Distributional curvatures on discrete surfaces with application to shells

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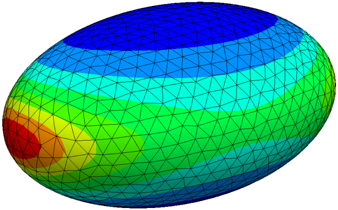
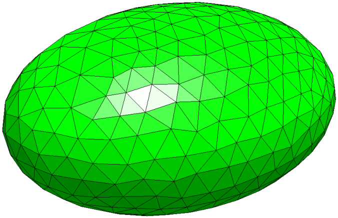
17th Austrian Numerical Analysis Day, Vienna, April 28th, 2023

Motivation

Approximate extrinsic/intrinsic curvature of non-smooth surfaces





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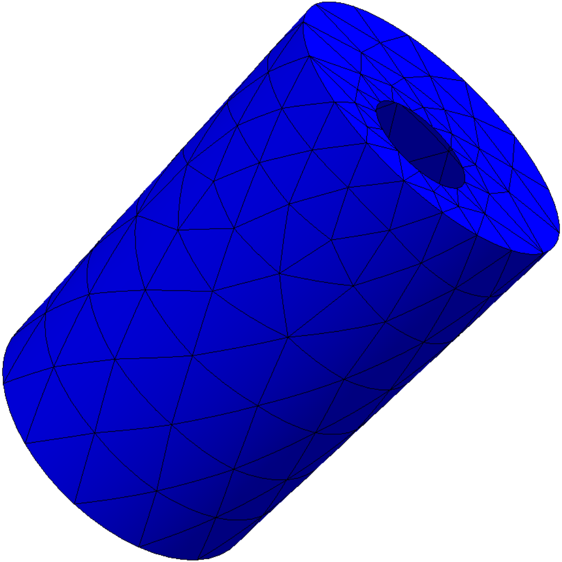

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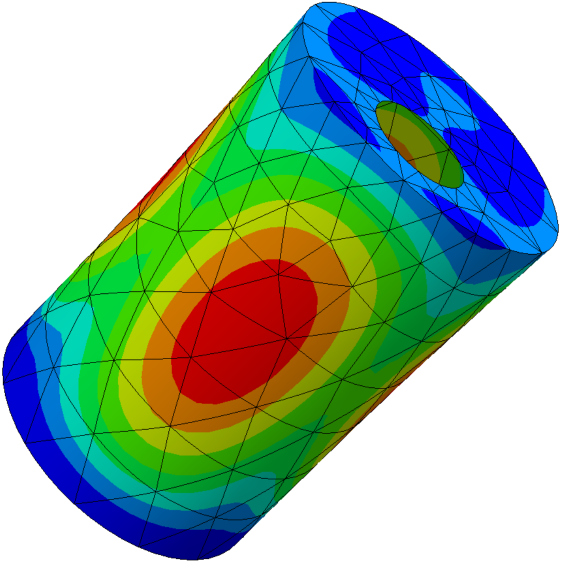

Contents

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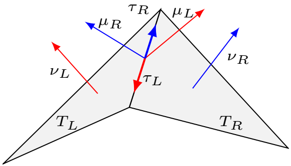
Distributional extrinsic and intrinsic curvature

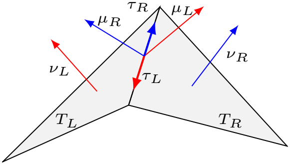
- Normal vector ν

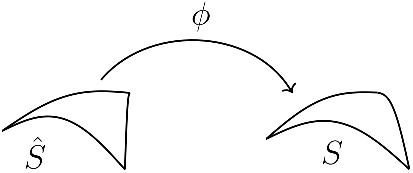
  Tangent vector τ

  Element normal vector $\mu = \nu \times \tau$



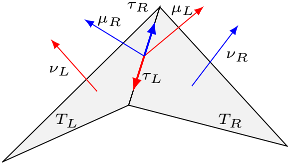
- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

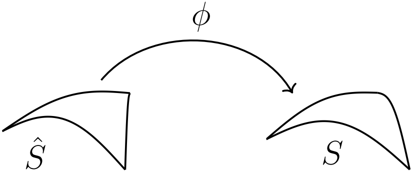




- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

- $\boldsymbol{F} = \nabla_{\hat{\tau}} \phi$, $J = \sqrt{\det(\boldsymbol{F}^\top \boldsymbol{F})}$





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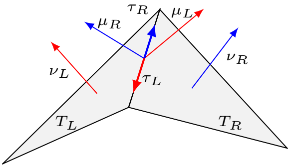
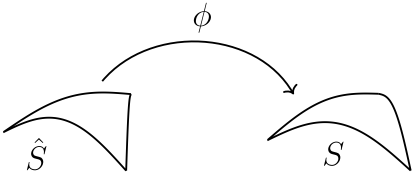
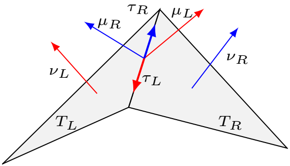


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$

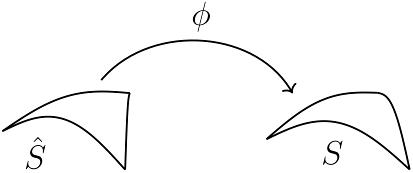


Differential geometry

- Normal vector $\hat{\nu}$
  Tangent vector $\hat{\tau}$
  Element normal vector $\hat{\mu} = \hat{\nu} \times \hat{\tau}$

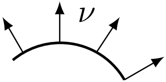


- $\boldsymbol{F} = \nabla_{\hat{\tau}}\phi$, $J = \|\operatorname{cof}(\boldsymbol{F})\|_F$
- $\nu \circ \phi = \frac{1}{J}\operatorname{cof}(\boldsymbol{F})\hat{\nu}$
  $\tau \circ \phi = \frac{1}{J_B}\boldsymbol{F}\hat{\tau}$
  $$\mu \circ \phi = \nu \circ \phi \times \tau \circ \phi = \frac{(\boldsymbol{F}^\dagger)^\top \hat{\mu}}{\|(\boldsymbol{F}^\dagger)^\top \hat{\mu}\|}$$

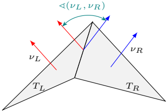


Distributional extrinsic curvature

- Change of normal vector measures curvature $\nabla\nu$

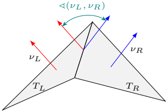


- Change of normal vector measures curvature $\nabla\nu$
- How to define $\nabla\nu$ for discrete surface?



Grinspun, Gingold, Reisman and Zorin: Computing discrete shape operators on general meshes, *Computer Graphics Forum 25*, 3 (2006), pp. 547–556.

Distributional extrinsic curvature



- Change of normal vector measures curvature $\nabla \nu$
- How to define $\nabla \nu$ for discrete surface?
  - Distributional Weingarten tensor

$$\langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}} = \sum_{T \in \mathscr{T}_h} \int_T \nabla \nu|_T : \boldsymbol{\sigma} \, dx + \sum_{E \in \mathcal{E}_h} \int_E \sphericalangle(\nu_L, \nu_R) \boldsymbol{\sigma}_{\mu\mu} \, ds$$

  - Measure jump of normal vector
  - Test function $\boldsymbol{\sigma}$ symmetric, normal-normal continuous $\Rightarrow$ Hellan–Herrmann–Johnson finite elements

N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *arXiv:2107.13794*.

$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, \nabla u \in [L^2(\Omega)]^d\}$$

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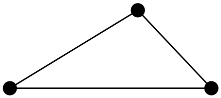
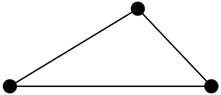
$$V_h^k := \mathcal{P}^k(\mathscr{T}_h) \cap C(\Omega)$$

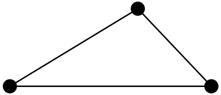


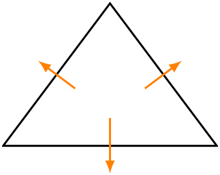
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathbf{div}\sigma \in L^2(\Omega)\}$$



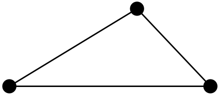
$$H(\mathrm{div}) := \{\sigma \in [L^2(\Omega)]^d \mid \mathbf{div}\sigma \in L^2(\Omega)\}$$

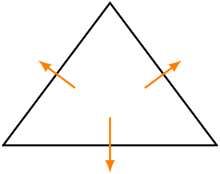
$BDM^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^d \mid \sigma_n$ is continuous over elements$\}$





$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \mid \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$

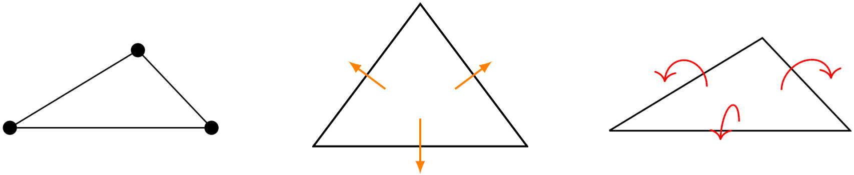




Hellan–Herrmann–Johnson finite elements

$$H(\mathrm{divdiv}) := \{\sigma \in [L^2(\Omega)]^{d\times d}_{sym} \,|\, \mathbf{divdiv}\sigma \in H^{-1}(\Omega)\}$$

$$M_h^k := \{\sigma \in [\mathcal{P}^k(\mathscr{T}_h)]^{d\times d}_{sym} \,|\, n^T \sigma n \text{ is continuous over elements}\}$$

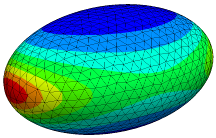


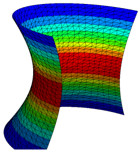
A. Pechstein and J. Schöberl: The TDNNS method for Reissner-Mindlin plates, *J. Numer. Math. (2017) 137, pp. 713-740.*

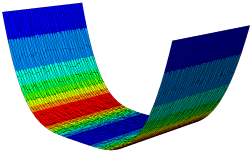
Lifting of distributional Weingarten tensor

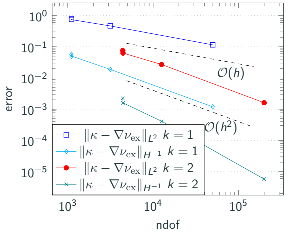
Find $\boldsymbol{\kappa} \in M_h^{k-1}$ for $\mathscr{T}_h$ curving order k s.t. for all $\boldsymbol{\sigma} \in M_h^{k-1}$

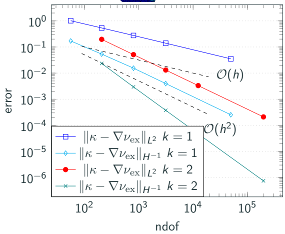
$$\int_{\mathscr{T}_h} \boldsymbol{\kappa} : \boldsymbol{\sigma} \, dx = \langle \nabla \nu, \boldsymbol{\sigma} \rangle_{\mathscr{T}}$$

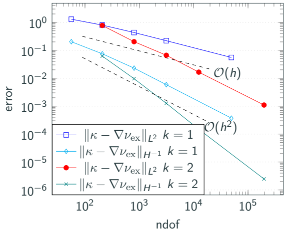




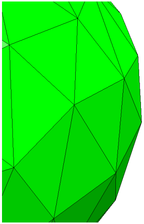




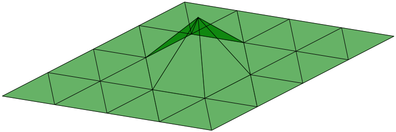




Gauss Theorema Egregium: Gauss curvature depends on metric



angle defect (DDG, Regge calculus)
metric $g = \nabla\Phi^\top\nabla\Phi$



REGGE: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

Distributional intrinsic curvature

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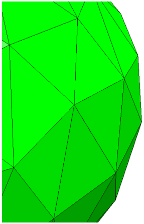

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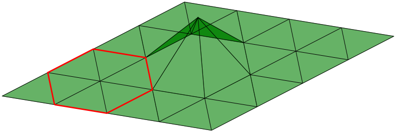

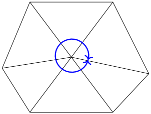

Regge: General relativity without coordinates, *Il Nuovo Cimento* (1955-1965), 19 (1961).

Distributional intrinsic curvature

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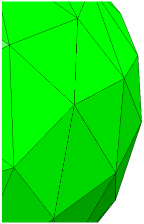

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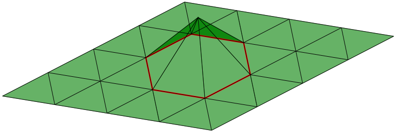

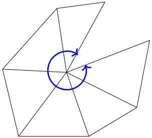

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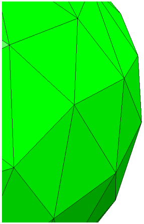

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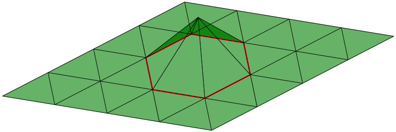
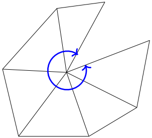
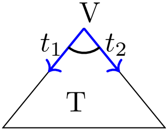


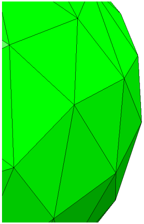




Let $g \in \mathrm{Reg}_h^k(\mathscr{T})$ and $\varphi \in V_h^{k+1}$

$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}} K_V(\varphi, g), \quad K_V(\varphi, g) = (2\pi - \sum_{T:V\subset T} \sphericalangle_V^T(g))\varphi(V)$$

Gauss Theorema Egregium: Gauss curvature depends on metric



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metric $g = \nabla\Phi^\top\nabla\Phi$

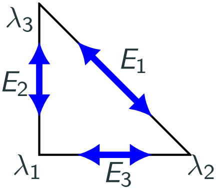
$\kappa_g = g(\nabla_\tau \tau, \mu)$

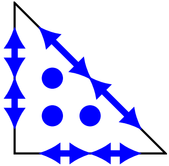
Let $g \in \mathrm{Reg}_h^k(\mathscr{T})$ and $\varphi \in V_h^{k+1}$

$$\langle (K\omega)(g), \varphi\rangle = \sum_{V\in\mathscr{V}} K_V(\varphi, g) + \int_{\mathscr{T}_h} K(g)\,\varphi\,\omega_T + \sum_{E\in\mathscr{E}} \int_E [\![\kappa(g)]\!]\varphi\,\omega_E$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *Found Comput Math* (2022).

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathscr{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$





$$\varphi_{E_i} = \nabla\lambda_j \odot \nabla\lambda_k, \qquad t_j^\top \varphi_{E_i} t_j = c_i \delta_{ij}, \qquad \varphi_{T_i} = \lambda_i \nabla\lambda_j \odot \nabla\lambda_k$$

$\mathcal{R}_h^k : C^0(\Omega) \to \mathrm{Reg}_h^k$ canonical interpolant

$$\int_E (g - \mathcal{R}_h^k g)_{tt}\, q\, dl = 0 \text{ for all } q \in \mathcal{P}^k(E)$$

$$\int_T (g - \mathcal{R}_h^k g) : Q\, da = 0 \text{ for all } Q \in \mathcal{P}^{k-1}(T, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

Lifting of distributional Gauss curvature

For $g \in \mathrm{Reg}_h^k$ find $K_h \in V_h^{k+1}$ such that for all $\varphi \in V_h^{k+1}$

$$\int_\Omega K_h \varphi\, \omega = \langle (K\omega)(g), \varphi \rangle.$$

Theorem (Gopalakrishnan, N., Schöberl, Wardetzky 2022)

Let $g_h = \mathcal{R}_h^k g \in \mathrm{Reg}_h^k$, $-1 \leq l \leq k-1$

$$\|K_h - K\|_{H_h^l} \leq C\, h^{-l+k} \left(|g|_{W^{k+1,\infty}} + |K|_{H^k} \right)$$

Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.

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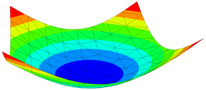
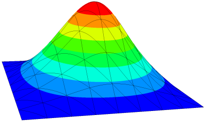


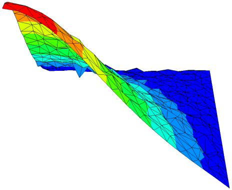


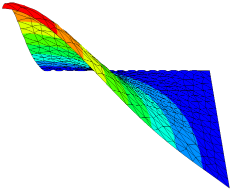
NGSolve

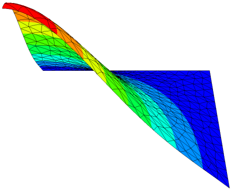
$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \qquad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

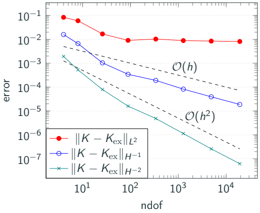
$$K(g) = \frac{81(1 - x^2)(1 - y^2)}{(9 + x^2(x^2 - 3)^2 + y^2(y^2 - 3)^2)^2}$$

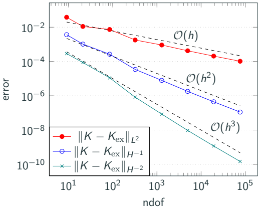
Analysis and example

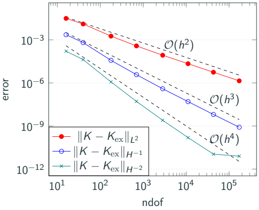












$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$

Nonlinear shells

Koiter shell

$$\mathcal{W}(u) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{24}\|\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu}\|^2_{\boldsymbol{M}}$$

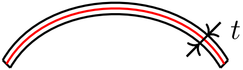
u . . . displacement of mid-surface

t . . . thickness

$\boldsymbol{M}$ . . . material tensor

$$\boldsymbol{F} = \nabla u + \boldsymbol{P} = \nabla\phi, \qquad \boldsymbol{P} = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{P}) = \frac{1}{2}(\nabla u^\top\nabla u + \nabla u^\top\boldsymbol{P} + \boldsymbol{P}\nabla u)$$



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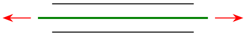

membrane energy

u . . . displacement of mid-surface

t . . . thickness

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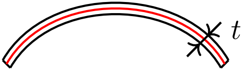

Koiter shell

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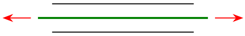
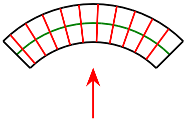
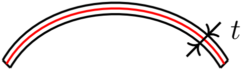


membrane energy



bending energy



Distributional curvature for Koiter shell

- Lifted curvature difference $\boldsymbol{\kappa}^{\mathrm{diff}}$ via three-field formulation

$$\mathcal{L}(u, \boldsymbol{\kappa}^{\mathrm{diff}}, \boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} + \frac{t^3}{12}\|\boldsymbol{\kappa}^{\mathrm{diff}}\|^2_{\boldsymbol{M}} - \langle f, u\rangle$$
$$+ \sum_{T\in\mathcal{T}_h}\int_T \left(\boldsymbol{\kappa}^{\mathrm{diff}} - (\boldsymbol{F}^T\nabla(\nu\circ\phi) - \nabla\hat{\nu})\right) : \boldsymbol{\sigma}\, dx$$
$$+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds$$

- Lagrange parameter $\boldsymbol{\sigma} \in M_h^k$ moment tensor
- Eliminate $\boldsymbol{\kappa}^{\mathrm{diff}} \to$ two-field formulation in $(u, \boldsymbol{\sigma})$

N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.

Formulation: Koiter shell

Shell problem

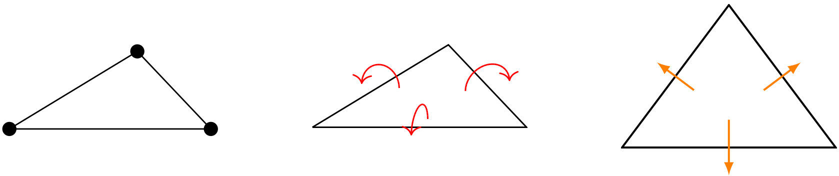
Find $u \in [V_h^k]^3$ and $\boldsymbol{\sigma} \in M_h^{k-1}$ for $(\boldsymbol{H}_\nu := \sum_i (\nabla^2 u_i)\nu_i)$

$$
\begin{aligned}
\mathcal{L}(u,\boldsymbol{\sigma}) =& \frac{t}{2}\|\boldsymbol{E}(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} - \langle f, u\rangle \\
&+ \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\sigma} : (\boldsymbol{H}_\nu + (1-\hat{\nu}\cdot\nu)\nabla\hat{\nu})\,dx \\
&+ \sum_{E\in\mathcal{E}_h}\int_E (\sphericalangle(\nu_L,\nu_R) - \sphericalangle(\hat{\nu}_L,\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\,ds
\end{aligned}
$$

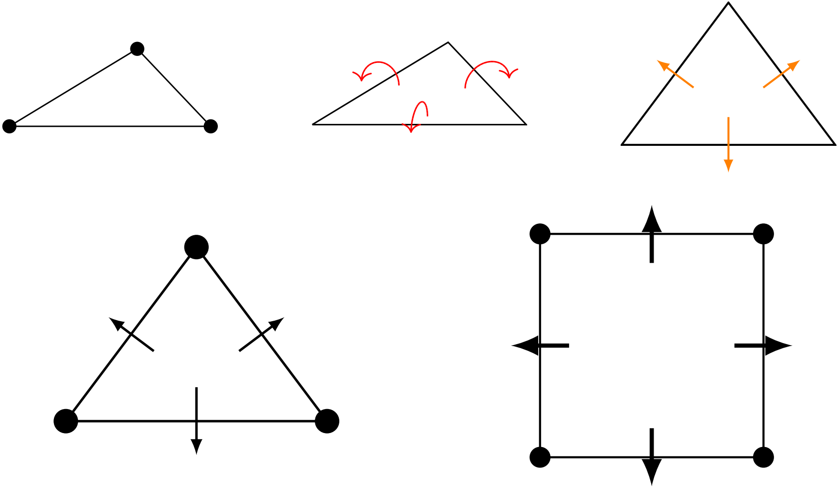
Use hybridization to eliminate $\boldsymbol{\sigma}$ $\rightarrow$ recover minimization problem

N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*

Shell element



Shell element



Membrane locking

Membrane locking

$$\mathcal{W}(u) = t\,E_{\text{mem}}(u) + t^3\,E_{\text{bend}}(u) - f \cdot u, \qquad f = t^3 \tilde{f}$$

Membrane locking

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$$\mathcal{W}(u) = t^{-2} E_{\text{mem}}(u) + E_{\text{bend}}(u) - \tilde{f} \cdot u, \qquad f = t^3 \tilde{f}$$

Enforces $E_{\text{mem}}(u) = 0$ in the limit $t \to 0$

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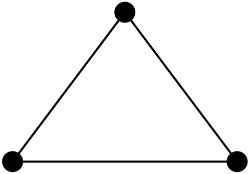

$$V_h = \mathcal{P}(\mathscr{T}_h) \cap C(\Omega) \subset H^1(\Omega)$$

Membrane locking

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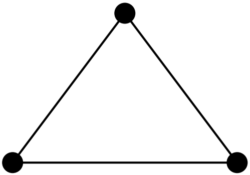
$$E_{\text{mem}}(u) = 0 \quad \not\Rightarrow \quad E_{\text{mem}}(u_h) = 0$$



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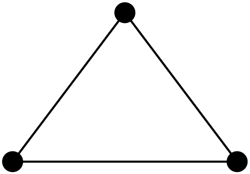

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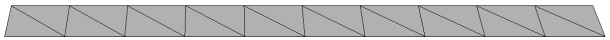
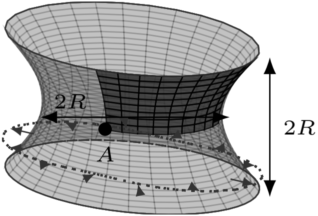
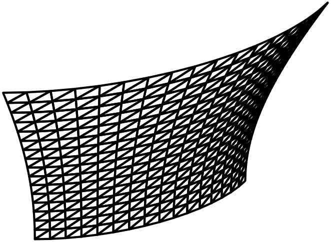
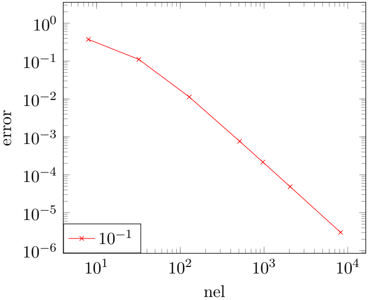


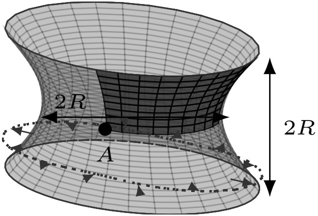
Hyperboloid with free ends

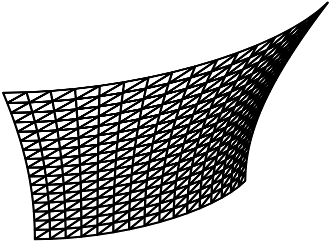


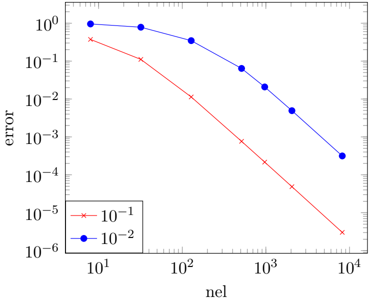




Hyperboloid with free ends

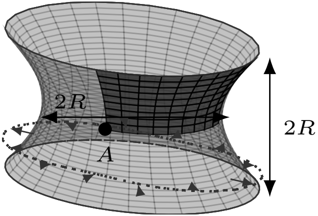


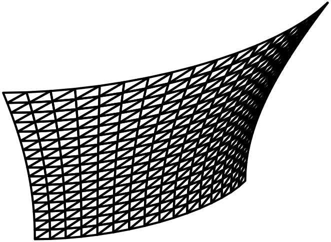


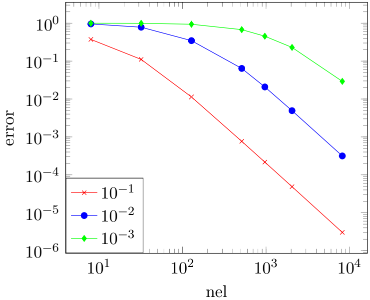


- Pre-asymptotic regime

Hyperboloid with free ends

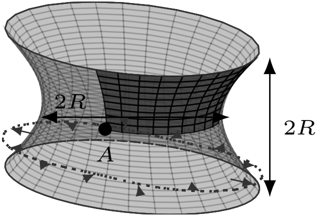


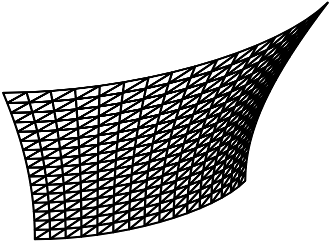


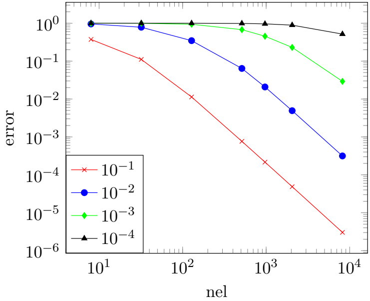


- Pre-asymptotic regime

Hyperboloid with free ends





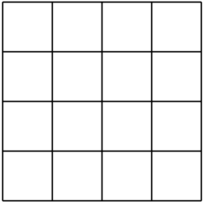


- Pre-asymptotic regime

Membrane locking

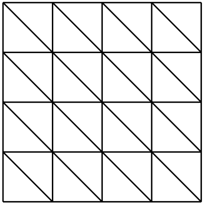
$$\frac{1}{t^2}\| \quad \boldsymbol{E}(u_h)\|^2_{\boldsymbol{M}}$$

$$\frac{1}{t^2}\|\Pi^k_{L^2}\boldsymbol{E}(u_h)\|^2_{\boldsymbol{M}}$$

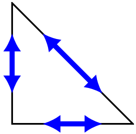


- Reduced integration for quadrilateral meshes

$$\frac{1}{t^2}\|\mathcal{I}_{\mathcal{R}}^{k}\boldsymbol{E}(u_h)\|_{\boldsymbol{M}}^2$$

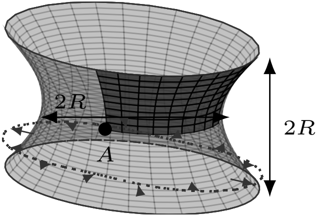


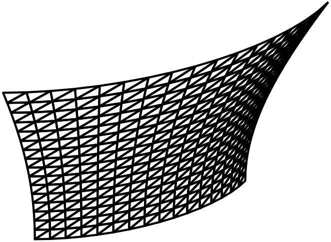
- Reduced integration for quadrilateral meshes
- Regge interpolant for triangles
- Connection to MITC shell elements

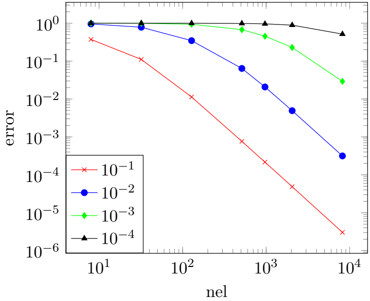


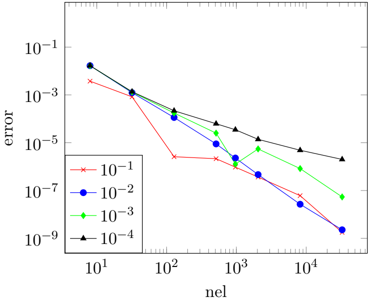
N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*

Hyperboloid with free ends



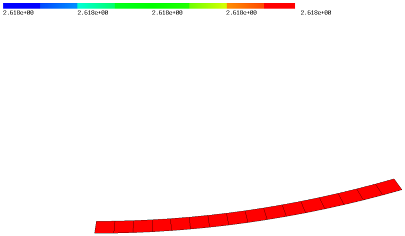


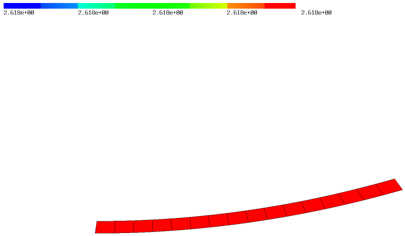




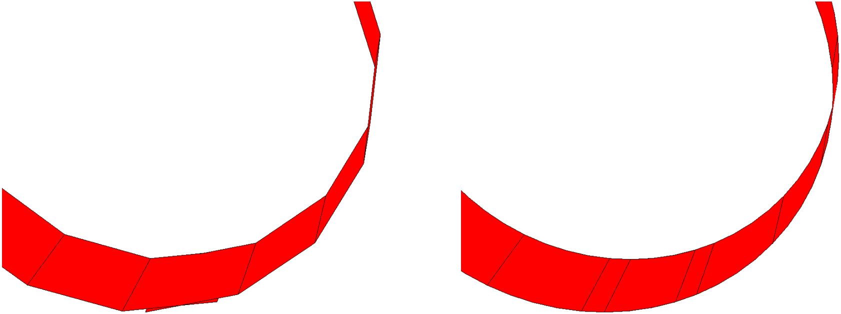
Numerical examples

Cantilever subjected to end moment



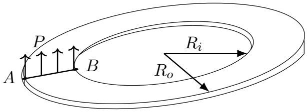


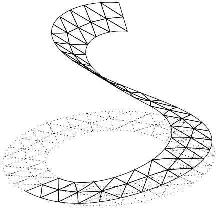
Cantilever subjected to end moment

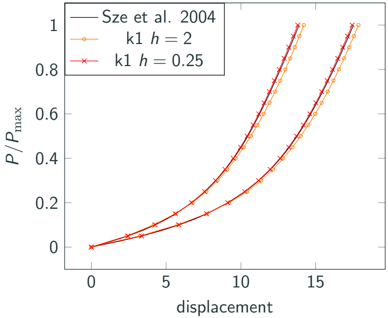


Cantilever subjected to end moment

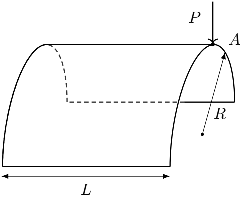
Slit annular plate

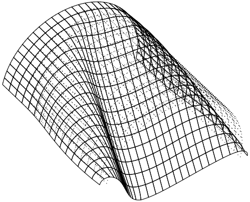


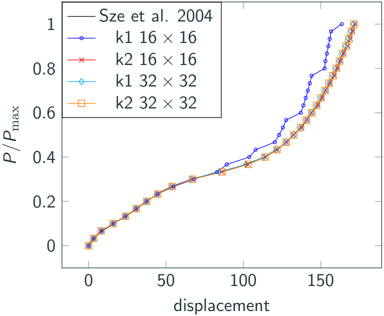




Pinched cylinder







Summary & outlook

- Distributional extrinsic/intrinsic curvature
- Application to nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

Summary & outlook

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- Distributional extrinsic/intrinsic curvature
- Application to nonlinear shells
- Hellan–Herrmann–Johnson and Regge finite elements for stress and strain/metric fields

- Naghdi shells

- Coupling for 3D elasticity
- Distributional curvature higher dimension $\rightarrow$ general relativity

Literature

- N., Schöberl: The Hellan–Herrmann–Johnson and TDNNS method for linear and nonlinear shells, *arXiv:2304.13806*.
- N., Schöberl: The Hellan–Herrmann–Johnson method for nonlinear shells, *Comput. Struct. 225 (2019).*
- N., Schöberl: Avoiding membrane locking with Regge interpolation, *Comput. Methods Appl. Mech. Engrg 373 (2021).*
- N.: Mixed Finite Element Methods For Nonlinear Continuum Mechanics And Shells, *PhD thesis, TU Wien (2021).*
- Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *arXiv:2206.09343*.
- N., Schöberl, Sturm, Numerical shape optimization of Canham-Helfrich-Evans bending energy, *arXiv:2107.13794*.

4th NGSolve UserMeeting

July 9-11, Portland, Oregon, USA

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Thank You for Your attention!