Tangential-Rotation and Normal-Normal-Momentum Continuous Mixed Finite Elements for Non-Linear Shell Models

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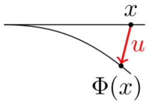
Contents

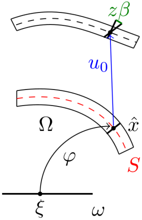
Continuum mechanics

Deformation	$\Phi : \Omega \to \mathbb{R}^3$
Displacement	$u := \Phi - id$
Deformation gradient	$F := \nabla\Phi$
Cauchy-Green strain tensor	$C := F^T F$
Green strain tensor	$E := \frac{1}{2}(C - I)$
Linearized strain tensor	$\epsilon(u) := \frac{1}{2}(\nabla u^T + \nabla u)$



$$\frac{||\Phi(x + \Delta x) - \Phi(x)||^2}{||\Delta x||^2} = \frac{\Delta x^T F^T F \Delta x}{||\Delta x||^2} + \mathcal{O}(||\Delta x||)$$

Thin-walled structures



- Model of reduced dimensions
- $\Omega = \left\{\varphi(\xi) + zn(\xi) : \xi \in \omega, z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$
- $u(\hat{x} + zn(\xi)) = u_0(\hat{x}) + z\beta(\hat{x})$

Geometric derivation

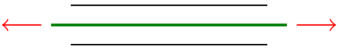
$$\nabla u(\hat{x} + zn) = \nabla_\tau u_0 + z\nabla_\tau \beta + \beta n^T$$

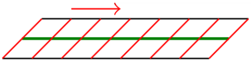
$$C_{\tau\tau} := P_\tau C P_\tau$$
$$C_{\tau n} := P_\tau C P_n$$
$$C_{nn} := P_n C P_n$$

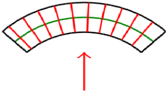
$$C_{\tau\tau} = (P_\tau + \nabla_\tau u_0^T)(P_\tau + \nabla_\tau u_0)$$

$$C_{\tau n} = P_\tau (I + \nabla_\tau u_0)^T (n + \beta) n^T$$

$$E_{\text{memb}} \approx ||C_{\tau\tau} - I_{\tau\tau}||^2_{L_2(S)}$$
$$E_{\text{shear}} \approx ||C_{\tau n}||^2_{L_2(S)}$$
$$E_{\text{bend}} \approx t^2 ||\epsilon_{\tau\tau}((I + \nabla_\tau u_0)^T \beta)||^2_{L_2(S)}$$



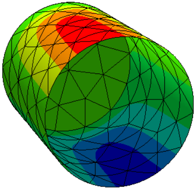


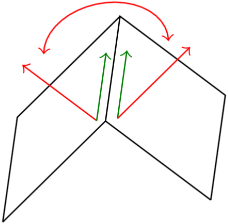


First variational problem

Find u and $\beta \in [H^1(S)]^3$ such that

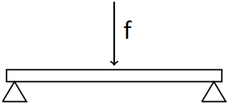
$$J(u,\beta) = ||C_{\tau\tau} - I_{\tau\tau}||^2_{L_2(S)} + ||C_{\tau n}||^2_{L_2(S)} + t^2||\epsilon_{\tau\tau}((I+\nabla_\tau u_0)^T\beta)||^2_{L_2(S)} - \int_S f\cdot u\,dx \to \min!$$





- β tangential continuous
- u continuous
- m normal-normal continuous

Kirchhoff plate and Hellan-Herrmann-Johnson method



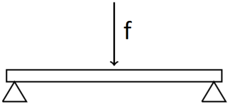
$$\begin{cases} \Delta^2 w = f & \text{in } \Omega \\ w = \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Find $w \in H_0^2(\Omega)$, such that

$$\int_\Omega \nabla^2 w : \nabla^2 v \, dx = \int_\Omega fv \, dx \quad \text{for all } v \in H_0^2(\Omega)\,.$$

$$\begin{aligned} \int_\Omega m : \tau \, dx - \int_\Omega \nabla^2 w : \tau \, dx &= 0 & \forall \tau \\ -\int_\Omega m : \nabla^2 v \, dx \qquad\qquad &= -\int_\Omega fv \, dx & \forall v \end{aligned}$$

Kirchhoff plate and Hellan-Herrmann-Johnson method



$$\begin{cases} \Delta^2 w = f & \text{in } \Omega \\ w = \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Find $w \in H_0^2(\Omega)$, such that

$$\int_\Omega \nabla^2 w : \nabla^2 v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in H_0^2(\Omega)\,.$$

$$\begin{aligned} \int_\Omega m : \tau \, dx - \int_\Omega \nabla^2 w : \tau \, dx &= 0 && \forall \tau \\ -\int_\Omega \operatorname{divdiv} m v \, dx &= -\int_\Omega f v \, dx && \forall v \end{aligned}$$

Derivation

$$\int_\Omega m : \tau \, dx + \sum_T \int_T \nabla w \cdot \operatorname{div} \tau \, dx - \int_{\partial T} \nabla w \cdot \tau_n \, ds = 0$$

$$\sum_T \int_T \operatorname{div}\, m \cdot \nabla v \, dx - \int_{\partial T} (\operatorname{div}\, m) \cdot nv \, ds \qquad = -\int_\Omega fv \, dx$$

$$\int_E \nabla w \, [\tau_n]_E \, ds = \int_E \nabla_n w \, [\tau_{nn}]_E + \nabla_\tau w \, [\tau_{n\tau}]_E \, ds = \int_E \nabla_\tau w \, [\tau_{n\tau}]_E \, ds$$

$$\int_{\partial T} (\operatorname{div}\, m) \cdot nv \, ds = 0 \quad \text{as well as} \quad \int_{\partial T} \nabla_\tau v \cdot m_{n\tau} \, ds = 0$$

Find $m \in H(\text{divdiv})$ and $w \in H^1(\Omega)$, such that

$$\int_\Omega m : \tau \, dx + \sum_T \int_T \nabla w \cdot \operatorname{div} \tau \, dx - \int_{\partial T} \nabla_\tau w \cdot \tau_{n\tau} \, ds = 0 \qquad \forall \tau,$$

$$\sum_T \int_T \operatorname{div}\, m \cdot \nabla v \, dx - \int_{\partial T} \nabla_\tau v \cdot m_{n\tau} \, ds \qquad = -\int_\Omega fv \, dx \quad \forall v.$$

[Comodi, 1989]

Reissner-Mindlin method

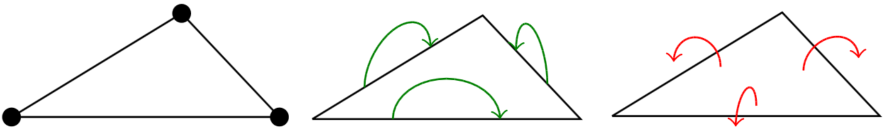
Find $m \in H(\text{divdiv})$, $\beta \in H(\text{curl})$ and $u \in H^1(\Omega)$, such that

$$\int_\Omega m : \tau \, dx + \sum_T \int_T \beta \cdot \text{div } \tau \, dx - \int_{\partial T} \beta_\tau \cdot \tau_{n\tau} \, ds = 0 \qquad \forall \tau,$$

$$\sum_T \int_T \text{div } m \cdot \delta \, dx - \int_{\partial T} \delta_\tau \cdot m_{n\tau} \, ds - \frac{1}{t^2} \int_\Omega (\nabla u - \beta)(\nabla v - \delta) \, dx = - \int_\Omega f \cdot \delta \, dx \qquad \forall v, \delta.$$

- Shear strain: $\nabla u - \beta$
- Shear stress: $\frac{1}{t^2}(\nabla u - \beta)$
- For $t \to 0$: $\nabla u - \beta \to 0$
- Error analysis similar

Finite Element Spaces



Boundary conditions: $u = u_D$ on Γ_D, $\beta = \beta_D$ on Γ_D, $m_n = m_N$ on Γ_N

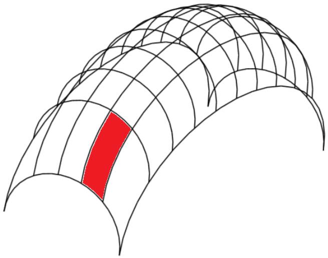
	Displacements	Rotations	Bending moments
Essential	u	β_τ	m_{nn}
Natural	$\frac{\partial u}{\partial n}$	β_n	$m_{n\tau}$

$V_h := \{v \in L_2(\Omega) : v_{|T} \in \mathcal{P}^{k+1}, v \text{ continuous}, v = 0 \text{ on } \Gamma_D\} \subset H^1$
$B_h := \{\delta \in [L_2(\Omega)]^n : \delta_{|T} \in \mathcal{P}^k, \delta_\tau \text{ continuous}, \delta_\tau = 0 \text{ on } \Gamma_D\} \subset H(\text{curl})$
$M_h := \{\tau \in L_2^{sym}(\Omega) : \tau_{|T} \in \mathcal{P}^k, \tau_{nn} \in \mathcal{P}^{\max(1,k-1)} \text{ continuous}, \tau_{nn} = 0 \text{ on } \Gamma_N\} \subset H(\text{divdiv})$

Finite Element Spaces

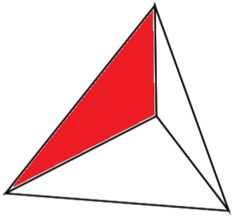
Finite Element Space M_{Surf}:

- 2D elements of M_h as face-elements



Finite Element Space V_h and B_h:

- Traces of 3D elements



Error estimates

Discrete norms

$$||\beta||_{B_h}^2 := \sum_T ||\epsilon(\beta)||_{L^2(T)}^2 + \sum_{F\in\mathcal{F}} \frac{1}{h_F} ||[\beta_n]||_{L^2(F)}^2$$

$$||m||_{M_h}^2 := \sum_T ||m||_{L^2(T)}^2 + \sum_{F\in\mathcal{F}} h_F ||m_{nn}||_{L^2(F)}^2$$

Elasticity

$$||m - m_h||_{M_h}^2 + ||\beta - \beta_h||_{B_h}^2 \leq C h^{2k} ||\beta||_{H^{k+1}(\Omega)}^2$$

Reissner-Mindlin

$$||m - m_h||_{M_h}^2 + ||\beta - \beta_h||_{B_h}^2 + \frac{1}{t^2}||(\nabla u - \beta) - (\nabla u_h - \beta_h)||_{L^2(\Omega)}^2 \leq C(m, \beta, u) h^{2k}$$

[Schöberl + Pechstein, 2011]

Minimization problem

Find $m \in V$ and $u \in Q$, such that

$$\begin{aligned} a(m,\tau) + b(\tau,u) &= 0 && \text{for all } \tau \in V\,, \\ b(m,v) &= f(v) && \text{for all } v \in Q\,. \end{aligned}$$

Lagrange-function

$$L(m,u) = \frac{1}{2}a(m,m) + b(m,u) - f(u)$$

- $L(m,u)$ convex in $m \Rightarrow J(u) := \min_m L(m,u)$ by solving linear problem
- $J(u)$ concave
- Find $\max_u J(u)$ with Newton's method

Shell model

Bilinear forms

$$a(m,\tau) := -\frac{1}{t^2}\int_S m : \tau \, dx$$
$$b(m,v) := \sum_{T\in S}\int_T \operatorname{div} m \cdot ((\nabla v)^T n)\, dx - \int_{\partial T} m_{n\tau} \cdot ((\nabla_\tau v)^T n)\, ds$$

- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ correspond to E_{bend}
- Adding $E_{\text{memb}} = ||C_{\tau\tau} - I_{\tau\tau}||^2_{L_2(S)}$

Shell model

First $E_{\text{shear}} = \frac{1}{t^2}||(\nabla u)^T n - \beta||^2_{L_2(\Omega)} = 0$

Find $u \in [V_h]^3$ and $m \in M_{\text{Surf}}$ such that

$$L(m, u) = -\frac{1}{2}a(m, m) + b(m, u) + E_{\text{memb}}(u) - f(u),$$
$$J(u) = \min_m L(u, m) \to \min!.$$

Adding $E_{\text{shear}} = \frac{1}{t^2}||(\nabla u)^T n - \beta||^2_{L_2(\Omega)}$

Find $u \in [V_h]^3$, $m \in M_{Surf}$ and $\beta \in B_h$, such that

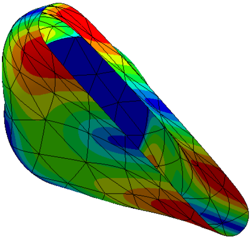
$$L(m, u, \beta) = -\frac{1}{2}a(m, m) + b(m, \beta) + E_{\text{memb}}(u) + E_{\text{shear}}(u, \beta) - f(u),$$
$$J(u, \beta) = \min_m L(u, \beta, m) \to \min!\,.$$

NGS-PY

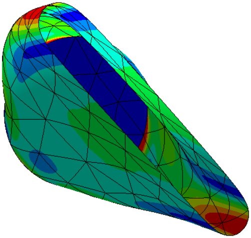
```
Etautau=Ptau(1/2*(C-I))

a=BilinearForm(fes,symmetric=True)
a+=SymbolicBFI(-100*InnerProduct(m,tau),BND)
a+=SymbolicBFI(-div(m)*delta-div(tau)*beta,BND)
a+=SymbolicBFI(mn*tang(delta)+taun*tang(beta),element_boundary=True)

a+=SymbolicEnergy(InnerProduct(Etautau,Etautau),BND)

a+=SymbolicEnergy(100*(ngradu-beta)*(ngradu-beta),BND)

a+=SymbolicEnergy(-y*uy,BND)
```

Apply Newton's method with automatic differentiation

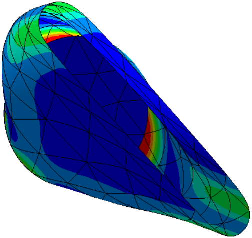
Examples



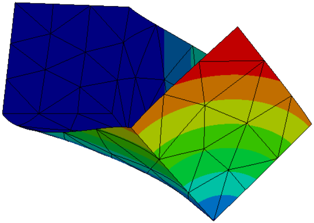
Rotation



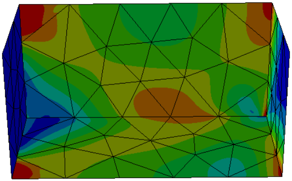
Moments



Shear stress



Displacement



Moments

Thank you for your attention!