# Analysis of intrinsic curvature approximations with Regge finite elements

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Der Wissenschaftsfonds.

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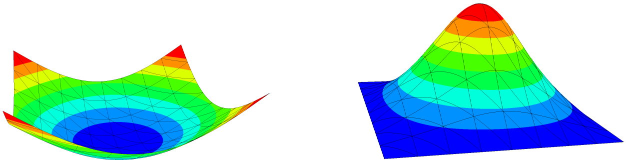
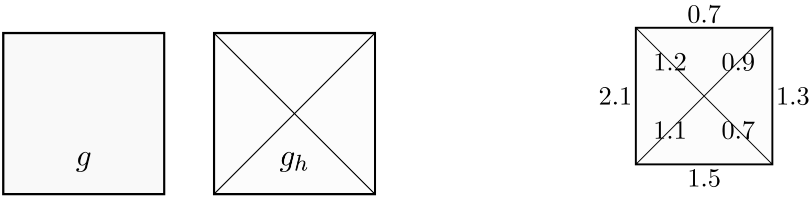
# Motivation

Curvature of approximated metric tensor $\|K_h(g_h) - K(g)\|_? \leq ?$





# Contents

# Differential Geometry

# Curvature (2D)

Riemannian manifold $(M, g)$

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Riemannian manifold $(M \subset \mathbb{R}^2, g)$

# Curvature (2D)

Riemannian manifold $(M, g)$

Levi-Civita connection $\nabla$

Riemann curvature tensor

$$\mathfrak{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{XY-YX}$$

$$R(X, Y, Z, W) = g(\mathfrak{R}(X, Y)Z, W)$$

# Curvature (2D)



Riemannian manifold $(M, g)$

Levi-Civita connection $\nabla$

Christoffel symbols: $\nabla_{\partial_j}\partial_k = \Gamma^l_{jk}\partial_l$

Riemann curvature tensor

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$$R(X, Y, Z, W) = g(\mathfrak{R}(X, Y)Z, W)$$

$$R_{ijkl} = \left(\partial_i\Gamma^p_{jk} - \partial_j\Gamma^p_{ik} + \Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq}\right) g_{lp}$$

# Curvature (2D)



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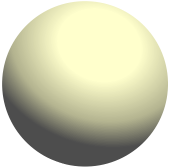
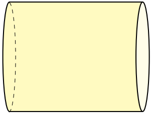
$$\Gamma^k_{ij}(g) = g^{kl}\underbrace{\frac{1}{2}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)}_{=\Gamma_{ijl}}$$

**Gauss curvature**:

$$K(g) = \frac{g(R(u,v)v,u)}{\|u\|_g \|v\|_g - g(u,v)^2} = \frac{R_{1221}}{\det g}$$



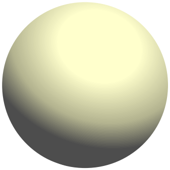


# Curvature

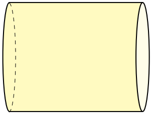


**Gauss curvature**:

$$K(g) = \frac{g(R(u,v)v,u)}{\|u\|_g \|v\|_g - g(u,v)^2} = \frac{R_{1221}}{\det g}$$

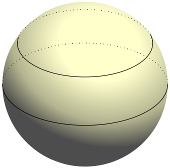


**Geodesic curvature**:



$$\kappa(g) = g(\nabla_{\hat{t}}\hat{t}, \hat{n}) = \frac{\sqrt{\det g}}{g_{tt}^{3/2}} \left(\partial_t t \cdot n + \Gamma^n_{tt}\right)$$

$$\hat{n} = \hat{t} \times \hat{\nu}$$



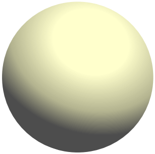
# Gauss–Bonnet theorem

## Gauss–Bonnet

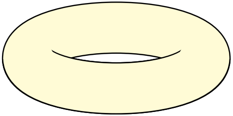
On manifold $M$:

$$\int_M K(g) + \int_{\partial M} \kappa(g) + \sum_V (\pi - \sphericalangle_V^M(g)) = 2\pi\chi_M$$

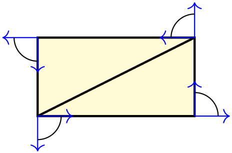
$$\chi_M(\mathcal{T}) = n_V - n_E + n_T$$



$\chi_M = 2$



$\chi_M = 0$



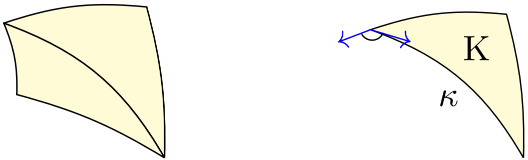
$\chi_M = 1$

**Gauss–Bonnet**

On triangle $T$:

$$\int_T K(g) + \int_{\partial T} \kappa(g) + \sum_{i=1}^{3} (\pi - \sphericalangle_{V_i}^{T}(g)) = 2\pi$$

$$\chi_T = 3 - 3 + 1 = 1$$



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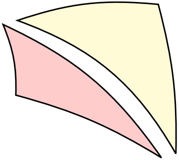

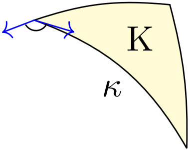

# Curvature operator and analysis

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**Lifted distributional curvature**

For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $K_h(g) \in V_h^{k+1}$ s.t. for all $\varphi \in V_h^{k+1}$

$$\int_{\mathcal{T}} K_h(g)\varphi = \sum_{T\in\mathcal{T}} \Big( K^T(\varphi, g) + \sum_{E\in\mathcal{E}_T} K_E^T(\varphi, g) + \sum_{V\in\mathcal{V}_T} K_V^T(\varphi, g) \Big)$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv preprint arXiv:2111.02512* (2021)

**Lifted distributional curvature**

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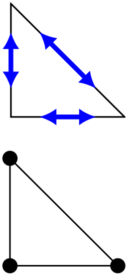
$$K^T(\varphi, g) = \int_T K(g)\,\varphi$$

$$K_E^T(\varphi, g) = \int_E \kappa(g)\varphi$$

$$K_V^T(\varphi, g) = \Big( \sphericalangle_V^T(\delta) - \sphericalangle_V^T(g) \Big)\,\varphi(V)$$



# Distributional curvature operator

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For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $K_h(g) \in V_h^{k+1}$ s.t. for all $\varphi \in V_h^{k+1}$

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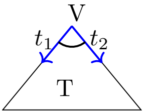


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$$\sphericalangle_V^T(g) = \arccos\left(\frac{t_1^\top g t_2}{\|t_1\|_g \|t_2\|_g}\right)$$

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### Lifted distributional curvature

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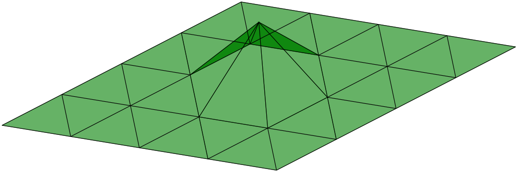
$$\int_{\mathcal{T}} K_h(g)\varphi = \sum_{T\in\mathcal{T}}\Big(K^T(\varphi,g) + \sum_{E\in\mathcal{E}_T} K_E^T(\varphi,g) + \sum_{V\in\mathcal{V}_T} K_V^T(\varphi,g)\Big)$$

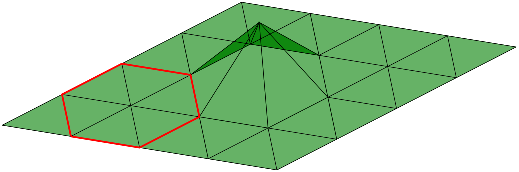
$$\int_{\mathcal{T}} K_h(g)\,\varphi\sqrt{\det g}\,da = \sum_{T\in\mathcal{T}}\Big(\int_T \frac{R_{1221}\,\varphi}{\sqrt{\det g}}\,da + \int_{\partial T}\frac{\sqrt{\det g}}{g_{tt}}(\partial_t t\cdot n + \Gamma_{tt}^n)\,\varphi\,dl + \sum_{V\in\mathcal{V}_T} K_V^T(\varphi,g)\Big)$$

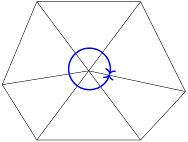
# Regge elements



T. Regge: General relativity without coordinates, *Il Nuovo Cimento (1955-1965), 19 (1961), pp. 558–571*

# Regge elements





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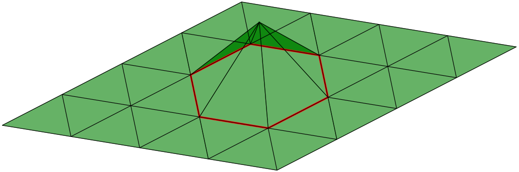

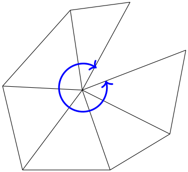

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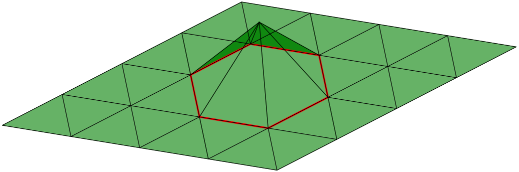

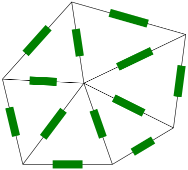

- metric tensor

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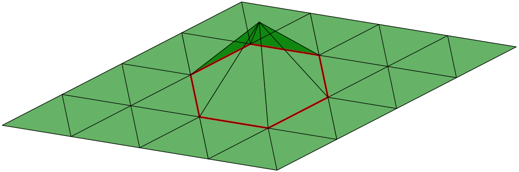
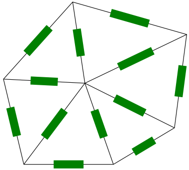
Sorkin, R.: Time-evolution problem in Regge calculus, *Phys. Rev. D 12* (1975), pp. 385–396





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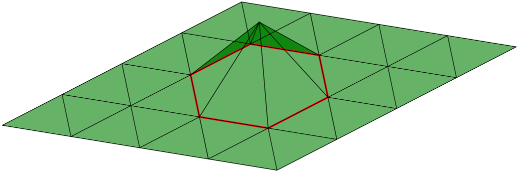
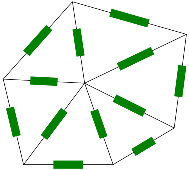
Cheeger, Müller, Schrader: On the curvature of piecewise flat spaces, *Communications in Mathematical Physics, 92(3) (1984), pp. 405–454*

# Regge elements



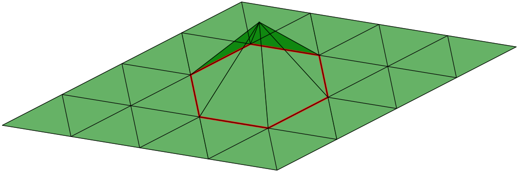


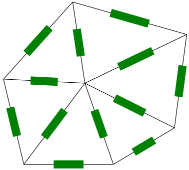
- metric tensor (tangential-tangential continuous)

$$\mathrm{Reg}_h^k = \{\varepsilon \in \mathcal{P}^k(\mathcal{T}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \,|\, [\![t^\top \varepsilon\, t]\!]_E = 0 \text{ for all edges } E\}$$

S. H. Christiansen: On the linearization of Regge calculus, *Numerische Mathematik 119, 4 (2011), pp. 613–640.*

# Regge elements





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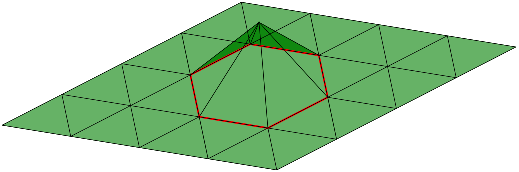
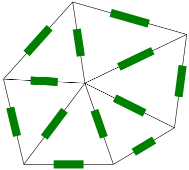
L. Li: Regge Finite Elements with Applications in Solid Mechanics and Relativity, *PhD thesis, University of Minnesota (2018).*





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N.: Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells, *PhD thesis, TU Wien (2021).*

**Consistency**

For $g \in C^2(M, \mathcal{S})$ there holds $\int_{\mathcal{T}} K_h(g)\, u_h = \int_{\mathcal{T}} K(g) u_h$, $u_h \in V_h^k$.

Gateaux derivative $D_g f(g)[\sigma] = \lim\limits_{t\to 0} \frac{f(g+t\sigma)-f(g)}{t}$

**Variation (B-K, G; GNSW )**

$\int_{\mathcal{T}} D_g(K_h(g))[\sigma]\, u_h = 0.5\langle \mathrm{div}_g \mathrm{div}_g S_g\sigma, u_h\rangle = -0.5\langle \mathrm{inc}_g\, \sigma, u_h\rangle$

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## Reformulation Gauss curvature

$$\int_{\mathcal{T}} K_h(g_h)\, u_h = \frac{1}{2} \int_0^1 b_h(\delta + t(g_h - \delta), g_h - \delta, u_h)\, dt, \; \forall u_h \in V_{h,0}^{k+1}$$

$$b_h(g_h, \sigma_h, u_h) = \langle \mathrm{div}_{g_h} \mathrm{div}_{g_h} S_{g_h} \sigma_h, u_h \rangle = -\langle \mathrm{inc}_{g_h}\, \sigma_h, u_h \rangle$$

$$\|K_h(g_h) - K(g)\|_? \leq h^?$$

**Convergence**

Let $g_h \in \mathrm{Reg}_h^k$ by the Regge interpolant of a smooth $g$. Then for sufficiently small $h$

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^k \|g\|_{H^{k+1}}.$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv preprint arXiv:2111.02512* (2021)

# First convergence result



$$\|K_h(g_h) - K(g)\|_? \leq h^?$$

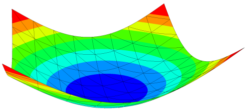
**Convergence**

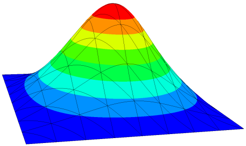
Let $g_h \in \mathrm{Reg}_h^0$ by the Regge interpolant of a smooth $g$. Then for sufficiently small $h$

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^0 \|g\|_{H^1} \quad .$$

Berchenko-Kogan, Gawlik: Finite element approximation of the Levi-Civita connection and its curvature in two dimensions, *arXiv preprint arXiv:2111.02512* (2021)

# Numerical example

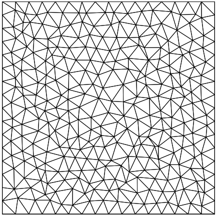


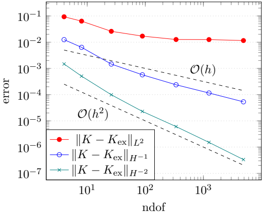


$$g = \begin{pmatrix} 1 + (\partial_x f)^2 & \partial_x f \partial_y f \\ \partial_x f \partial_y f & 1 + (\partial_y f)^2 \end{pmatrix} \qquad f = \frac{1}{2}(x^2 + y^2) - \frac{1}{12}(x^4 + y^4)$$

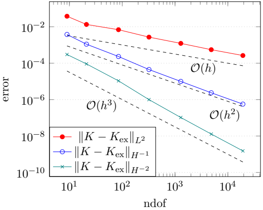
$$K(g) = \frac{81(1 - x^2)(1 - y^2)}{(9 + x^2(x^2 - 3)^2 + y^2(y^2 - 3)^2)^2}$$

NGSolve

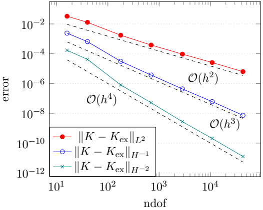





$k = 0$

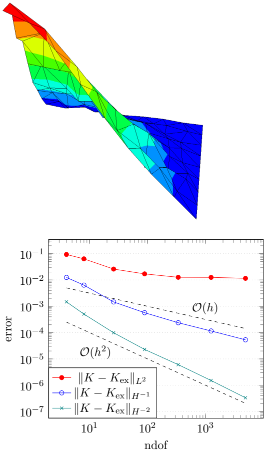



$k = 1$




$k = 2$

# Numerical example

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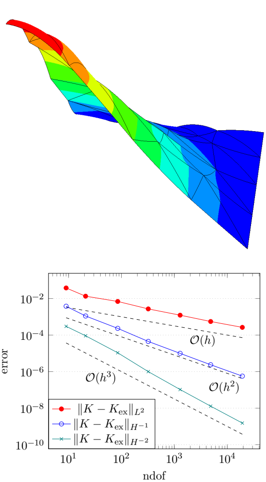
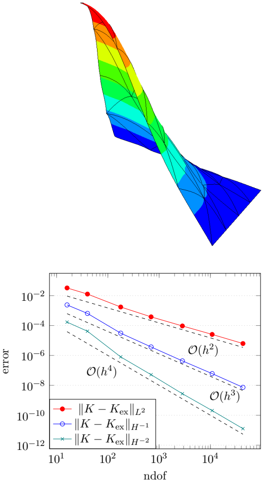



$k = 1$

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# Improved error analysis

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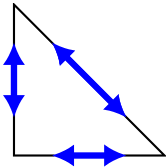

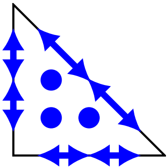

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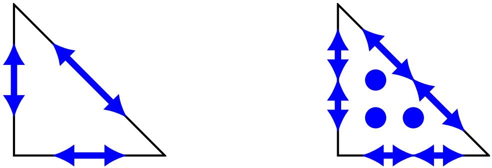
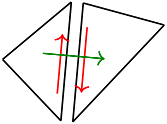


$\mathrm{inc}_g\,\sigma = \mathrm{curl}_g\,\mathrm{curl}_g\,\sigma$



For $g,\ \sigma \in \mathrm{Reg}_h^k$ and $\varphi \in \mathcal{P}^k(\mathcal{T},\mathbb{R}^2)$ normal continuous the distributional covariant curl is

$$\langle \mathrm{curl}_g \sigma, \varphi\rangle = \int_{\mathcal{T}} \frac{1}{\sqrt{\det g}}(\mathrm{curl}_g\,\sigma)(\varphi) - \int_{\partial\mathcal{T}} \frac{1}{\sqrt{\det g}} g(\varphi, n_g)\sigma(n_g, t_g)$$

$$= \sum_{T\in\mathcal{T}} \int_T \frac{\mathrm{curl}\,\sigma_i\varphi^i + \sigma_{ij}\varepsilon^{ik}\Gamma_{kl}^j\varphi^l}{\sqrt{\det g}}\,dx - \int_{\partial T} \frac{g_{tt}\sigma_{nt} - g_{nt}\sigma_{tt}}{\sqrt{\det g}\,g_{tt}}\varphi_n\,ds.$$

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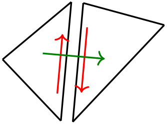

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- Standard distributional curl

$$\langle \mathrm{curl}_\delta\,\sigma, \varphi\rangle = \sum_{T\in\mathcal{T}} \int_T \mathrm{curl}\,\sigma \cdot \varphi\,da - \int_{\partial T} \sigma_{nt}\varphi_n\,dl$$

- Smooth $g$ and $\sigma$ leads to classical covariant curl

**Lemma**

For $k \in \mathbb{N}$, $\sigma_h = \mathcal{R}_h^k \sigma$, $\sigma \in W^{1,\infty}(\Omega, \mathbb{S})$, $v_h \in \mathcal{P}^k(\mathcal{T}, \mathbb{R}^2)$, and $g \in W^{1,\infty}(\Omega, \mathbb{S})$ there holds

$$\langle \mathrm{curl}_g(\sigma - \sigma_h), v_h \rangle \leq C\big(\|\sigma - \sigma_h\|_{L^2} + h|\sigma - \sigma_h|_{H_h^1}\big)\|v_h\|_{L^2(\Omega)},$$

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$$|\langle \operatorname{curl}_g \sigma_h, v_h \rangle - \langle \operatorname{curl}_{g_h} \sigma_h, v_h \rangle| \leq C(\|g - g_h\|_{L^\infty} + h\|g - g_h\|_{W_h^{1,\infty}}) \|\sigma_h\|_{H_h^1} \|v_h\|_{L^2}$$

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For $g,\ \sigma \in \mathrm{Reg}_h^k$ and $u \in \mathcal{P}^{k+1}(\mathcal{T})$ continuous the distributional covariant incompatibility operator

$$\langle \mathrm{inc}_g \sigma, u\rangle = \langle \mathrm{curl}_g\, \sigma, \mathrm{rot}\, u\rangle = \sum_{T\in\mathcal{T}} \int_T \mathrm{inc}_g\, \sigma\, u$$
$$- \int_{\partial T} u\, g(\mathrm{curl}_g\, \sigma - \mathrm{grad}_g\, \sigma(n_g, t_g), t_g) - \sum_{V\in\mathcal{V}_T} [\![\sigma(n_g, t_g)]\!]_V^T u(V)$$

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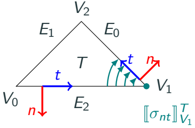
$$\langle \mathrm{inc}_\delta\,\sigma, u\rangle = \sum_{T\in\mathcal{T}} \int_T \mathrm{inc}\,\sigma\, u - \int_{\partial T} u(\mathrm{curl}\,\sigma - \nabla\sigma_{nt})\cdot t - \sum_{V\in\mathcal{V}_T} [\![\sigma_{nt}]\!]_V^T u(V)$$

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**Corollary**

Let $k \in \mathbb{N}$, $g$ a smooth metric tensor, $\sigma \in W^{1,\infty}(\Omega, \mathbb{S})$, $\sigma_h = \mathcal{R}_h^k \sigma$, and $u_h \in V_{h,0}^{k+1}$. Then

$$\langle \mathrm{inc}_g(\sigma - \sigma_h), u_h \rangle \leq C \big( \|\sigma - \sigma_h\|_{L^2} + h \|\sigma - \sigma_h\|_{H_h^1} \big) \|\nabla u_h\|_{L^2}.$$

**Corollary**

Let $k \in \mathbb{N}$, $\sigma_h \in \mathrm{Reg}_h^k$, $u_h \in V_{h,0}^{k+1}$, and $g_h = \mathcal{R}_h^k g$, $g \in W^{1,\infty}(\Omega, \mathbb{S})$. Then

$$|\langle \mathrm{inc}_g\, \sigma_h, u_h \rangle - \langle \mathrm{inc}_{g_h}\, \sigma_h, u_h \rangle| \leq C(\|g - g_h\|_{L^\infty} + h\|g - g_h\|_{W_h^{1,\infty}}) \|\sigma_h\|_{H_h^1} \|\nabla u_h\|_{L^2}.$$

# Improved error analysis

**Theorem (Gopalakrishnan, N., Schöberl, Wardetzky)**

Let $k \in \mathbb{N}$, $g$ be a smooth metric tensor with $K(g) \in H^k(\Omega)$, and $g_h = \mathcal{R}_h^k g$ the Regge interpolant. Then there holds for the lifted Gauss curvature $K_h(g_h) \in V_{h,0}^{k+1}$ for sufficiently small $h$

$$\|K_h(g_h) - K(g)\|_{H^{-1}} \leq Ch^{k+1}(\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}).$$

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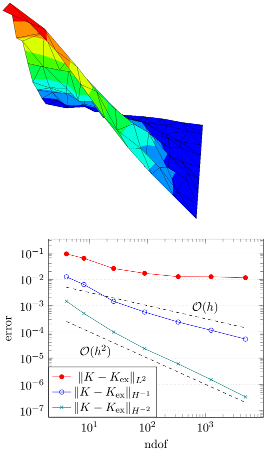
**Corollary**

There holds for $0 \leq l \leq k$

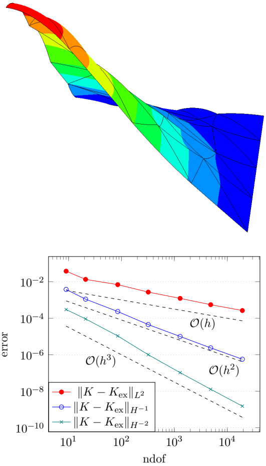
$$\|K_h(g_h) - K(g)\|_{L^2} \leq C h^{k} (\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}),$$
$$|K_h(g_h) - K(g)|_{H_h^l} \leq C h^{k-l} (\|g\|_{W^{k+1,\infty}} + |K(g)|_{H^k}).$$

# Numerical example

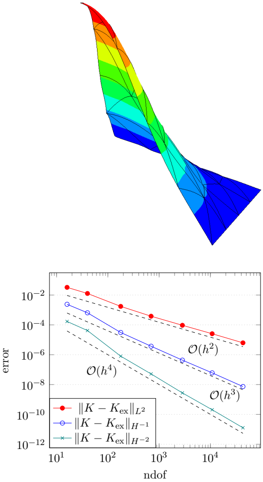





$k = 0$




$k = 1$

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# Extension to 3D

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- Riemann curvature tensor $R_{ijkl}$ has 6 independent entries
- Curvature operator $Q : M \to \mathbb{S}$

$$\langle Q(u \times v), w \times z \rangle = \langle R(u, v)z, w \rangle \text{ for all } u, v, w, z \in \mathbb{R}^3$$

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- No Gauss–Bonnet theorem in 3D

**Lifted distributional curvature**

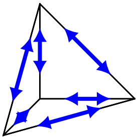
For $g \in \mathrm{Reg}_h^k(\mathcal{T})$ find $Q_h(g) \in \mathrm{Reg}_h^k(\mathcal{T})$ s.t. for all $v \in \mathrm{Reg}_h^k(\mathcal{T})$

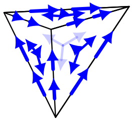
$$\int_{\mathcal{T}} Q_h(g) : v = \sum_{T\in\mathcal{T}} \Big( K^T(v,g) + \sum_{F\in\mathcal{F}_T} K_F^T(v,g) + \sum_{E\in\mathcal{E}_T} K_E^T(v,g) \Big)$$

$$K^T(v,g) = \int_T Q(g) : v$$

$$K_F^T(v,g) = \int_F ? : v$$

$$K_E^T(v,g) = \left( \sphericalangle_E^T(\delta) - \sphericalangle_E^T(g) \right) v_{t_E t_E}$$





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$$\int_{\mathcal{T}} Q_h(g) : v\sqrt{\det g}\,dx = \sum_{T\in\mathcal{T}} \Big( \int_T \frac{Q(g) : v}{\sqrt{\det g}}\,dx + \int_{\partial T} \frac{\sqrt{\det g}}{\mathrm{cof}(g)_{nn}} ((n\otimes n)\times\Gamma^n_{\bullet\bullet}) : v\,da + \sum_{E\in\mathcal{E}_T} K_E^T(v,g) \Big)$$

$$\mathrm{cof}(A) = \det(A)A^{-\top}, \quad (A\times B)_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}A_{km}B_{ln}$$

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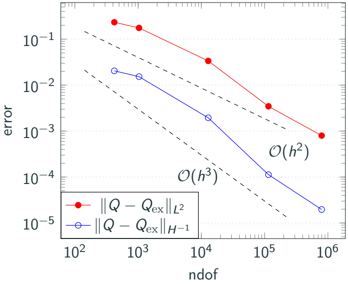
$$2\mathrm{D} : \int_{\partial T} \frac{\sqrt{\det g}}{g_{tt}} \Gamma^n_{tt}\, v\, dl$$

# Numerical examples (3D)



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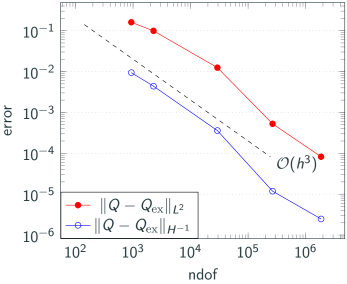


$k = 3$

# Summary



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- Optimal convergence rates

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GOPALAKRISHNAN, N., SCHÖBERL, WARDETZKY: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, *in preparation*

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**Thank You for Your attention!**

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