Distributional differential operators on Riemannian manifolds with smooth and Regge metrics

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Motivation

Analysis of curvatures from Regge metrics involves distributional covariant operators

Riemann curvature tensor	Incompatibility operator $-\mathrm{Inc}, \quad \mathrm{curl}^T \mathrm{curl}$
Einstein tensor	Ein operator $\mathrm{ein} = J \,\mathrm{def}\, \mathrm{div}\, J - 0.5 \Delta\, J$
Scalar curvature	$\mathrm{div}\,\mathrm{div}\,\mathbb{S}, \qquad \mathbb{S}\sigma = \sigma - \mathrm{tr}(\sigma)\, I$
Gauss curvature	$-\mathrm{inc} = \mathrm{div}\,\mathrm{div}\,\mathbb{S}$

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- Well-understood in Euclidean setting (and smooth manifolds)
- Possible for tangential-tangential continuous metrics?

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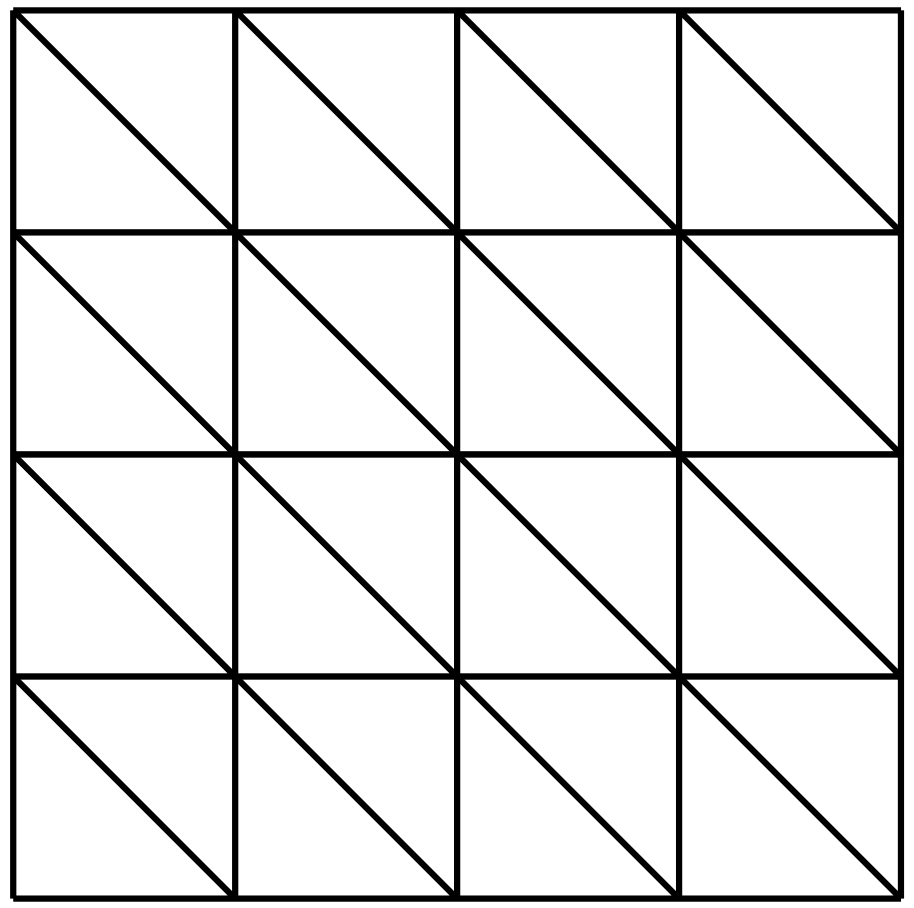
Distributional Euclidean differential operators



1. $C_0^\infty(\Omega)$ space of test functions ⇒ distributional derivatives

$$\langle \nabla f, \Psi \rangle = -\int_\Omega f \,\mathrm{div}\, \Psi \, dx, \qquad f \in C^\infty(\mathcal{T}), \quad \Psi \in C_0^\infty(\Omega, \mathbb{R}^N)$$

2. Integration by parts element-wise

$$-\sum_{T \in \mathcal{T}} \int_T f \,\mathrm{div}\, \Psi \, dx = \sum_{T \in \mathcal{T}} \int_T \nabla f \cdot \Psi \, dx - \sum_{E \in \mathcal{E}} \int_E [\![f]\!] \, \Psi \cdot n \, ds$$

$$| \langle \nabla f, \Psi \rangle | \leq C(f) \, \| \Psi \|_{H(\mathrm{div})}$$

3. Density: $C_0^\infty(\Omega, \mathbb{R}^3)$ dense in $H(\mathrm{div})$ ⇒ $\langle \nabla f, v \rangle$ well-defined for $v \in H(\mathrm{div})$

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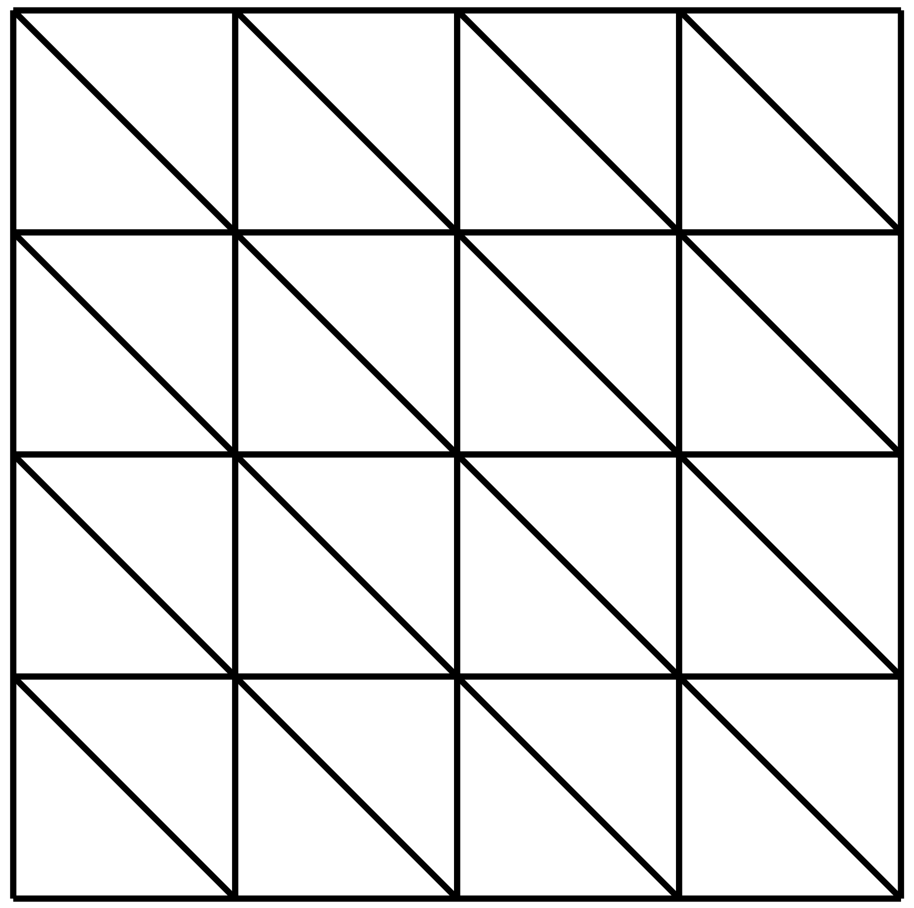

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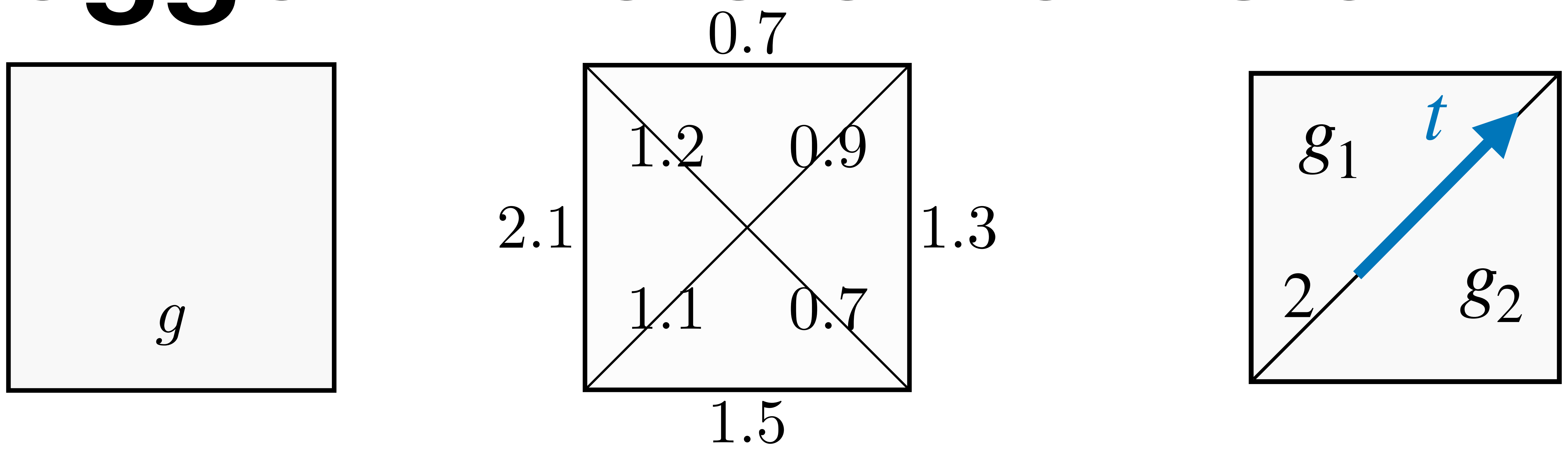
- Extension to smooth Riemannian manifolds via charts
- Test functions and density results for non-smooth (tt-continuous) metrics?

Regge finite elements & metric



$$\int_E g_1(t,t)\,ds = \int_E g_2(t,t)\,ds = 2$$

$$g_h = g_1 \cup g_2$$

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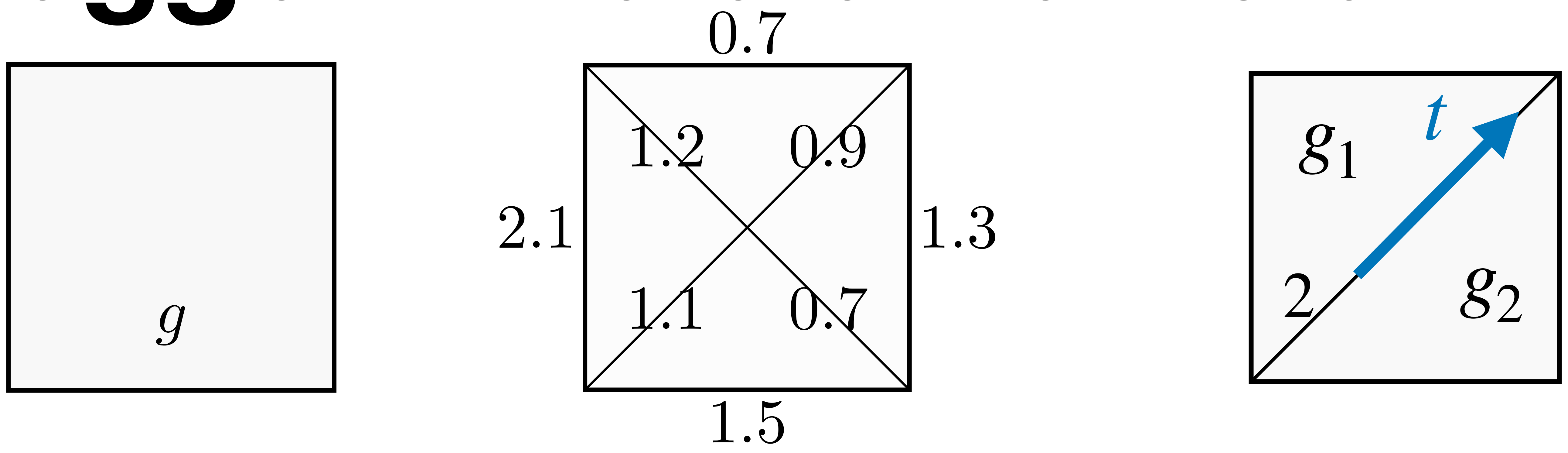

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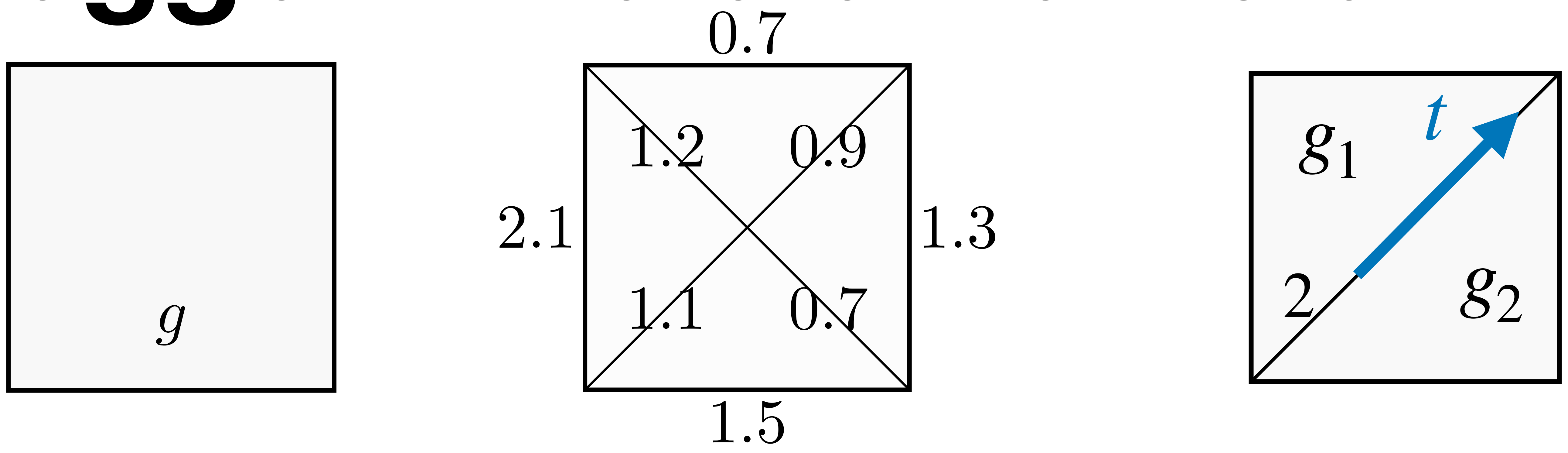

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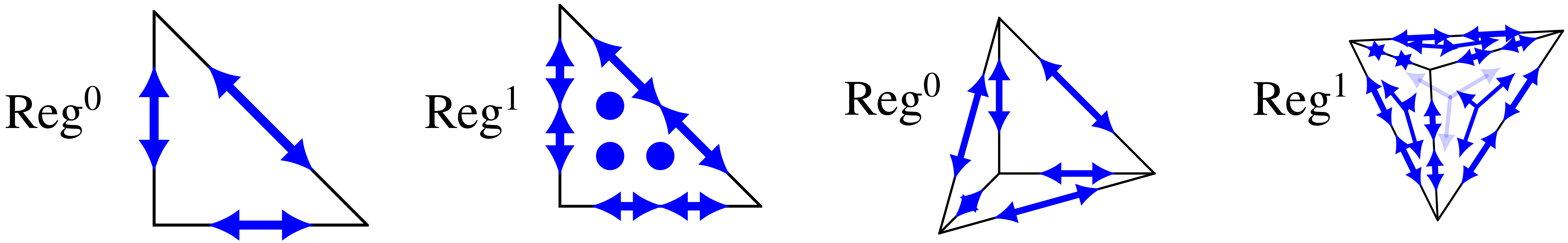
$$\mathrm{Reg}^k := \left\{ \sigma \in \mathscr{P}^k(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \mid \sigma \text{ is tangential-tangential continuous} \right\}$$

$$H(\mathrm{curl}\,\mathrm{curl}) := \left\{ \sigma \in L^2(\Omega, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \mid \mathrm{curl}^T\mathrm{curl}(\sigma) \in H^{-1} \right\}$$



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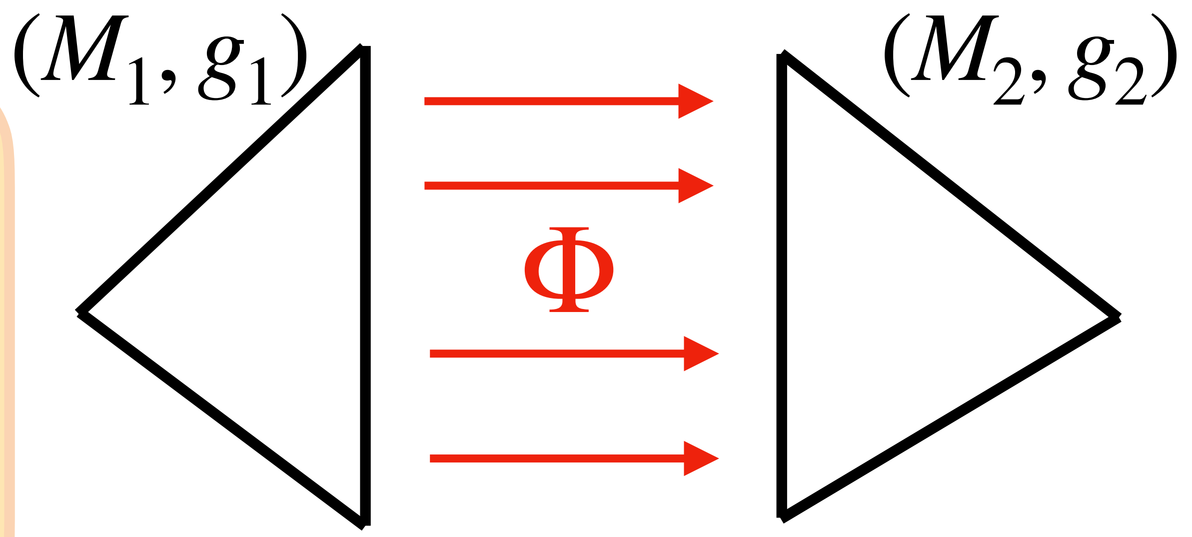
Gluing isometric Riemannian manifolds

Def.: Let (M_1, g_1) and (M_2, g_2) Riemannian manifolds with boundary and $\Phi : \partial M_1 \to \partial M_2$ an isometry. We call (M, g) with $M = M_1 \cup M_2, g = g_1 \cup g_2$ a glued Riemannian manifold.



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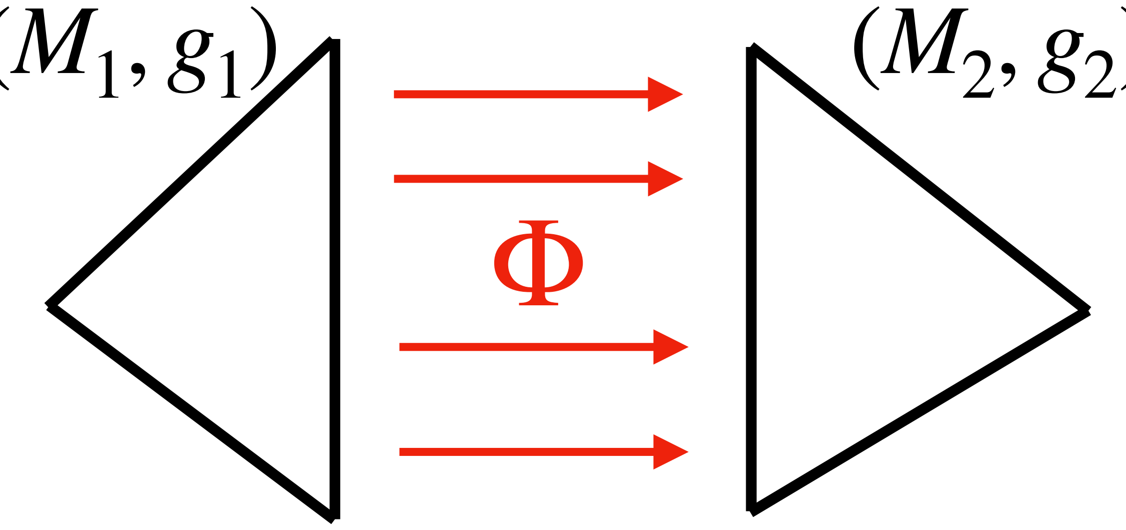

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- g is tangential-tangential continuous $\quad g_1(X, Y) = g_2(\Phi_*(X), \Phi_*(Y)), \quad \forall X, Y \in \mathfrak{X}(\partial M_1)$
- Triangulation $\mathscr{T}$ of M with Regge metric $g_h \in \mathrm{Reg}^k$ yields glued Riemannian manifold (M, g_h)

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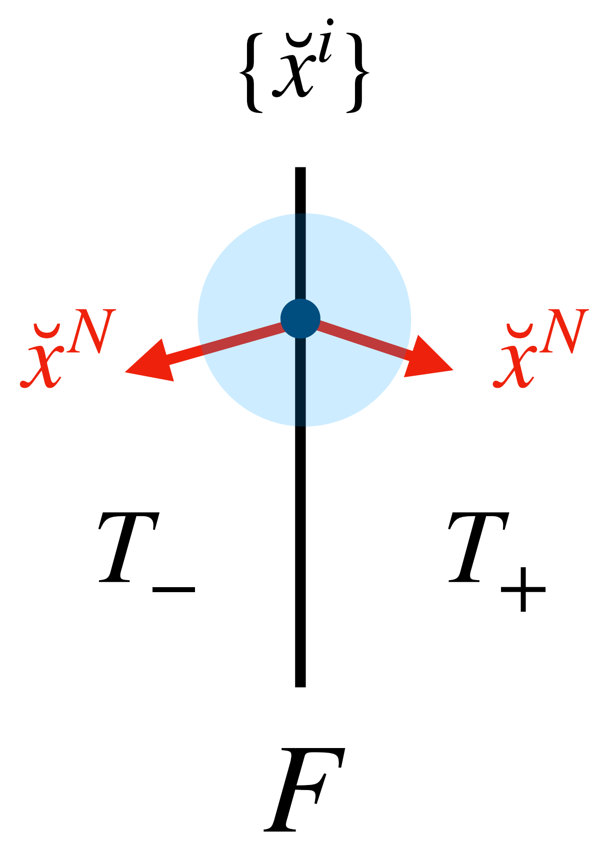
Fermi coordinates

Def.: Let $F \in \mathring{\mathscr{F}}$ and $z \in F$ an interior point. Let $\{x^1, \ldots, x^{N-1}\}$ coordinates on F. Let U_z an open neighborhood of z and $d_g(\cdot\,,\cdot)$ the distance function generated by g on M. For $p \in U_z$ let $\pi(p) = \mathrm{argmin}_{q \in F} d_g(p, q)$ be the projection of p onto F. The Fermi coordinates $\{\breve{x}^i\}$ are defined by

$$\breve{x}^N(p) := \pm\, d_g(\pi(p), p) \text{ if } p \in T_\pm, \qquad \breve{x}^i(p) := x^i(\pi(p)) \text{ for } i = 1, \ldots, N-1.$$



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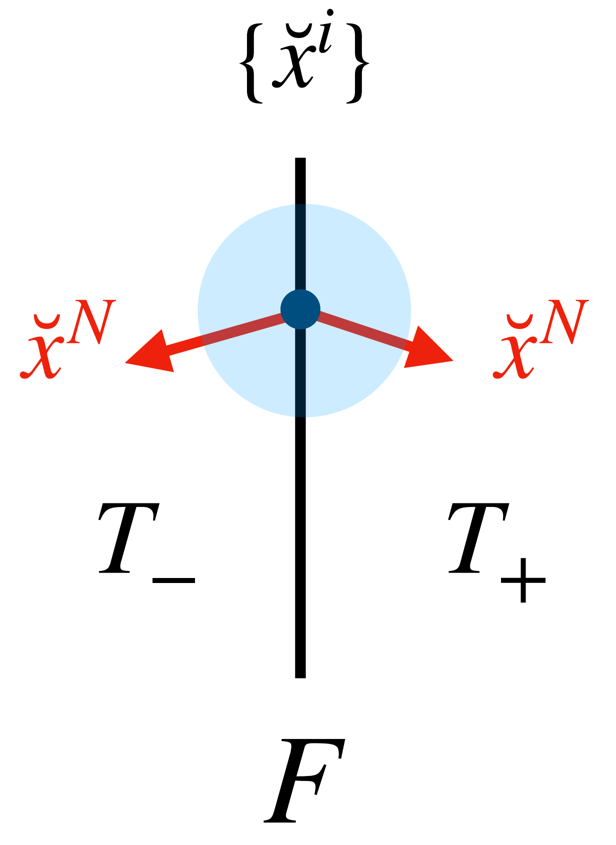

g is continuous in Fermi coordinates over F

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,N-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{N-1,1} & \cdots & g_{N-1,N-1} & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

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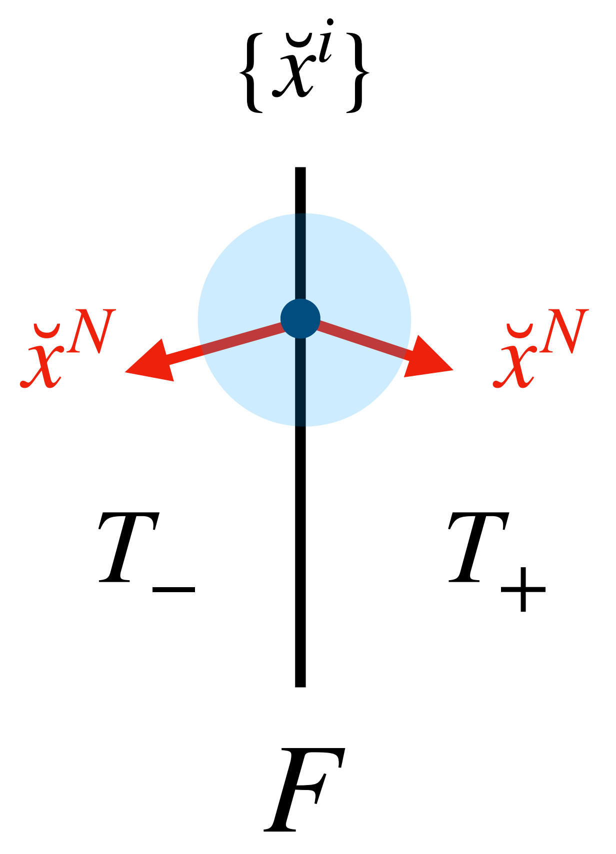

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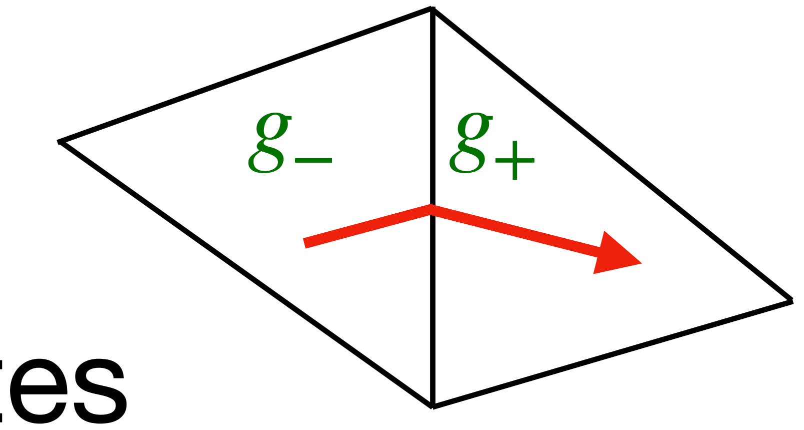
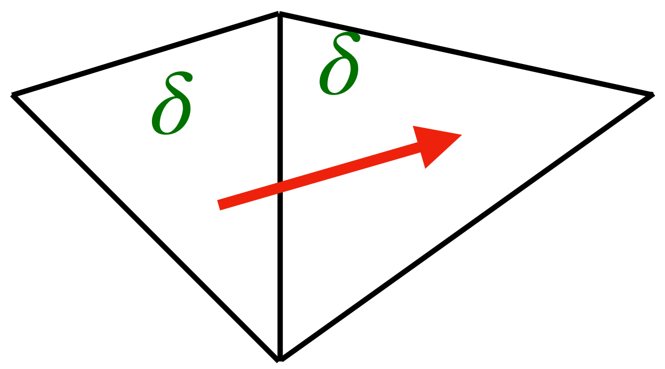
For $g \in \mathrm{Reg}^0$ Fermi coordinates yield Euclidean metric





Natural smooth glued structure of manifolds

Lemma: Let M, N be two smooth N-dimensional manifolds which can be glued together and have compatible smooth structures. Then there exists a smooth structure on $M \cup N$.

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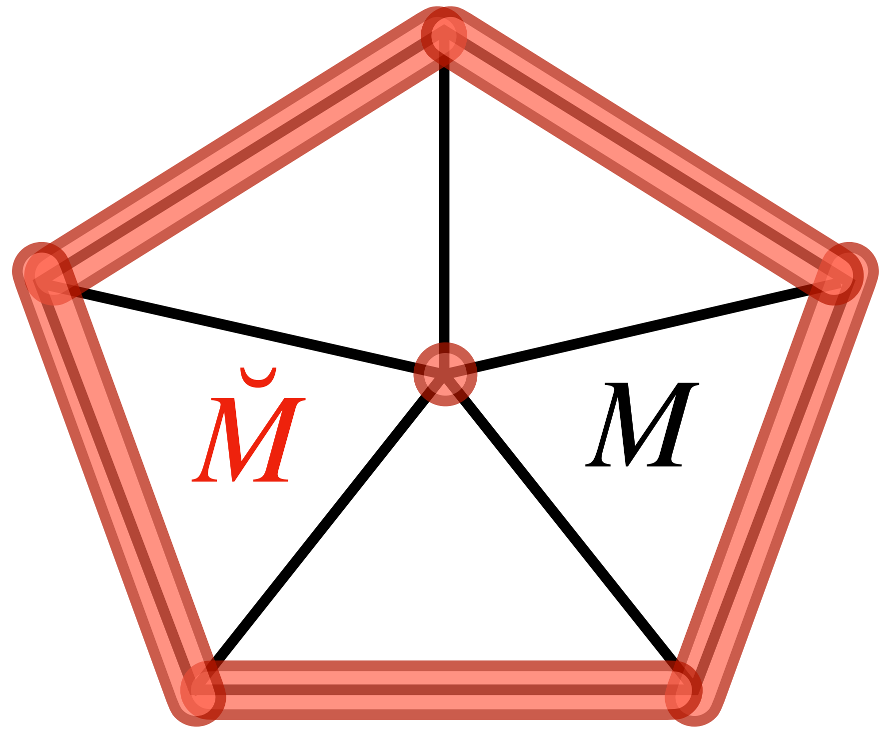
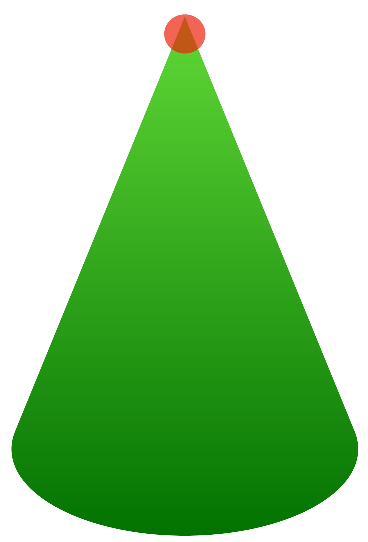
Def.: Let M be a glued Riemannian manifold. Denote by $\breve{M}$ the (abstract) punctured manifold of M by removing all interior codimenion 2 interior boundaries (called bones) and the boundary ∂M.





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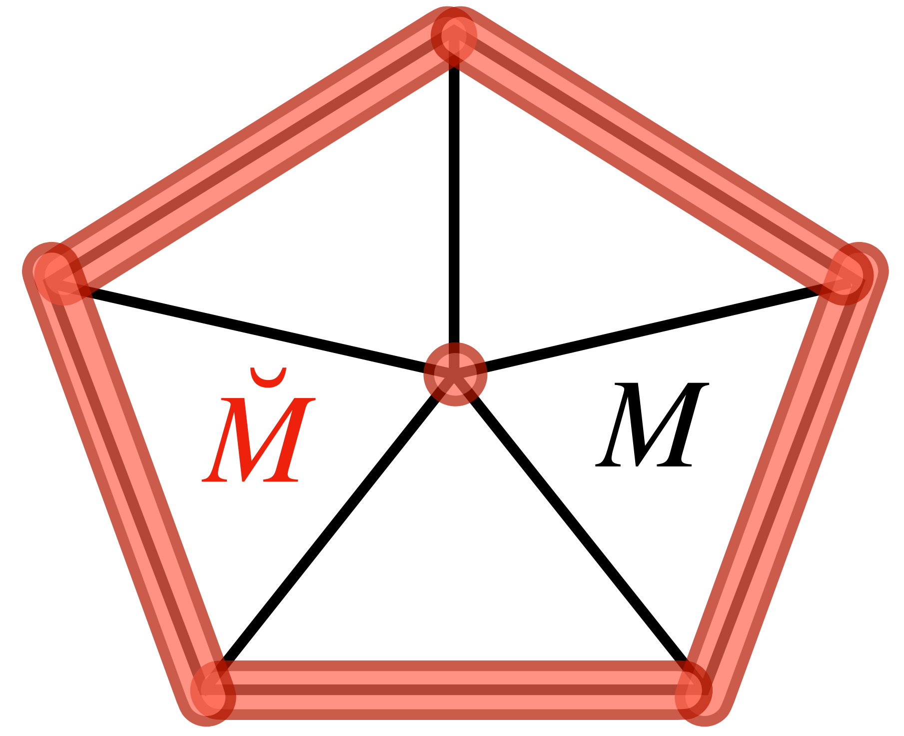
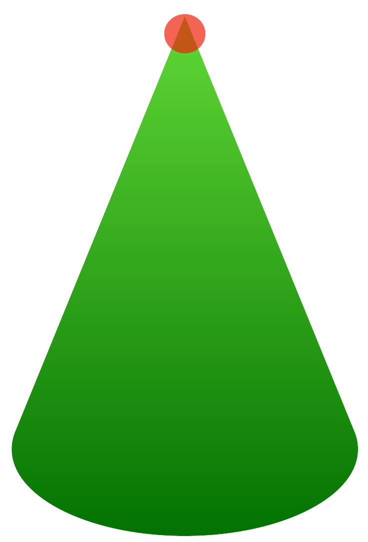
- On $\breve{M}$ exist smooth functions, vector-fields, and k-forms
- Use it to define test functions and Sobolev spaces on M
- $L^p(M) := \{f : M \to \mathbb{R} \mid ||f||_{L^p} < \infty\}, \ ||f||^p_{L^p} := \int_M |f|^p \, \omega$





Test functions

Def.: The space of smooth k-forms is given by $C^\infty\Lambda^k(M) := \{\alpha \in L^\infty\Lambda^k(M) \mid \alpha$ smooth on $\breve{M}, d\alpha \in L^\infty\Lambda^{k+1}(M)\}$ and the set of test functions $C_0^\infty\Lambda^k(M)$ denotes all smooth k-forms with compact support in $M \backslash \partial M$.

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1. Density in $L^p\Lambda^k(M)$
2. Integration by parts ⇒ weak derivatives
3. Definition of Sobolev spaces
4. Density in Sobolev spaces

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Properties

Lemma: Let $\alpha, \beta \in L^1\Lambda^k(M)$. If for all $\Psi \in C_0^\infty\Lambda^k(M)$

$$\int_M g(\alpha, \Psi)\,\omega = \int_M g(\beta, \Psi)\,\omega$$

then $\alpha = \beta$ almost everywhere.

Lemma: $C_0^\infty\Lambda^k(M)$ is dense in $L^p\Lambda^k(M)$ for all $p \in [1, \infty)$.

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Co-derivative $\delta : C^\infty\Lambda^k(M) \to C^\infty\Lambda^{k-1}(M), \quad \delta = (-1)^k \star^{-1} d \star$

Lemma: There holds for $\alpha \in C^\infty\Lambda^{k-1}(M)$ and $\beta \in C^\infty\Lambda^{N-k}(M)$ the integration by parts formula

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$$C^\infty\Lambda^{k,\star}(M) := \{\star^{-1}\alpha \mid \alpha \in C^\infty\Lambda^{N-k}(M)\}, \quad \delta : C^\infty\Lambda^{k,\star}(M) \to C^\infty\Lambda^{k+1,\star}(M)$$

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Sobolev spaces on glued Riemannian manifolds

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$W^{1,p}\Lambda^k(M) := \{\alpha \in L^p\Lambda^k(M) \,|\, d\alpha \in L^p\Lambda^{k+1}(M)\}$ with norm
$\|\alpha\|^p_{W^{1,p}\Lambda^k} := \|\alpha\|^p_{L^p\Lambda^k} + \|d\alpha\|^p_{L^p\Lambda^{k+1}}$. We set $H\Lambda^k(M) := W^{1,2}\Lambda^k(M)$.

Theorem: $C^\infty\Lambda^k(M) \cap W^{1,p}\Lambda^k(M)$ is dense in $W^{1,p}\Lambda^k(M)$ for $p \in [1,3)$.

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- Works for $g \in \mathrm{Reg}^0$
- General metric: WIP

Sobolev spaces on glued Riemannian manifolds

Cor.: Rellich-Kondrachov: $H\Lambda^k(M) \subset\subset L^2\Lambda^k(M)$.

Cor.: Poincaré inequality: For all $\alpha \in H\Lambda^k(M)$ there holds with the mean $\bar{\alpha}$
$\|\alpha - \bar{\alpha}\|_{L^2\Lambda^k} \leq C\|d\alpha\|_{L^2\Lambda^{k+1}}$.

Def.: $\mathring{H}\Lambda^k(M)$ is the closure of $C_0^\infty\Lambda^k(M)$ in $H\Lambda^k(M)$. $H^{-1}\Lambda^k(M)$ denotes the dual space of $\mathring{H}\Lambda^k(M)$.
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Lemma: Define $W^{l,p}\Lambda^{k,\bar{\star}}(\partial M) := \{\bar{\star}^{-1}\alpha \mid \alpha \in W^{l,p}\Lambda^{N-1-k}(\partial M)\}$. There exists a trace operator $\mathrm{Tr} : H\Lambda^k(M) \to H^{-1/2}\Lambda^{k,\bar{\star}}(\partial M)$ for all $\alpha \in H\Lambda^k(M)$ such that

$$\|\mathrm{Tr}\alpha\|_{H^{-1/2}\Lambda^{k,\bar{\star}}(\partial M)} \leq C\|\alpha\|_{H\Lambda^k}.$$

H(div) and H(curl)

- For Riemannian manifolds we can identify smooth 1-forms with vector fields

$$X^\flat = g(X, \cdot\,) \in \Lambda^1, \qquad \alpha^\sharp = g^{-1}(\alpha, \cdot\,) \in \mathfrak{X}$$

- Not possible for glued Riemannian manifolds

$$\alpha \in C^\infty \Lambda^1(M) \not\Rightarrow \alpha^\sharp \in C^\infty \mathfrak{X}(M)$$

- Covariant derivatives depend on metric: $\operatorname{div} X = \star\, d \star X^\flat, \operatorname{curl} X = (\star d X^\flat)^\sharp$

- Idea: Relate $H(\operatorname{div}, M)$ and $H(\operatorname{curl}, M)$ with $H\Lambda^{N-1}(M)$ and $H\Lambda^1(M)$

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$$H(\operatorname{curl}, M) := \{X \in L^2\mathfrak{X}(M) \mid \operatorname{curl} X \in L^2\mathfrak{X}(M)\}$$

Lemma: There holds $H(\operatorname{div}, M) = \{(\star^{-1}\alpha)^\sharp \;\mid\; \alpha \in H\Lambda^{N-1}(M)\}$ and $H(\operatorname{curl}, M) = \{\alpha^\sharp \;\mid\; \alpha \in H\Lambda^1(M)\}$.

H(div) and H(curl)

- $H^1(M) := \{f \in L^2(M) \mid \nabla f \in L^2\mathfrak{X}(M)\} = H\Lambda^0(M)$
- Integration by parts

$$\int_M g(\nabla f, u)\,\omega = -\int_M f \operatorname{div} u\,\omega + \int_{\partial M} f\, g(u, n)\,\omega_{\partial M}, \qquad f \in H^1(M), u \in H(\operatorname{div}, M)$$

$$\int_M g(\operatorname{curl} u, v)\,\omega = \int_M g(u, \operatorname{curl} v)\,\omega + \int_{\partial M} g(u, v \times n)\,\omega_{\partial M}, \qquad u, v \in H(\operatorname{curl}, M)$$

H(div) and H(curl)

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- Traces and continuity

$$\|g(u,n)\|_{H^{-1/2}(\partial M)} \le C\|\mathrm{Tr}(\star u^\flat)\|_{H^{-1/2}\Lambda^{N-1,\bar{\star}}(\partial M)} = C\|\mathrm{Tr}\alpha\|_{H^{-1/2}\Lambda^{N-1,\bar{\star}}(\partial M)} \le C\|\alpha\|_{H\Lambda^{N-1}} \le C\|u\|_{H(\mathrm{div})}$$

$$\|u \times n\|_{H^{-1/2}(\partial M)} \le C\|u\|_{H(\mathrm{curl})}$$

Distributional differential operators revisited

1. $C_0^\infty \Lambda^{N-1}(M)$ space of test functions

$$\langle \nabla f, \Psi \rangle = -\int_M f \,\mathrm{div}\, \Psi\, \omega, \qquad f \in C^\infty(\mathcal{T}), \quad \Psi = (\star^{-1}\Phi)^\sharp, \Phi \in C_0^\infty \Lambda^{N-1}(M)$$

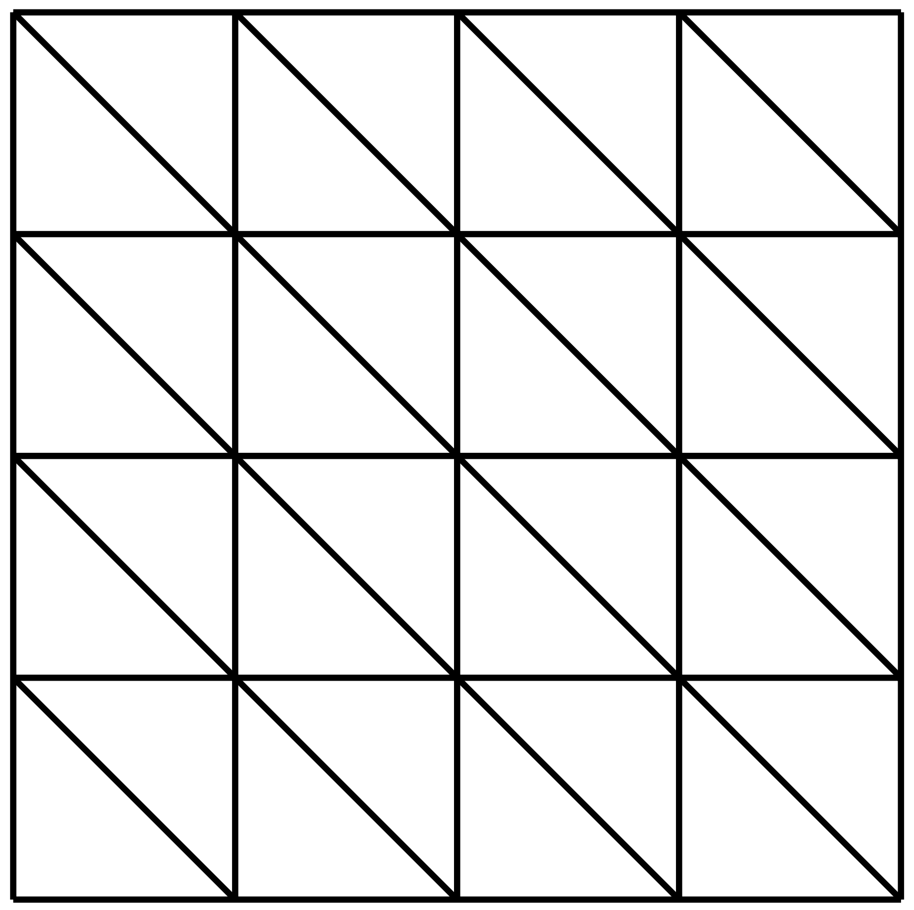
2. Integration by parts element-wise

$$-\sum_{T\in\mathcal{T}} \int_T f \,\mathrm{div}\, \Psi\, \omega = \sum_{T\in\mathcal{T}} \int_T g(\nabla f, \Psi)\, \omega - \sum_{E\in\mathcal{E}} \int_E [\![f]\!]\, g(\Psi, n)\, \omega_{\partial M}$$

$$|\langle \nabla f, \Psi \rangle| \leq C(f)\, \|\Psi\|_{H(\mathrm{div})} \leq C(f) \|\Phi\|_{H\Lambda^{N-1}}$$

3. Density: $C_0^\infty \Lambda^{N-1}(M)$ dense in $H\Lambda^{N-1}(M)$

$\Rightarrow \langle \nabla f, v \rangle$ well-defined for $v \in H(\mathrm{div}, M)$



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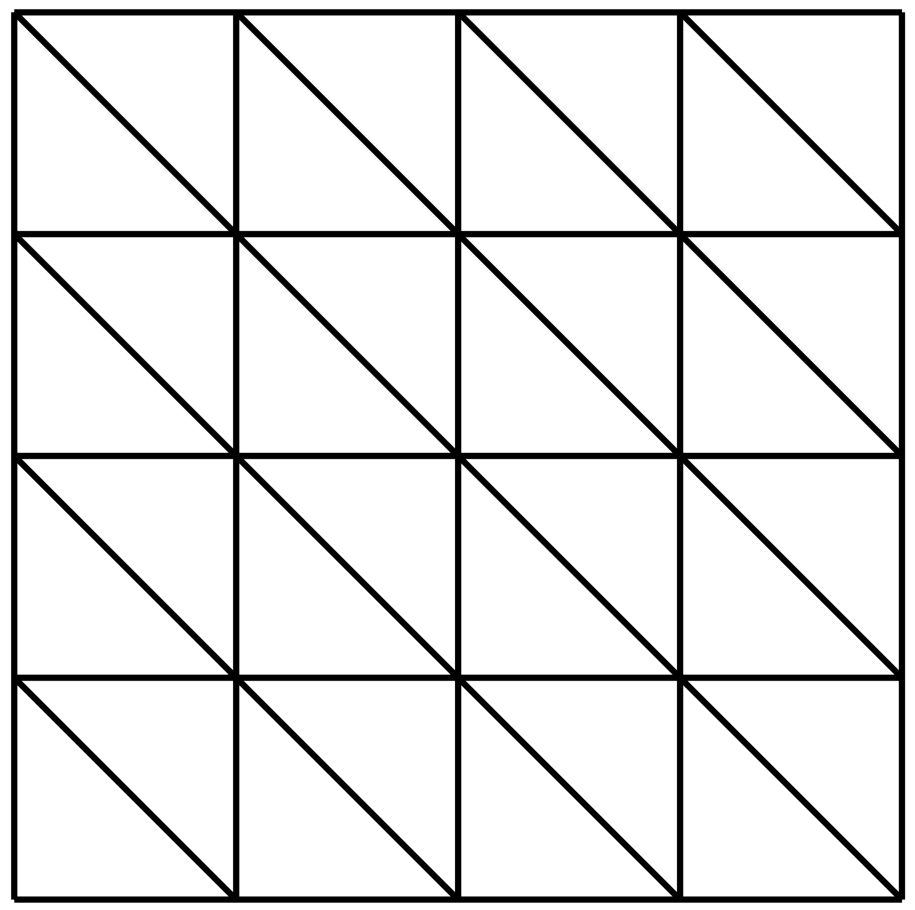


- $\langle \mathrm{div} u, f \rangle$ for $u \in C^\infty(\mathcal{T}, \mathbb{R}^{\mathbb{N}}), f \in H^1(M)$
- $\langle \mathrm{curl} u, v \rangle$ for $u \in C^\infty(\mathcal{T}, \mathbb{R}^3), v \in H(\mathrm{curl}, M)$

Implementation

- Chart (U, ϕ). Define on parameter space

$$H_\delta(\mathrm{div}, \Phi(U)) := \{w = w^i \partial_i : \Phi(U) \to \mathbb{R}^N \mid w^i \in C^\infty(\Phi(U \cap \mathscr{T})), [\![\delta(w, n_\delta)]\!] = 0\}$$

$$H_\delta(\mathrm{curl}, \Phi(U)) := \{w = w^i \partial_i : \Phi(U) \to \mathbb{R}^3 \mid w^i \in C^\infty(\Phi(U \cap \mathscr{T})), [\![w \times_\delta n_\delta]\!] = 0\}$$

- Define operator

$$Q_g w = \frac{1}{\sqrt{\det g}} w^i \partial_i, \qquad w \in H_\delta(\mathrm{div}, \Phi(U))$$

- $w \in H_\delta(\mathrm{div}, \Phi(U))$ iff $Q_g w \in H(\mathrm{div}, U)$

$$[\![g(Q_g w, n)]\!] = 0 \Leftrightarrow [\![\delta(w, n_\delta)]\!] = 0$$

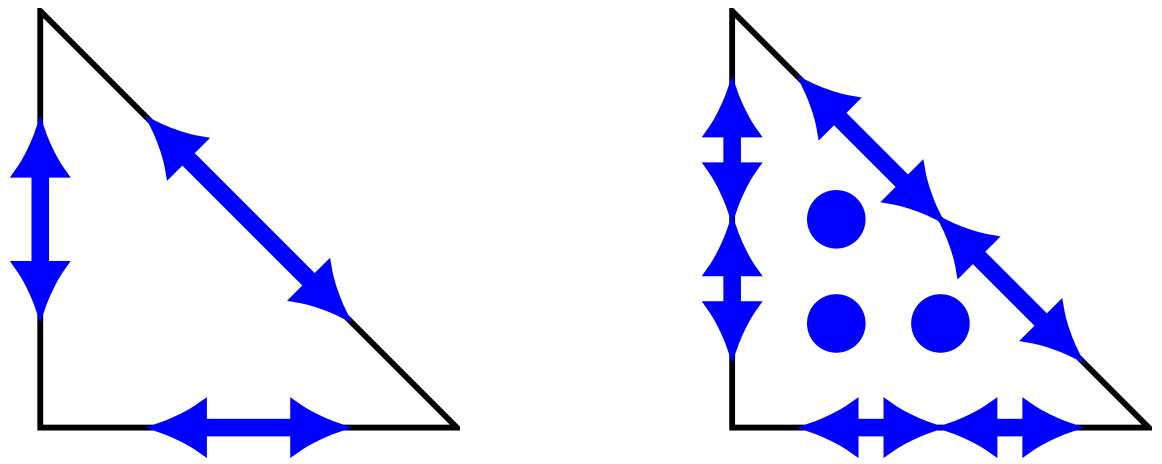
$u \in H_\delta(\mathrm{curl}, \Phi(U))$ iff $u \in H(\mathrm{curl}, U)$

$$[\![u \times n]\!] = 0 \Leftrightarrow [\![u \times_\delta n_\delta]\!] = 0$$

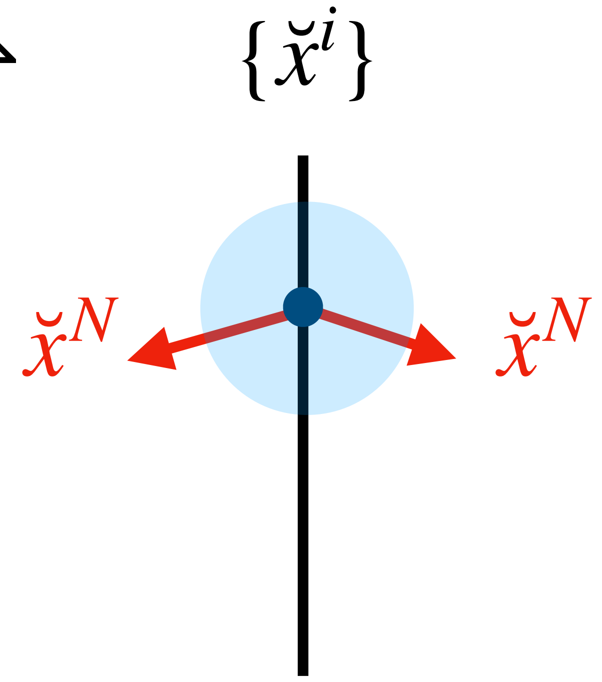
- Tangential vectors depend on tt-components of metric

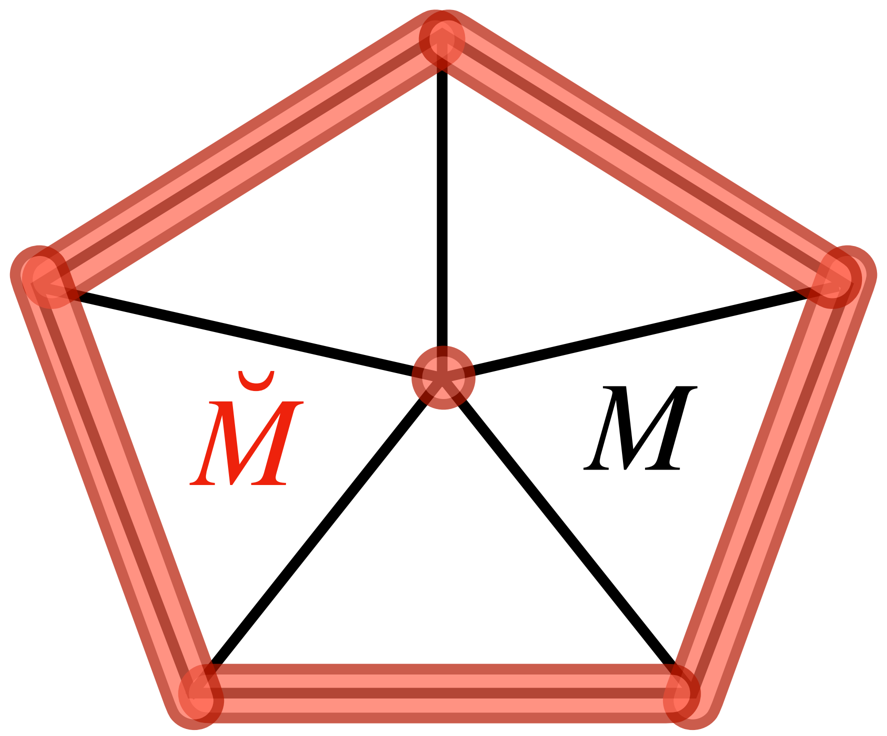
Gopalakrishnan, N., Schöberl, Wardetzky: Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics, SMAI J. Comput. Math., 2023.

Summary & Outlook



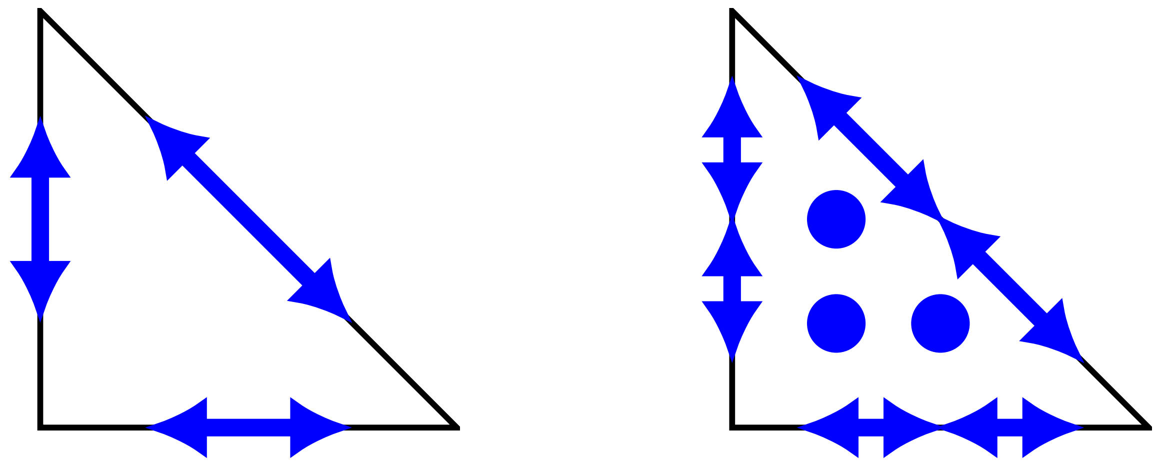
- Sobolev spaces on glued Riemannian manifolds
- Glueing of manifolds, smooth structure, Fermi coordinates
- Definition of test functions for k-forms
- Density and integration by parts





Summary & Outlook

- Sobolev spaces on glued Riemannian manifolds



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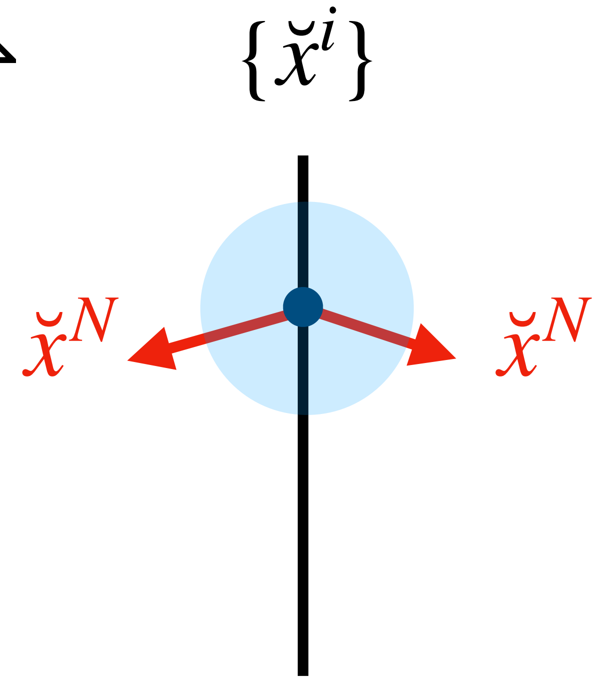

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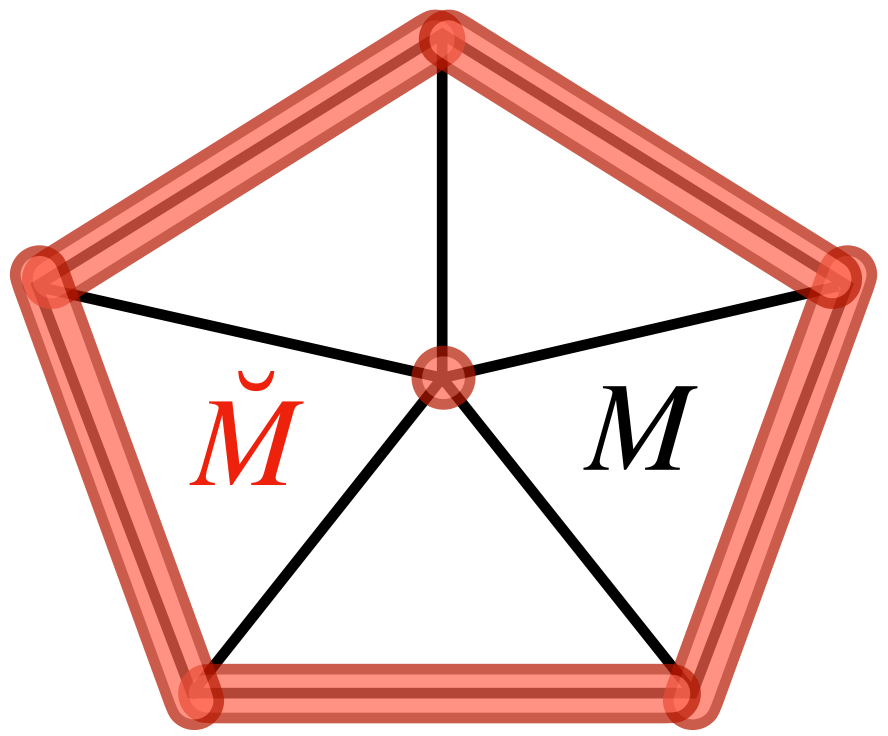
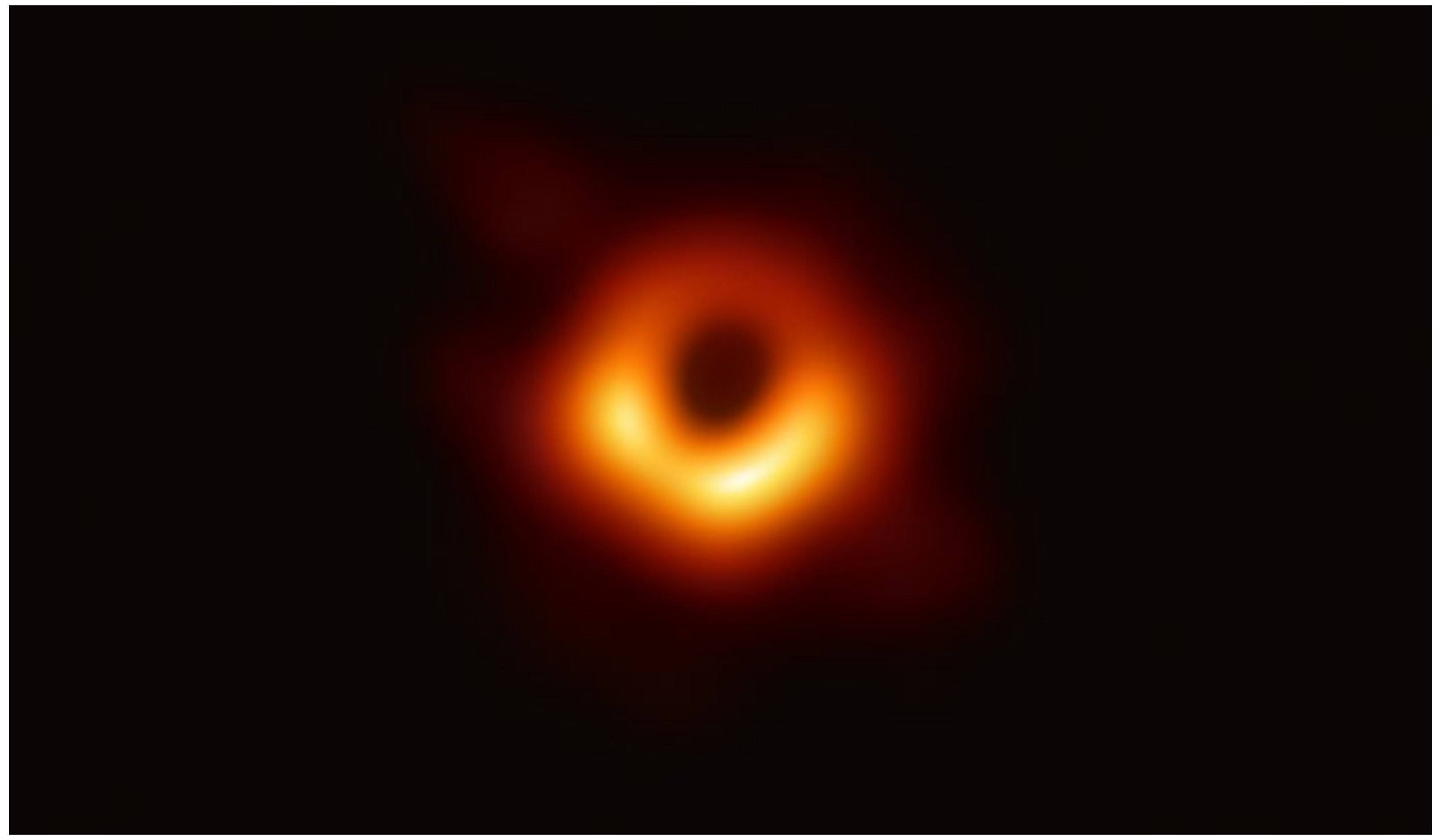


- Analysis on discrete/approximated Riemannian manifolds ($g_h \to g$)
- Polyhedral (and curved) surfaces included (discrete differential geometry + FEEC)
- Long-term goal: Application to geometric flows and numerical relativity



By Event Horizon Telescope (EHT)

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- Li: Finite Elements with Applications in Solid Mechanics and Relativity, PhD thesis, 2018.
- Gawlik: High-Order Approximation of Gaussian Curvature with Regge Finite Elements, SIAM J. Numer. Anal., 2020.
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Thank you for your attention!