Fluid-Structure Interaction with H(div)-Conforming HDG and a new H(curl)-Conforming Method for Non-Linear Elasticity

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14th Austrian Numerical Analysis Day, Klagenfurt, May 3-4, 2018

Contents

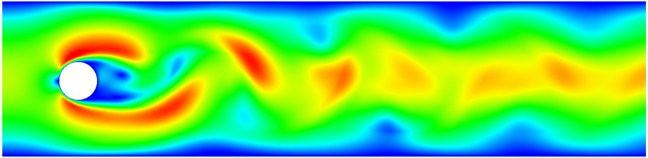
H(div)-conforming HDG for Navier-Stokes equations

Navier-Stokes equations

$$u(x,t)\ldots \quad \text{velocity}$$
$$p(x,t)\ldots \quad \text{pressure}$$

Navier-Stokes

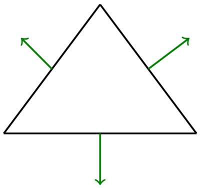
$$\begin{aligned} \frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p &= f \\ \mathrm{div}(u) &= 0 \end{aligned}$$



LEHRENFELD *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*. 2010

LEHRENFELD AND SCHÖBERL *High order exactly divergence-free Hybrid Discontinuous Galerkin Methods for unsteady incompressible flows*. 2015

- Normal continuous elements for velocity



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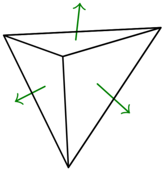
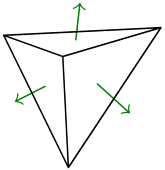
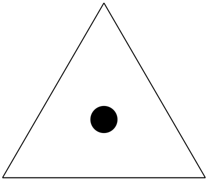
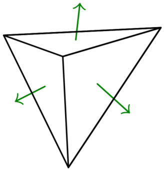


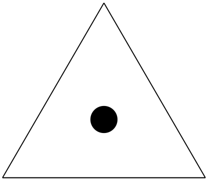
- Normal continuous elements for velocity
- Discontinuous pressure

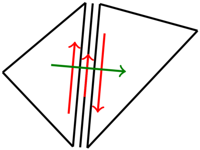




- Normal continuous elements for velocity
- Discontinuous pressure
- Facet variables for tangential part in hybrid fashion

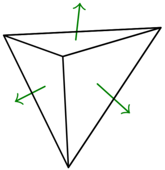


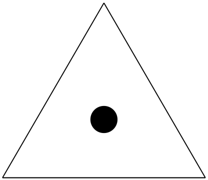


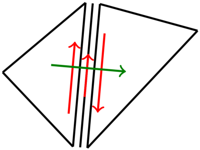


- Normal continuous elements for velocity
- Discontinuous pressure
- Facet variables for tangential part in hybrid fashion
- Solution exact divergence free

$$\int_\Omega \mathrm{div}(u)\, q\, dx = 0 \quad \forall q \in Q_h \Rightarrow \mathrm{div}(u) = 0$$







Arbitrary Lagrangian Eulerian for HDG

- Standard ALE with deformation $\Phi = id + d$

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$$\frac{\partial \hat{u}}{\partial t} + ((\hat{u} - \dot{d}) \cdot \nabla)\hat{u} - \nu \Delta \hat{u} + \nabla p = 0$$

- Mesh velocity $\dot{d}$ from differentiating $\hat{u}(\hat{x}, t) = u(\Phi(\hat{x}, t), t)$

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- Piola transformation to ensure normal continuity

$$u = P[\hat{u}] := \frac{1}{\det(F)} F \hat{u}, \qquad F = I + \nabla d$$

Arbitrary Lagrangian Eulerian for HDG

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- Piola transformation to ensure normal continuity

$$u = P[\hat{u}] := \frac{1}{\det(F)} F \hat{u}, \qquad F = I + \nabla d$$

- Second additional term from differentiating $u \circ \Phi = \frac{1}{\det(F)} F \hat{u}$

$$(\nabla \dot{d} F^{-1} - \mathrm{tr}(\nabla \dot{d} F^{-1})) P[\hat{u}]$$

Neunteufel *Advanced numerical methods for fluid-structure interaction*. 2017

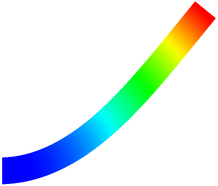
H(curl)-conforming discretization for elastic wave equation

Elastic wave equation

$$F = I + \nabla d, \qquad C = F^T F$$
$$\Sigma = \mu(C - I) + \frac{\lambda}{2}\mathrm{tr}(C - I)I$$

Elastic wave

$$\rho \frac{\partial^2 d}{\partial t^2} - \mathrm{div}(F\Sigma) = g$$

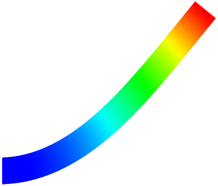


Elastic wave equation

$$F = I + \nabla d, \qquad C = F^T F$$
$$\Sigma = \mu(C - I) + \frac{\lambda}{2}\mathrm{tr}(C - I)I$$

Elastic wave

$$\begin{aligned} \dot{d} &= u \\ \rho\dot{u} - \mathrm{div}(F\Sigma) &= g \end{aligned}$$

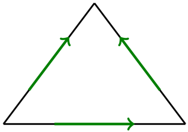


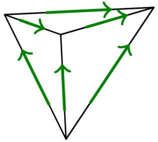
H(curl)-conforming discretization for elastic waves

Find $(d, u) \in [H^1(\Omega)]^n \times H(curl, \Omega)$ such that

$$\int_\Omega \frac{\partial d}{\partial t} \cdot v \, dx = \int_\Omega u \cdot v \, dx \qquad \forall v \in H(curl, \Omega)$$

$$\int_\Omega \rho \frac{\partial u}{\partial t} \cdot w \, dx = -\int_\Omega (F\Sigma) : \nabla w \, dx \quad \forall w \in [H^1(\Omega)]^n$$





H(curl)-conforming discretization for elastic waves

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Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx && \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega \frac{\partial p}{\partial t} \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega \frac{\partial p}{\partial t} \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}$$

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$$\int_\Omega p \cdot w \, dx = -\int_\Omega (F\Sigma) : \nabla w \, dx \qquad \forall w \in [H^1(\Omega)]^n$$

$$P = ?$$

H(curl)-conforming discretization for elastic waves

Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

$$
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\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega p \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega p \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}
$$

$$P = H(curl, \Omega)^*$$

H(curl)-conforming discretization for elastic waves

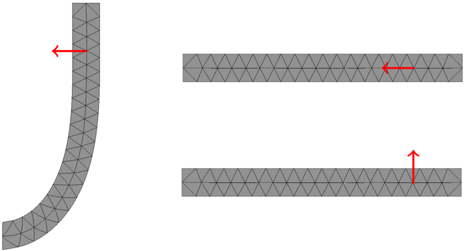
Find $(d, u, p) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times P$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot q \, dx &= \int_\Omega u \cdot q \, dx & \forall q \in P \\
\int_\Omega \rho \frac{\partial u}{\partial t} \cdot v \, dx &= \int_\Omega p \cdot v \, dx & \forall v \in H(curl, \Omega) \\
\int_\Omega p \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx & \forall w \in [H^1(\Omega)]^n
\end{aligned}$$

$$P = H(curl, \Omega)^*$$

NGSolve

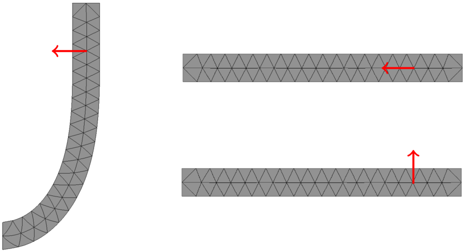
Transformation to material coordinates



Transformation to material coordinates

- Covariant transformation from global to material velocity

$$u = F^{-T}\hat{u}$$



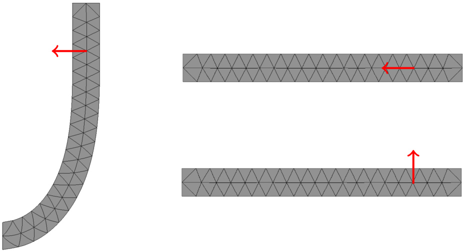
Transformation to material coordinates

- Covariant transformation from global to material velocity

$$u = F^{-T}\hat{u}$$

- Dual transformation for p

$$p = F\hat{p}$$



H(curl)-conforming discretization for elastic waves

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Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (F q)\, dx &= \int_\Omega \hat{u} \cdot q \, dx && \forall q \in H(curl, \Omega)^* \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T} \hat{u}) \cdot (F^{-T} v)\, dx &= \int_\Omega \hat{p} \cdot v \, dx && \forall v \in H(curl, \Omega) \\
\int_\Omega (F \hat{p}) \cdot w \, dx &= -\int_\Omega (F\Sigma) : \nabla w \, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}$$

H(curl)-conforming discretization for elastic waves

Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega)^{dc} \times H(curl, \Omega)^{*,dc}$ such that

$$\begin{aligned}
\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx &= \int_\Omega \hat{u} \cdot q\, dx && \forall q \in H(curl, \Omega)^{*,dc} \\
\int_\Omega \rho \frac{\partial}{\partial t}(F^{-T}\hat{u}) \cdot (F^{-T}v)\, dx &= \int_\Omega \hat{p} \cdot v\, dx && \forall v \in H(curl, \Omega)^{dc} \\
\int_\Omega (F\hat{p}) \cdot w\, dx &= -\int_\Omega (F\Sigma) : \nabla w\, dx && \forall w \in [H^1(\Omega)]^n
\end{aligned}$$

- Static condensation for discontinuous $\hat{u}$ and $\hat{p}$
- Further discretization in 2d and 3d

Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

$$\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx = \int_\Omega \hat{u} \cdot q\, dx \quad \forall q \in H(curl, \Omega)^*$$

$$\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1} \dot{C} C^{-1} \hat{u} \cdot v + \frac{1}{2J^2} C\, \mathrm{curl}(\hat{u}) \cdot (\hat{u} \times v))\, dx = \int_\Omega \hat{p} \cdot v\, dx \quad \forall v \in H(curl, \Omega)$$

$$\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx = 0 \quad \forall w \in [H^1(\Omega)]^n$$

NEUNTEUFEL *Advanced numerical methods for fluid-structure interaction.* 2017

Find $(d, \hat{u}, \hat{p}) \in [H^1(\Omega)]^n \times H(curl, \Omega) \times H(curl, \Omega)^*$ such that

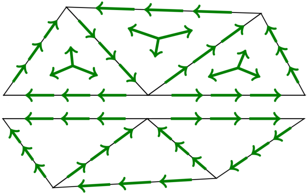
$$\int_\Omega \frac{\partial d}{\partial t} \cdot (Fq)\, dx = \int_\Omega \hat{u} \cdot q\, dx \quad \forall q \in H(curl, \Omega)^*$$

$$\int_\Omega \rho(F^{-T}\dot{\hat{u}} \cdot F^{-T} v - \frac{1}{2} C^{-1}\dot{C}C^{-1}\hat{u} \cdot v - \frac{1}{2J^2}\mathrm{curl}(\hat{u})\mathrm{rot}(\hat{u}) \cdot v)\, dx = \int_\Omega \hat{p} \cdot v\, dx \quad \forall v \in H(curl, \Omega)$$

$$\int_\Omega (F\hat{p}) \cdot w + (F\Sigma) : \nabla w\, dx = 0 \quad \forall w \in [H^1(\Omega)]^n$$

Neunteufel *Advanced numerical methods for fluid-structure interaction*. 2017

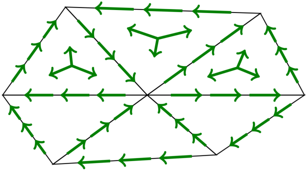
Interface conditions



- Continuity of displacement and tangential continuity of velocity fulfilled

$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

Interface conditions



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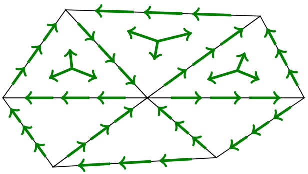
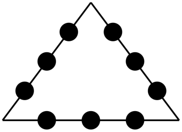




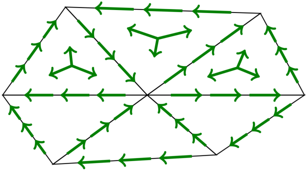
- Continuity of displacement and tangential continuity of velocity fulfilled

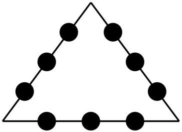
$$d^s = d^f, \quad u_\tau^s = u_\tau^f \qquad \text{on } \Gamma_I$$

- Normal continuity by Lagrange multiplier

$$\int_{\Gamma_I} (u^f - u^s)_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$

Interface conditions





- Continuity of displacement and tangential continuity of velocity fulfilled

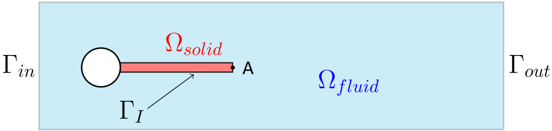
$$d^s = d^f, \quad u^s_\tau = u^f_\tau \qquad \text{on } \Gamma_I$$

- Normal continuity by Lagrange multiplier

$$\int_{\Gamma_I} (u^f - \frac{\partial d^s}{\partial t})_n \lambda = 0 \quad \forall \lambda \in L^2(\Gamma_I)$$

Numerical results

Benchmark (Turek/Hron)

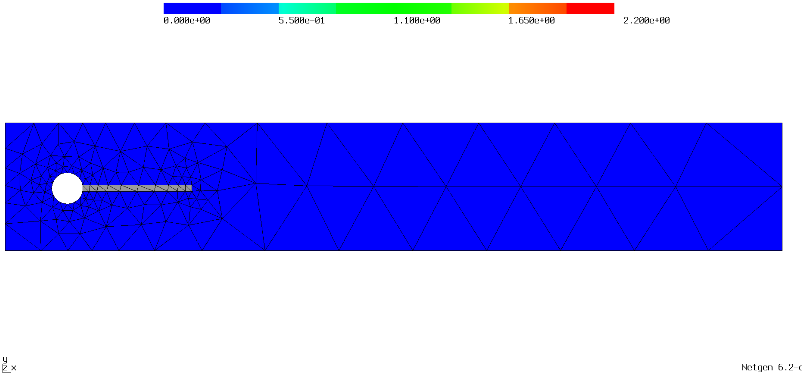


- Parabolic inflow
- Y-displacement of *A*

Turek and Hron *Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow*. 2006

Benchmark (Turek/Hron)

Video



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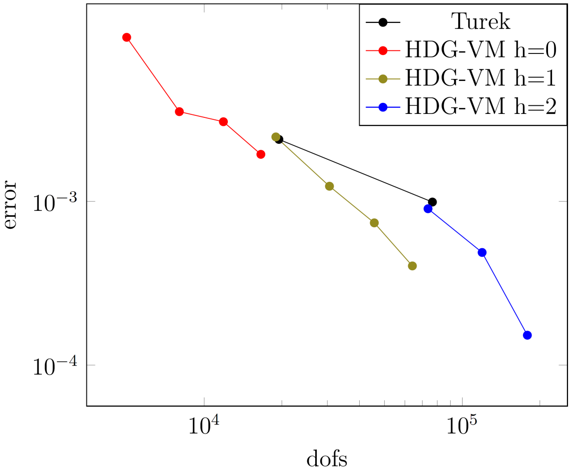

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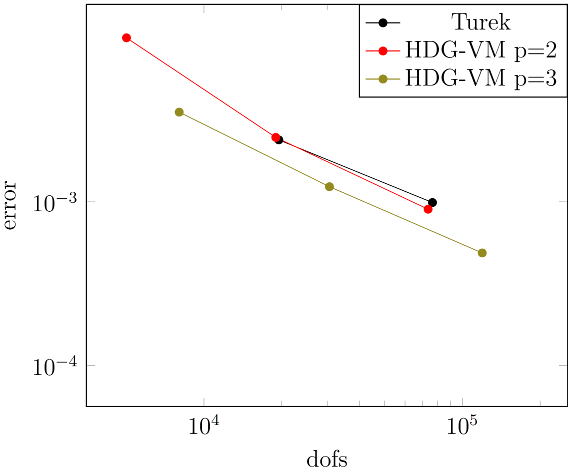

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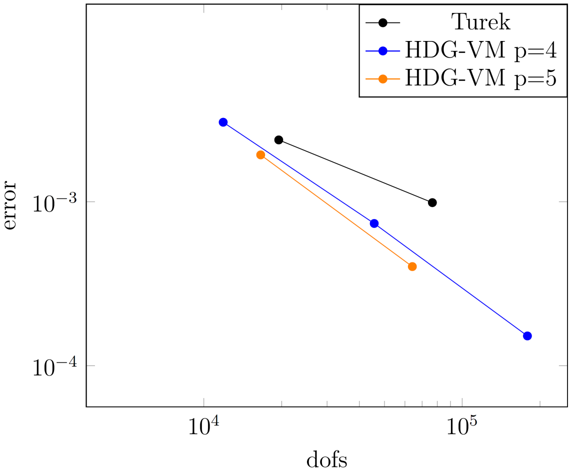

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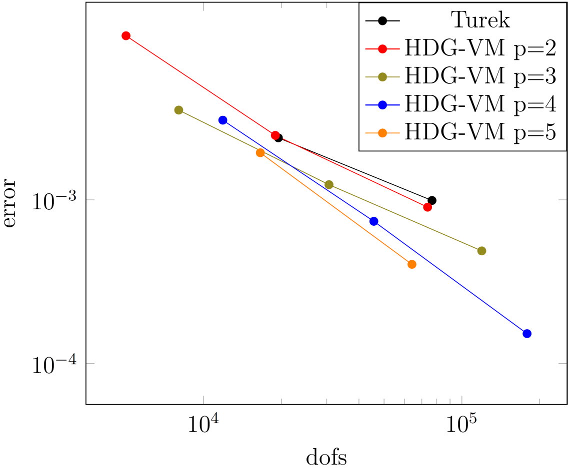

Summary

- ALE for H(div)-conforming HDG Navier-Stokes

Summary

- ALE for H(div)-conforming HDG Navier-Stokes
- New spatial discretization for elastic wave equation

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- ALE for H(div)-conforming HDG Navier-Stokes
- New spatial discretization for elastic wave equation
- Coupling of both equations

Current work

Current work

- 3D

Current work

- 3D
- Preconditioner

Current work

- 3D
- Preconditioner
- Splitting methods

Current work

- 3D
- Preconditioner
- Splitting methods

Thank you for your attention!