
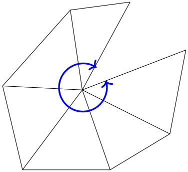


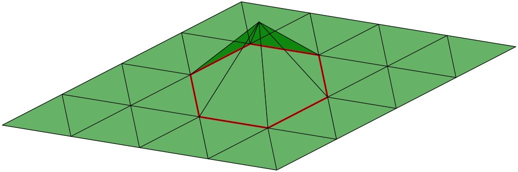


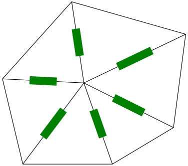
Regge elements





T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*



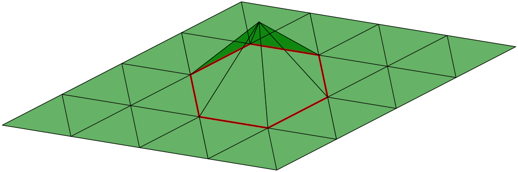


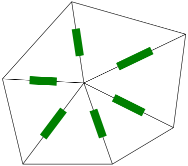
- metric tensor

T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*

Cheeger, Müller, Schrader: On the curvature of piecewise flat spaces *Communications in Mathematical Physics, 92(3) (1984), pp. 405–454*

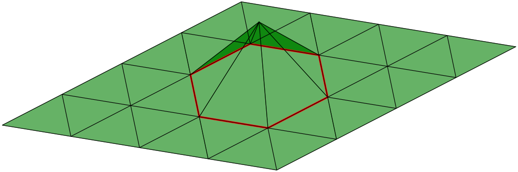
Regge elements

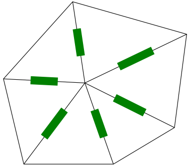




- metric tensor (tangential-tangential continuous)

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*





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L. LI: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*

Distributional incompatibility operator

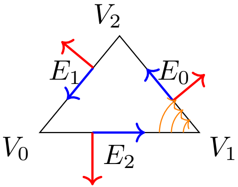
Distributional incompatibility operator

For $\eta \in \mathrm{Reg}(\mathcal{T})$ and $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \mathrm{inc}\,\eta, \varphi \rangle = \sum_{T\in\mathcal{T}} \Bigg(\int_T \mathrm{inc}\,\eta\,\varphi\,da + \int_{\partial T} (\partial_t \eta)_{nt}\,\varphi - (\mathrm{curl}\,\eta)\cdot t\,\varphi$$

$$+ \eta_{nn}\, n\cdot\partial_t t\,\varphi\,dl + \sum_{V\in\mathcal{V}_T} [\![\eta_{nt}]\!]_V^T\,\varphi(V) \Bigg)$$

$$[\![\eta_{nt}]\!]_{V_i}^T = \left(\eta_{nt}|_{E_{i-1}} - \eta_{nt}|_{E_{i+1}}\right)(V_i)$$



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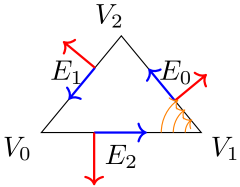

Distributional incompatibility operator

For $\eta \in \mathrm{Reg}(\mathcal{T})$ and $\varphi \in V_h^k$,

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$$+ \eta_{nn}\; n\cdot\partial_t t\,\varphi\; dl + \sum_{V\in\mathcal{V}_T^{\mathrm{int}}} [\![\eta_{nt}]\!]_V^T\,\varphi(V)\Big)$$

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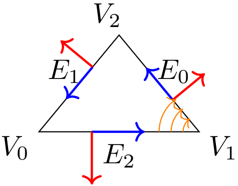


Lifted distributional incompatibility operator

For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $\mathrm{inc}_h(g) \in V_h^{k+1}$ s.t. for all $v \in V_h^{k+1}$

$$\int_\Omega \mathrm{inc}_h(g)\, v \; da = \langle \mathrm{inc}\, g, v \rangle.$$

Lemma

Let $g \in H^2(\Omega)$, $g_h := \mathcal{R}_h^0 g$, and $\mathrm{inc}_h(g_h) \in \mathring{V}_h^k$ the lifted inc. Then there holds

$$\|\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-1}} \leq ch\, \|\mathrm{inc}(g)\|_{L^2},$$
$$\|\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-2}} \leq ch^2\, \|\mathrm{inc}(g)\|_{L^2}.$$

Let $\eta \in H^2(\mathcal{S}, \Omega)$ and define $\eta_D := \operatorname{inc}(\eta)$,
$\eta_N := (\partial_t \eta)_{nt} - \operatorname{curl} \eta \cdot t + \eta_{nn} n \cdot \dot{t}$, and $\eta_V^T := [\![\eta_{nt}^T]\!]_V$.

Lifting with boundary terms

Find $\operatorname{inc}_h g \in V_h^k(\mathcal{T})$ such that $\operatorname{inc}_h g = \eta_D$ on Γ_D and

$$\int_{\mathcal{T}} \operatorname{inc}_h g \, v_h \, da = \langle \operatorname{inc} g, v_h \rangle + \sum_{V \in \mathcal{V}_T} \sum_{V \in \mathcal{V}_T^{\mathrm{bnd}} \cap \Gamma_N} ([\![g_{nt}]\!]_V^T - \eta_V^T) v_h(V)$$
$$+ \int_{\Gamma_N} ((\partial_t g)_{nt} - \operatorname{curl} g \cdot t + g_{nn} n \cdot \dot{t} - \eta_N) v_h \, dl$$

for all $v_h \in V_{h,\Gamma_D}^k(\mathcal{T}) := \{w \in V_h^k(\mathcal{T}) : w|_{\Gamma_D} = 0\}$

Let $\eta \in H^2(\mathcal{S},\Omega)$ and define $\eta_D := \mathrm{inc}(\eta)$,
$\eta_N := (\partial_t \eta)_{nt} - \mathrm{curl}\,\eta \cdot t + \eta_{nn} n \cdot \dot{t}$, and $\eta_V^T := [\![\eta_{nt}^T]\!]_V$.

Optimal rates

Let $k \in \mathbb{N}$, $g \in H^{4+k}(\Omega,\mathbb{S})$, and $g_h = \mathcal{R}_h^k g$ its Regge interpolant. Further take $\mathrm{inc}_h(g_h) \in V_h^{k+1}$ as lifted inc with η_D, η_N, and η_V^T computed in terms of g. Then

$$\Big(h^2\|\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{L^2} + h\|\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-1}} \\ + \|\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-2}}\Big) \leq h^{4+k}\|\mathrm{inc}(g)\|_{H^{2+k}}.$$

Let $\eta \in H^2(\mathcal{S},\Omega)$ and define $\eta_D := \mathrm{inc}(\eta)$,
$\eta_N := (\partial_t \eta)_{nt} - \mathrm{curl}\,\eta \cdot t + \eta_{nn} n \cdot \dot{t}$, and $\eta_V^T := [\![\eta_{nt}^T]\!]_V$.

Optimal rates

Let $k = 0$, $g \in H^4(\Omega, \mathbb{S})$, and $g_h = \mathcal{R}_h^0 g$ its Regge interpolant. Further take $\mathrm{inc}_h(g_h) \in V_h^1$ as lifted inc with η_D, η_N, and η_V^T computed in terms of g. Then

$$\big(h^2 \|\,\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{L^2} + h\|\,\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-1}} \\ + \|\,\mathrm{inc}(g) - \mathrm{inc}_h(g_h)\|_{H^{-2}}\big) \leq h^4 \|\,\mathrm{inc}(g)\|_{H^2}.$$

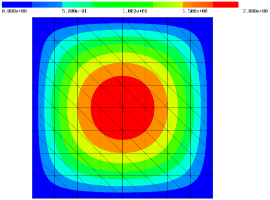
Linearization

Let $g \in \mathrm{Reg}_h^k(\mathcal{T})$ and $g = \delta + \eta + \mathcal{O}(\varepsilon^2)$ with $\eta = \mathcal{O}(\varepsilon)$, $\partial_i g = \mathcal{O}(\varepsilon)$, and $\partial_{ij}^2 g = \mathcal{O}(\varepsilon)$. Then, there holds with $\varphi \in V_h^{k+1}$

$$\int_{\mathcal{T}} K_h(g)\varphi = \frac{1}{2}\int_{\mathcal{T}} \mathrm{inc}_h(g)\varphi + \mathcal{O}(\varepsilon^2) \text{ for } \varepsilon \to 0.$$

- S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*
- E. S. Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, *SIAM 119, 58(3) (2020), pp. 1801–1821.*

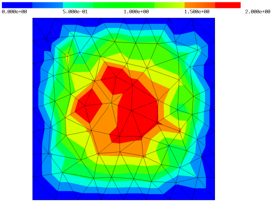
Numerical examples (inc)

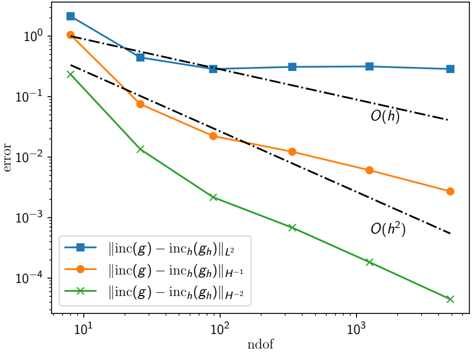


$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ \frac{x^2+y^2}{2} - \frac{x^4+y^4}{12} \end{bmatrix}$$

$$g = (D\Phi)' D\Phi$$

Numerical examples (inc)





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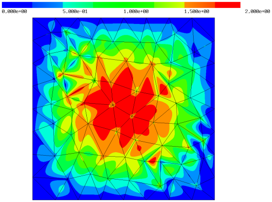

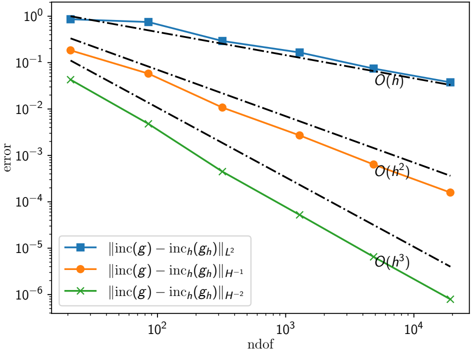

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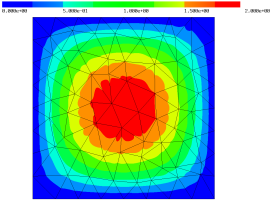

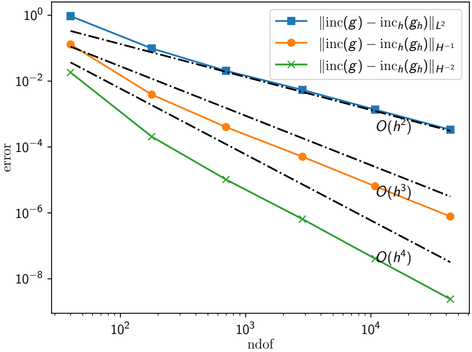

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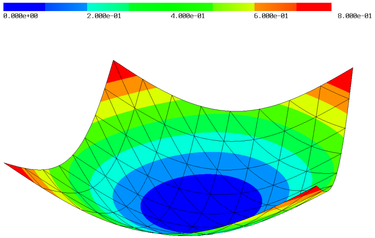
Numerical examples (curvature)

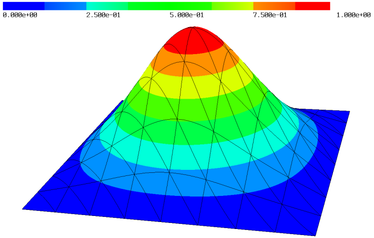
$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ \frac{x^2+y^2}{2} - \frac{x^4+y^4}{12} \end{bmatrix}$$

$$g = (D\Phi)' D\Phi$$

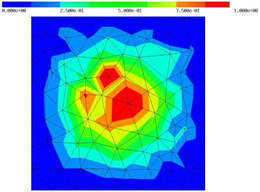
$$K(g) =$$

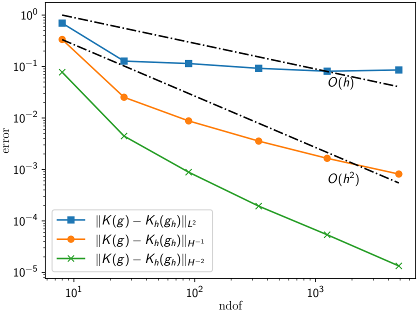
$$\frac{81(1-x^2)(1-y^2)}{(9+x^2(x^2-3)^2+y^2(y^2-3)^2)^2}$$





Numerical examples (curvature)





$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ \frac{x^2+y^2}{2} - \frac{x^4+y^4}{12} \end{bmatrix}$$

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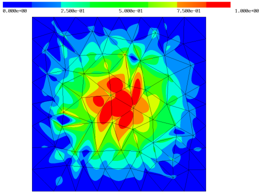
GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements

Numerical examples (curvature)



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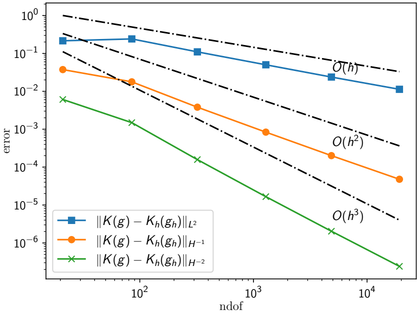
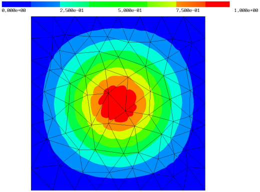
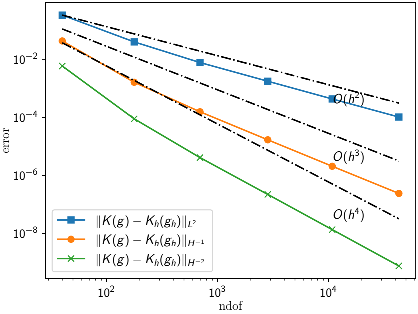


GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements

Numerical examples (curvature)





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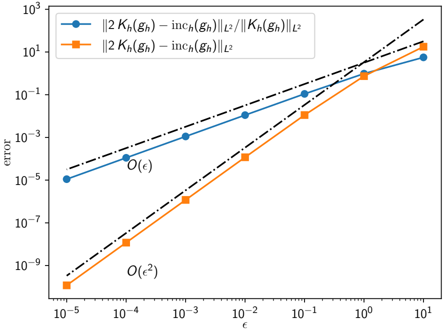
GAWLIK: High-Order Approximation of Gaussian Curvature with Regge Finite Elements

Numerical example (linearization)

$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ \sqrt{\epsilon}\left(\frac{x^2+y^2}{2} - \frac{x^4+y^4}{12}\right) \end{bmatrix}$$

$$g = (D\Phi)' D\Phi = \delta + \mathcal{O}(\varepsilon)$$

$$K(g) = \mathcal{O}(\varepsilon)$$



Extension to 3D

Extension to 3D

- Riemann curvature tensor R_{ijkl} has 6 independent entries
- Curvature operator $Q : M \to \mathbb{S}$

$$\langle Q(u \times v), w \times z \rangle = \langle R(u,v)z, w \rangle \text{ for all } u, v, w, z \in \mathbb{R}^3$$

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- Motivation:
  - Einstein field equation in general relativity
  - Generalize Regge calculus

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- Motivation:
  - Einstein field equation in general relativity
  - Generalize Regge calculus
- No Gauss–Bonnet theorem in 3D

Lifted distributional curvature

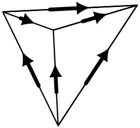
For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $Q_h(g) \in \mathrm{Reg}_h^k(\mathcal{T})$ s.t. for all $v \in \mathrm{Reg}_h^k(\mathcal{T})$

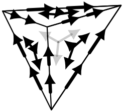
$$\int_{\mathcal{T}} Q_h(g) : v = \sum_{T\in\mathcal{T}} \Big(K^T(v,g) + \sum_{F\in\mathcal{F}_T^{\mathrm{int}}} K_F^T(v,g) + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(v,g) \Big)$$

$$K^T(v,g) = \int_T Q(g) : v$$

$$K_F^T(v,g) = \int_F {\color{red}?} : v$$

$$K_E^T(v,g) = \left(\sphericalangle_E^T(\delta) - \sphericalangle_E^T(g) \right) v_{t_E t_E}$$





Curvature operator (3D)

Lifted distributional curvature

For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $Q_h(g) \in \mathrm{Reg}_h^k(\mathcal{T})$ s.t. for all $v \in \mathrm{Reg}_h^k(\mathcal{T})$

$$\int_{\mathcal{T}} Q_h(g) : v = \sum_{T\in\mathcal{T}} \Big(K^T(v,g) + \sum_{F\in\mathcal{F}_T^{\mathrm{int}}} K_F^T(v,g) + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(v,g) \Big)$$

$$\int_{\mathcal{T}} Q_h(g) : v\sqrt{\det g}\, dx = \sum_{T\in\mathcal{T}} \Big(\int_T \frac{Q(g):v}{\sqrt{\det g}}\, dx + \int_{\partial T\setminus\partial\Omega} \frac{\sqrt{\det g}}{\mathrm{cof}(g)_{nn}} ((n\otimes n)\times \Gamma^n_{\cdot\cdot}) : v\, da + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(v,g) \Big)$$

$$\mathrm{cof}(A) = \det(A)A^{-\prime}, \quad (A\times B)_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}A_{km}B_{ln}$$

Curvature operator (3D)

Lifted distributional curvature

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$$\int_{\mathcal{T}} Q_h(g) : v \sqrt{\det g}\, dx = \sum_{T\in\mathcal{T}} \Big(\int_T \frac{Q(g) : v}{\sqrt{\det g}}\, dx$$

$$+ \int_{\partial T\setminus\partial\Omega} \frac{\sqrt{\det g}}{\mathrm{cof}(g)_{nn}} ((n\otimes n)\times \Gamma^n_{\cdot\cdot}) : v\, da + \sum_{E\in\mathcal{E}_T^{\mathrm{int}}} K_E^T(v,g) \Big)$$

$$2\mathrm{D} : \int_{\partial T\setminus\partial\Omega} \frac{\sqrt{\det g}}{g_{tt}} \Gamma^n_{tt}\, v\, dl$$

Computation of boundary term via “Gauss–Bonnet”?

- In 2D with Stokes' theorem

$$2\pi - \int_\gamma \kappa_g = \int_{\partial R} \omega_2^1 = \int_R d\,\omega_2^1 = \int_R K\,\theta^1 \wedge \theta^2 = \int_R K \,\mathrm{vol}$$

ω_2^1, Ω_2^1 connection and curvature form
structural equation: $\Omega_2^1 = d\,\omega_2^1 = K\,\theta^1 \wedge \theta^2$

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$$2\pi - \int_\gamma \kappa_g = \int_{\partial R} \omega^1_2 = \int_R d\,\omega^1_2 = \int_R K\,\theta^1 \wedge \theta^2 = \int_R K\,\mathrm{vol}$$

  ω^1_2, Ω^1_2 connection and curvature form
  structural equation: $\Omega^1_2 = d\,\omega^1_2 = K\,\theta^1 \wedge \theta^2$

- 3D: $Q \in C^1(R, \mathbb{S})$
  Bianchi: $\mathrm{div}\,Q = 0 = d\,Q \Rightarrow \exists q \in C^1(R, \mathbb{S}) :\ d\,q = Q$

$$\int_R Q\,\mathrm{vol} = \int_{\partial R} q \overset{?}{=} \int_{\partial\Omega} \frac{\sqrt{\det g}}{\mathrm{cof}(g)_{nn}} ((n \otimes n) \times \Gamma^{n}_{\cdot\cdot})\, da$$

Distributional inc

For $g \in \mathrm{Reg}_h^k$, $\varphi \in \mathcal{D}(\Omega, \mathbb{S})$, and $\mathrm{inc}(g) = \mathrm{curl}((\mathrm{curl}\, g)')$

$$\langle \mathrm{inc}\, g, \varphi \rangle = \sum_{T \in \mathcal{T}} \int_T \mathrm{inc}(g) : \varphi \, dx + \int_{\partial T \setminus \partial\Omega} (n \times \varphi)_{FF} : \mathrm{curl}(g)_{FF}^{\top}$$
$$+ \mathrm{rot}_F((n \times g)_{Fn}) : \varphi_{FF} \, da - \sum_{E \in \mathcal{E}^{\mathrm{int}}} \int_E [\![g_{Fn}]\!]_E^T \varphi_{t_E t_E} \, dl$$

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

Hauret, Hecht: A Discrete Differential Sequence for Elasticity Based upon Continuous Displacements, *SIAM 119, 35(1) (2013), pp. B291–B314.*

Incompatibility operator (3D)

Distributional inc

For $g \in \mathrm{Reg}_h^k$, $\varphi \in \mathrm{Reg}_h^k$, and $\mathrm{inc}(g) = \mathrm{curl}((\mathrm{curl}\, g)')$

$$\langle \mathrm{inc}\, g, \varphi \rangle = \sum_{T \in \mathcal{T}} \int_T \mathrm{inc}(g) : \varphi \, dx + \int_{\partial T \setminus \partial\Omega} (n \times \varphi)_{FF} : \mathrm{curl}(g)_{FF}^\top + \mathrm{rot}_F((n \times g)_{Fn}) : \varphi_{FF} \, da - \sum_{E \in \mathcal{E}^{\mathrm{int}}} \int_E [\![g_{Fn}]\!]_E^T \varphi_{t_E t_E} \, dl$$

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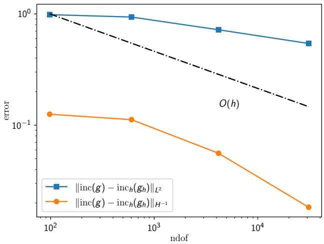
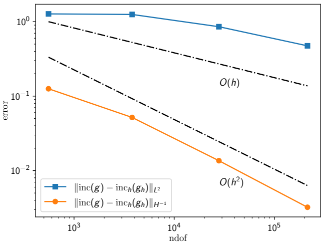
Linearization (3D)

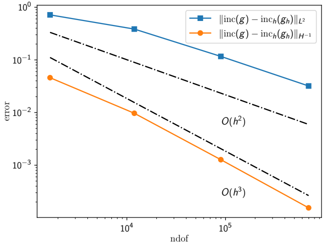
Let $g \in \mathrm{Reg}_h^k(\mathcal{T})$ and $g = \delta + \eta + \mathcal{O}(\varepsilon^2)$ with $\eta = \mathcal{O}(\varepsilon)$, $\partial_i g = \mathcal{O}(\varepsilon)$, and $\partial_{ij}^2 g = \mathcal{O}(\varepsilon)$. Then, there holds with $v \in \mathrm{Reg}_h^k(\mathcal{T})$

$$\int_{\mathcal{T}} Q_h(g) : v = \frac{1}{2} \int_{\mathcal{T}} \mathrm{inc}_h(g) : v + \mathcal{O}(\varepsilon^2) \text{ for } \varepsilon \to 0.$$

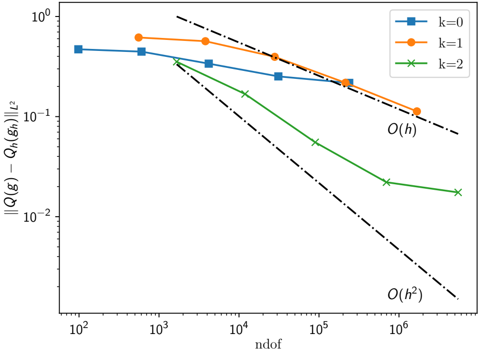
Numerical examples (3D inc)



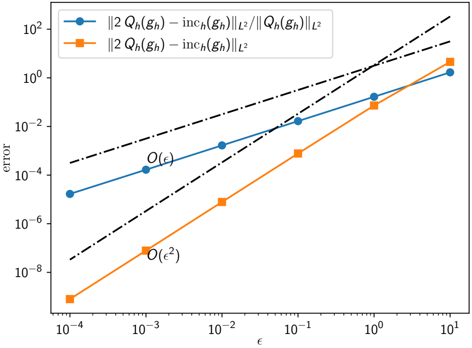




Numerical examples (3D curvature)



Numerical examples (3D linearization)



To prove

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For $g \in C^2(\Omega, \mathbb{S})$, $v \in C_0^\infty(\Omega)$, $g_h = \mathcal{R}_h^0 g$, $v_h = \mathcal{I}_h^1 v$.

$$|\langle K(g) - K_h(g_h), v_h\rangle| \leq \begin{cases} c(g) h \|v\|_{H^1} \\ c(g) h^2 \|v\|_{H^2} \end{cases}$$

Sense of measures (linear) with Lipschitz Killing fields

Gawlik: $\frac{d}{dt}\big|_{t=0} \langle \kappa(g + t\sigma), v\rangle_{g+t\sigma} = 0.5 \langle \mathrm{div}_g \mathrm{div}_g S_g \sigma, v\rangle_g$

To prove

For $g \in C^2(\Omega, \mathbb{S})$, $v \in C_0^\infty(\Omega)$, $g_h = \mathcal{R}_h^0 g$, $v_h = \mathcal{I}_h^1 v$.

$$|\langle K(g) - K_h(g_h), v_h\rangle| \leq \begin{cases} c(g)h\|v\|_{H^1} \\ c(g)h^2\|v\|_{H^2} \end{cases}$$

$$\begin{aligned}\langle K(g) - K_h(g_h), v_h\rangle &= \sum_{i=1}^N \alpha_i \langle K(g) - K_h(g_h), \varphi_i\rangle \\ &= \sum_{i=1}^N \alpha_i \left(\int_\Omega K(g)\varphi_i \, dx - (2\pi - \sum_{T \in \mathcal{T}_{V_i}} \sphericalangle_{V_i}^T(g_h)) \right)\end{aligned}$$

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Summary

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- Lifting of distributional curvature
- Relation to distributional incompatibility operator
- Optimal (numerical) convergence rates

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Thank You for Your attention!