# Distributional curvature approximation from Regge metrics

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Portland State UNIVERSITY

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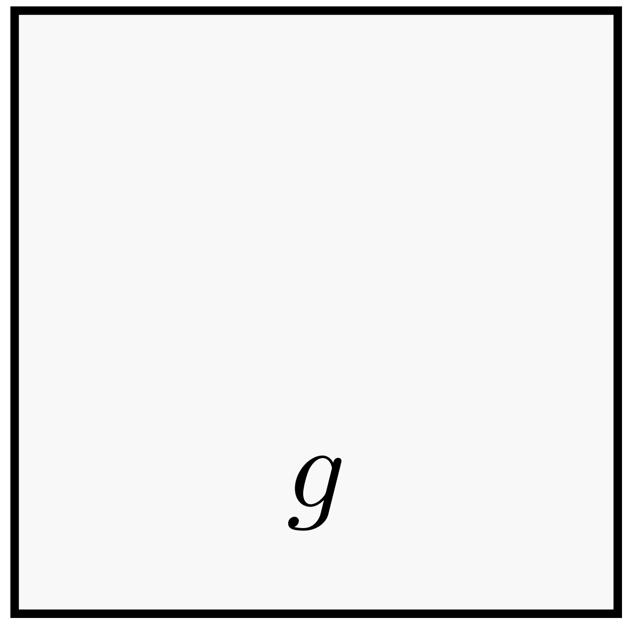
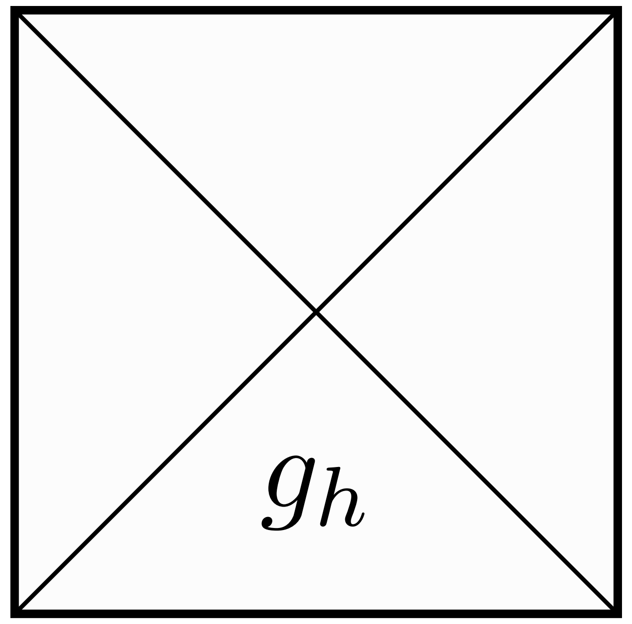
# Motivation

- Riemannian manifold $(\Omega, g), \Omega \subset \mathbb{R}^N, g$ metric tensor
- Approximation $g_h$ of $g$ on a triangulation
- How to approximate $g$?
- How to compute discrete curvature? Convergence?



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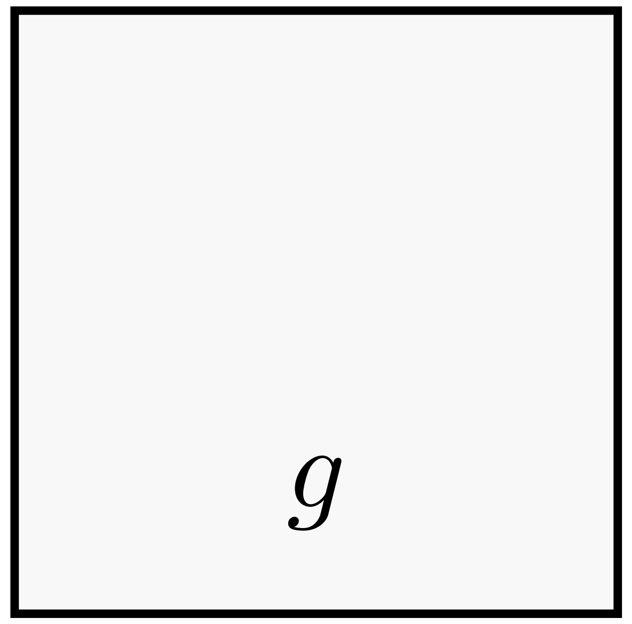


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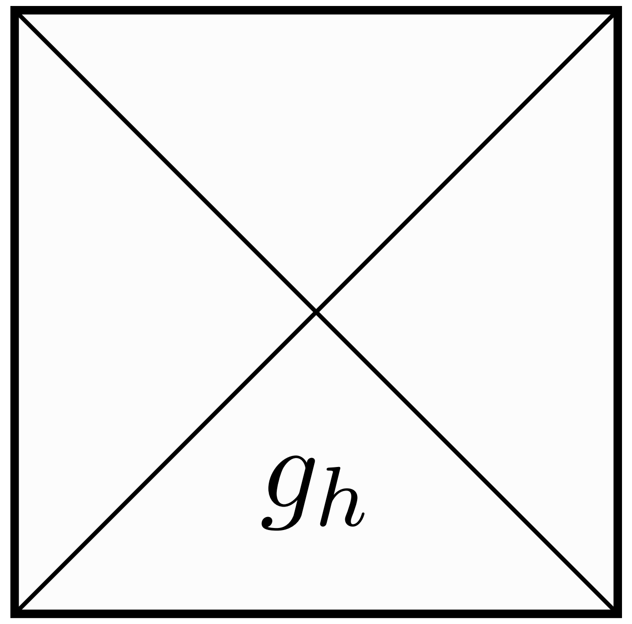


$g_h \in ?$ $\qquad \mathfrak{R}(g_h) = ?$ $\qquad \|\mathfrak{R}(g_h) - \mathfrak{R}(g)\|_? \leq \mathcal{O}(h^?)$

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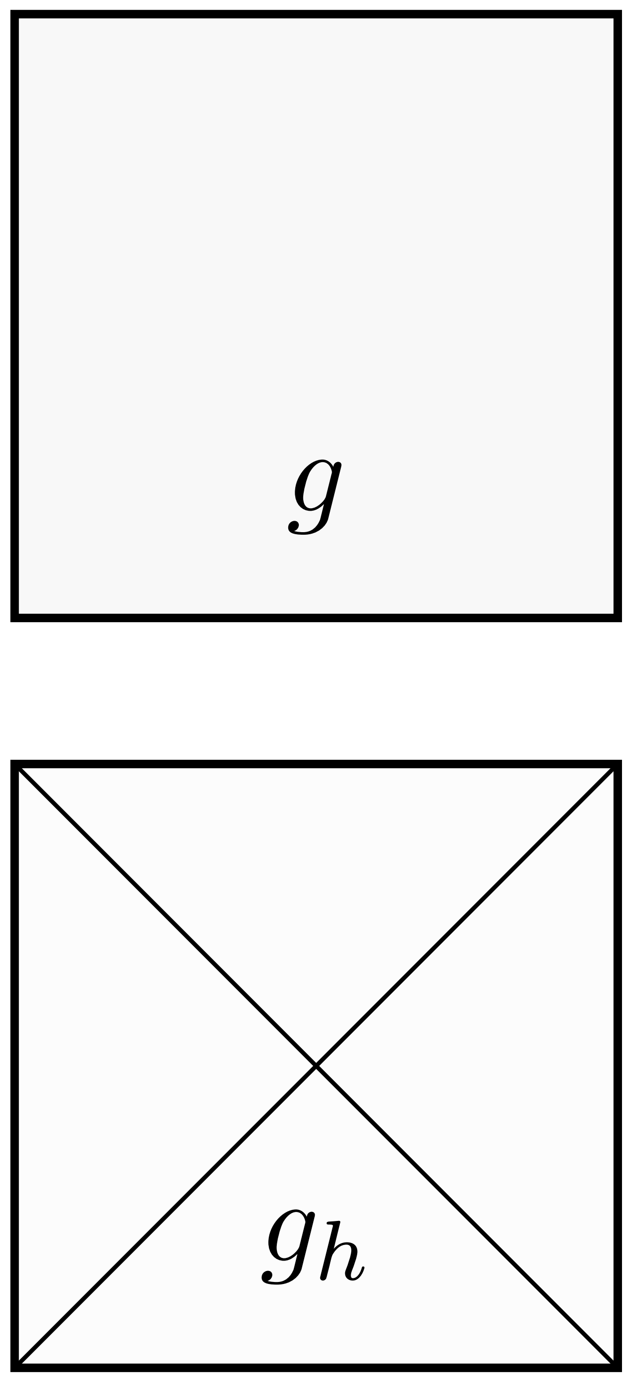


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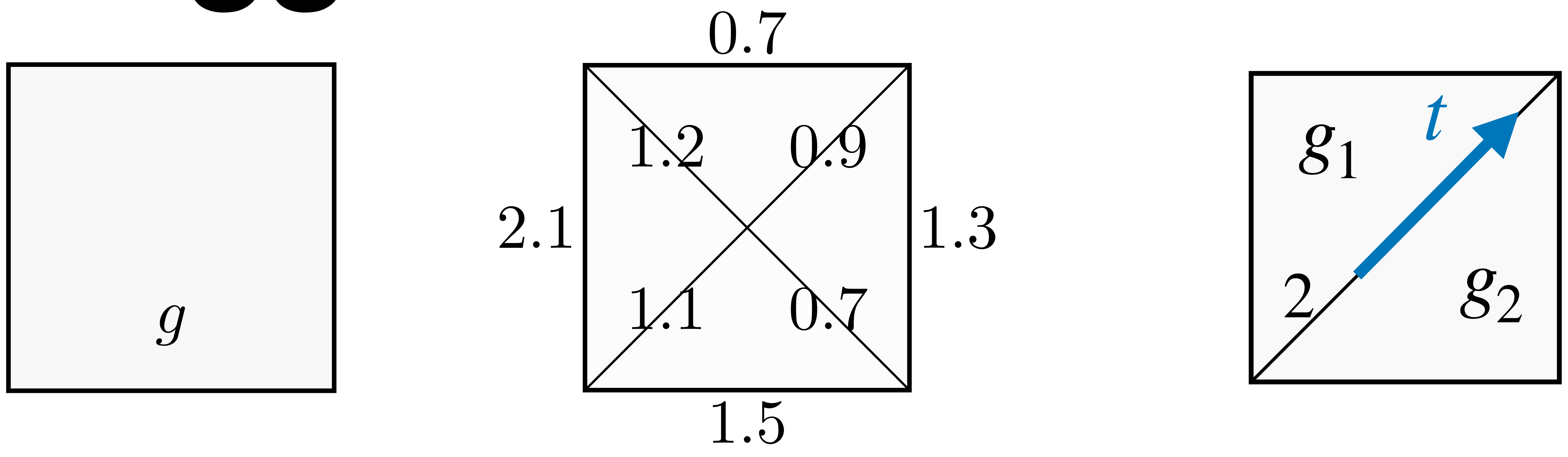
- Application in discrete differential geometry
- Possible extension to geometric flows and numerical relativity

# Regge finite elements & metric




$$\int_E g_1(t,t)\,ds = \int_E g_2(t,t)\,ds = 2$$

$$g_h = g_1 \cup g_2$$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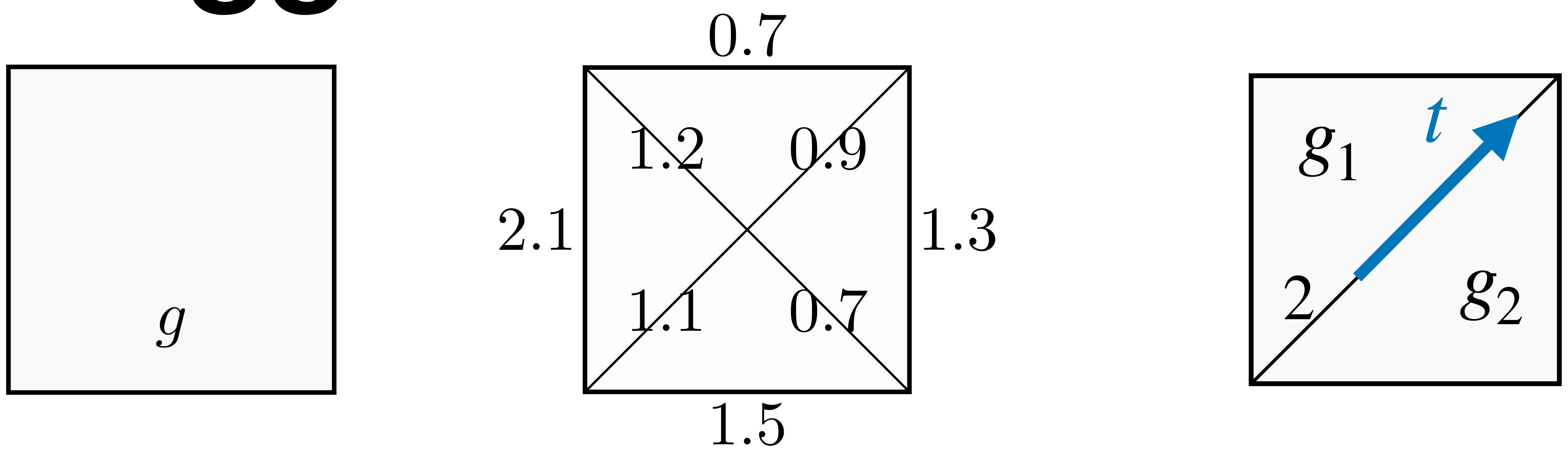


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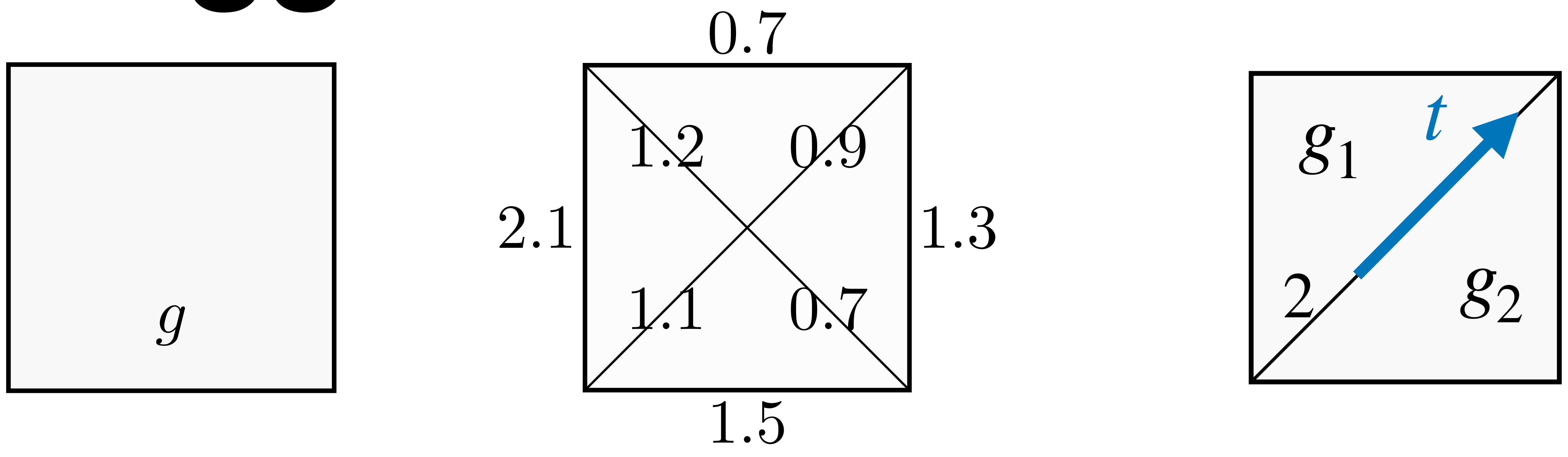


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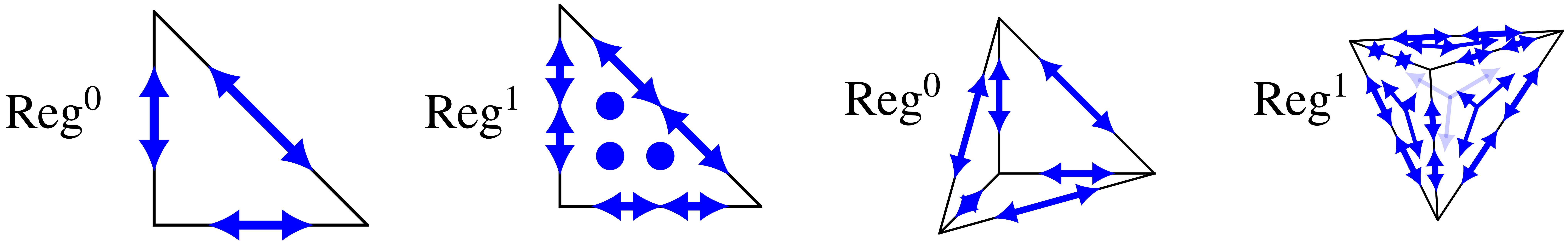
$$\mathrm{Reg}^k := \left\{ \sigma \in \mathscr{P}^k(\mathscr{T}, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \mid \sigma \text{ is tangential-tangential continuous} \right\}$$

$$H(\mathrm{curl\,curl}) := \left\{ \sigma \in L^2(\Omega, \mathbb{R}^{N\times N}_{\mathrm{sym}}) \mid \mathrm{curl}^T\mathrm{curl}(\sigma) \in H^{-1} \right\}$$





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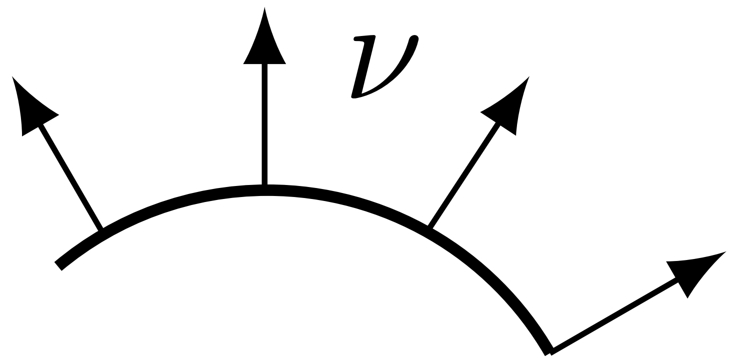
# Curvature

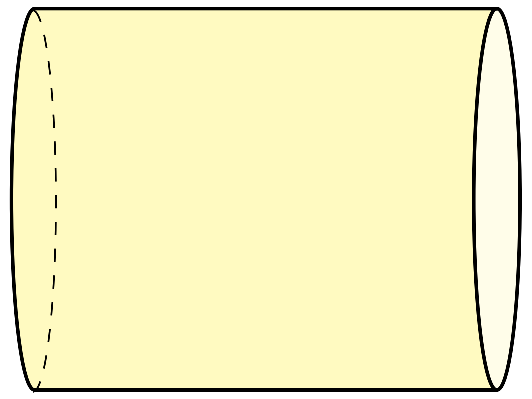



$\nabla\nu$ … shape operator, Weingarten tensor → extrinsic curvature

- Gauss Theorema Egregium: $K = f(g)$ → intrinsic curvature

$K = \frac{1}{r^2}, \quad H = \frac{1}{r}$

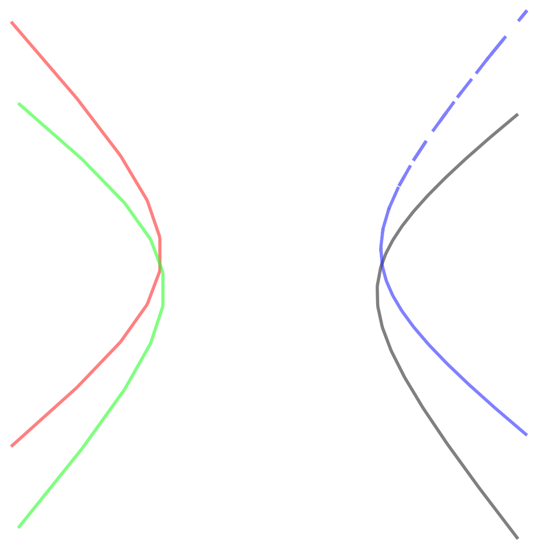
- $K = \kappa_1 \kappa_2, \qquad H = \frac{1}{2}(\kappa_1 + \kappa_2)$



$K = 0, \quad H = \frac{1}{2r}$

- $\mathfrak{R}(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$

Levi-Civita connection $\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$



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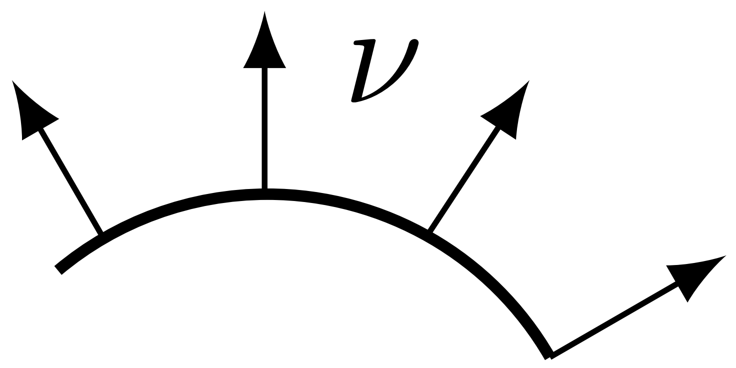


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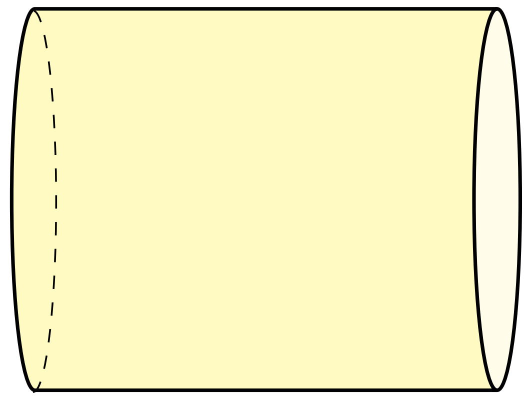

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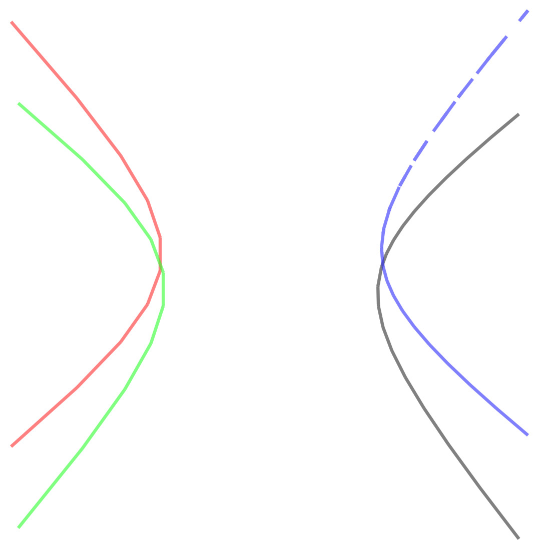

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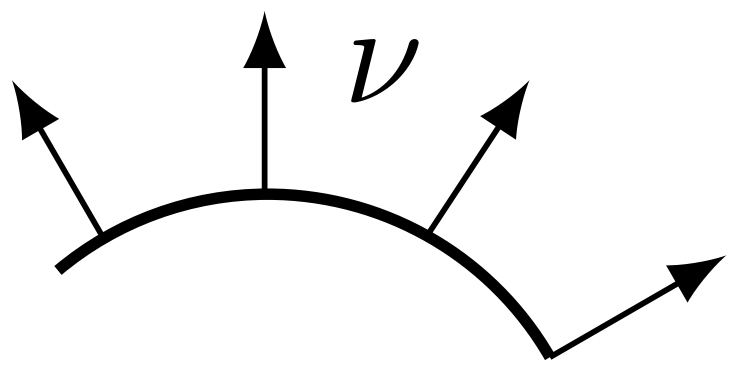


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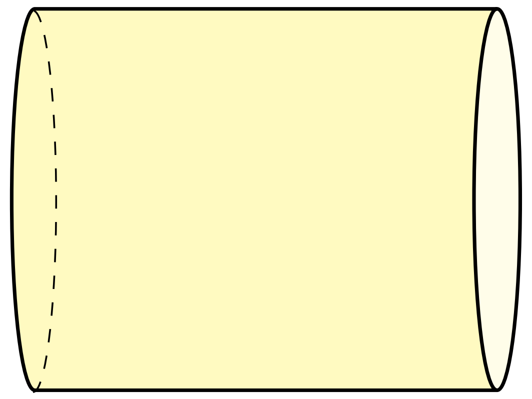

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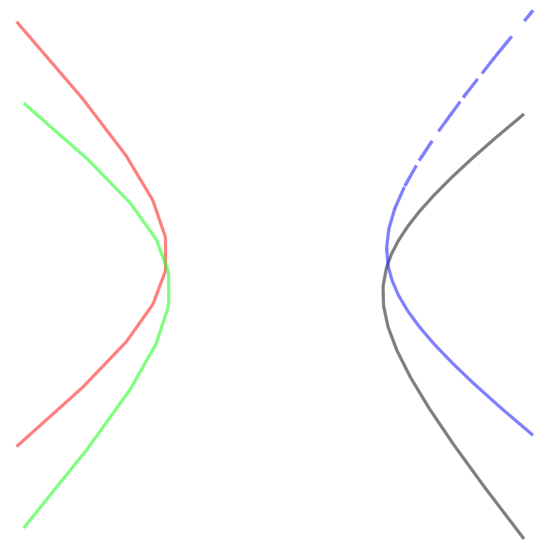

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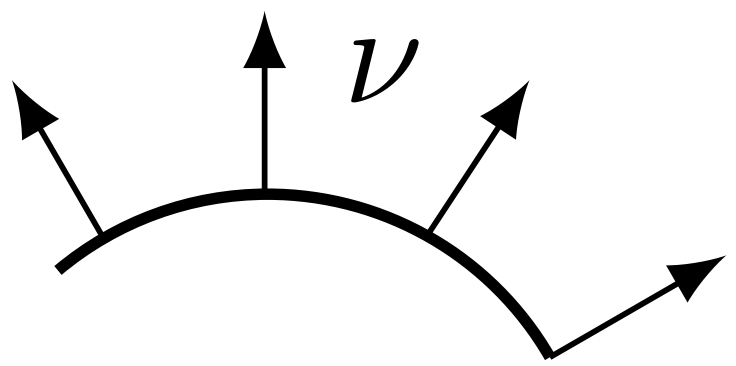


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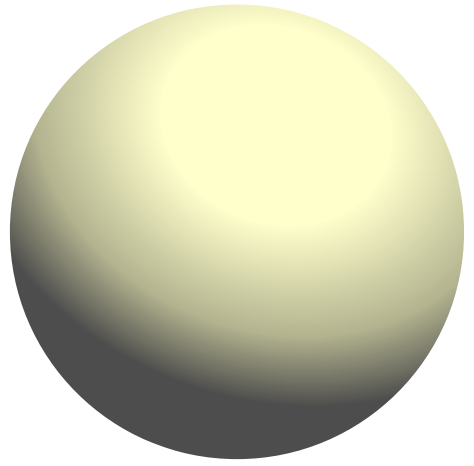

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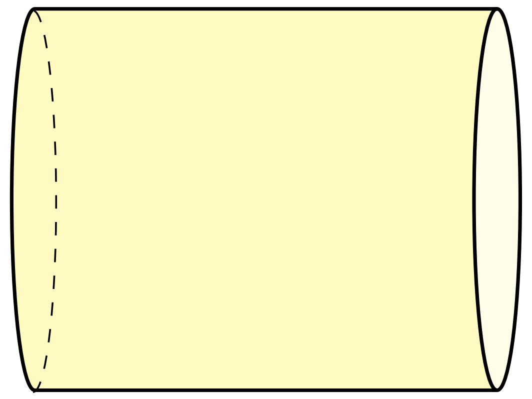

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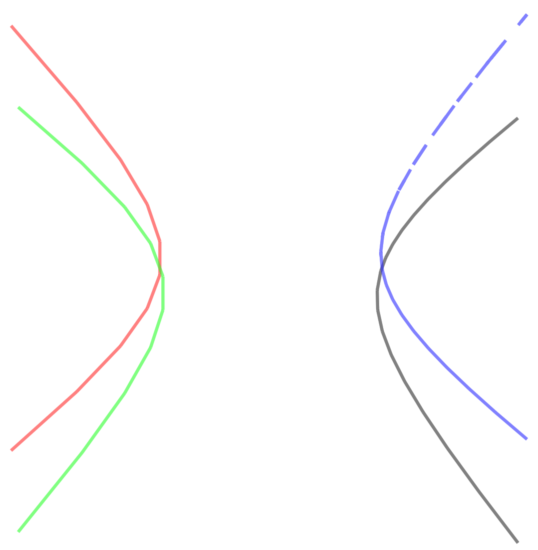

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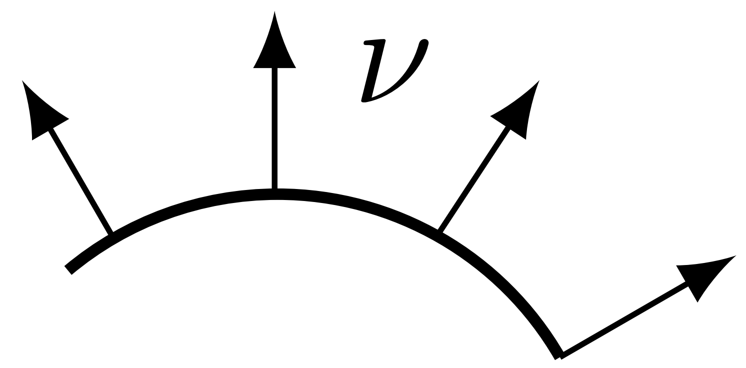


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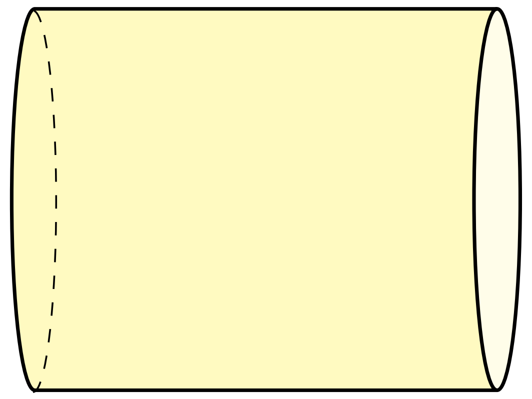

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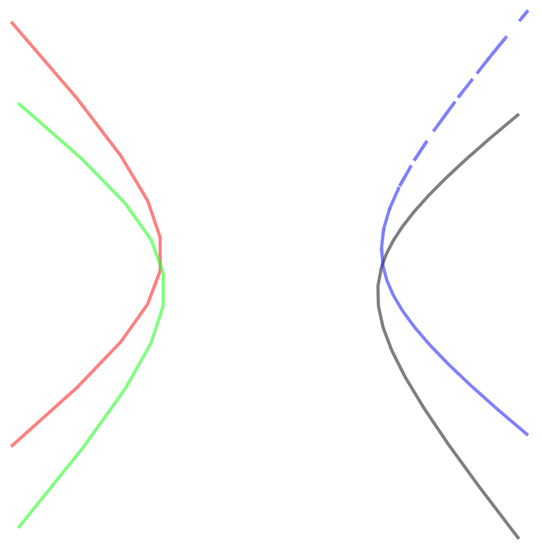

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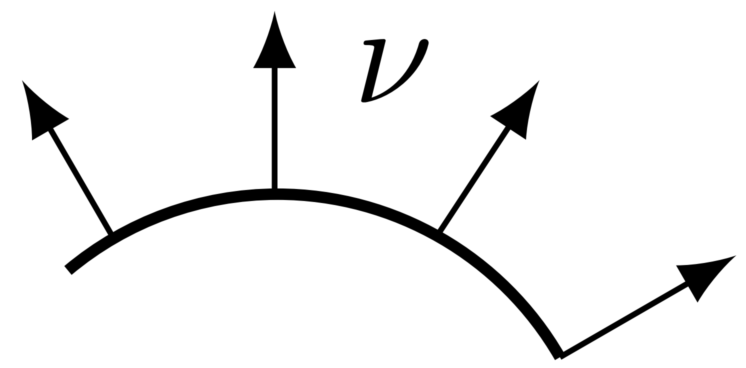


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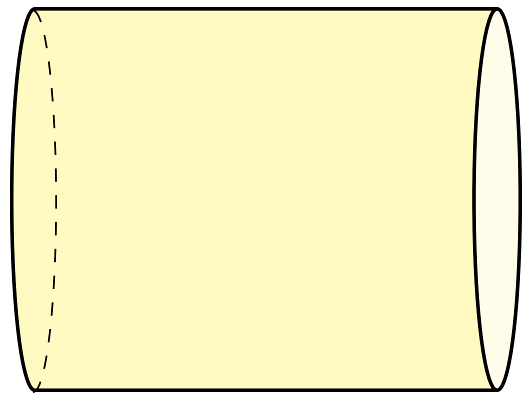

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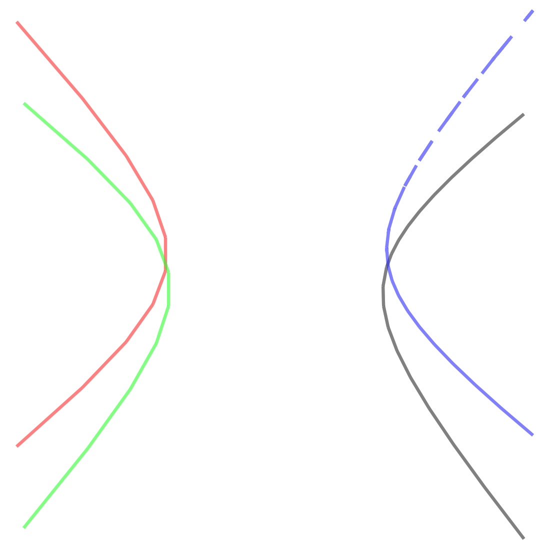

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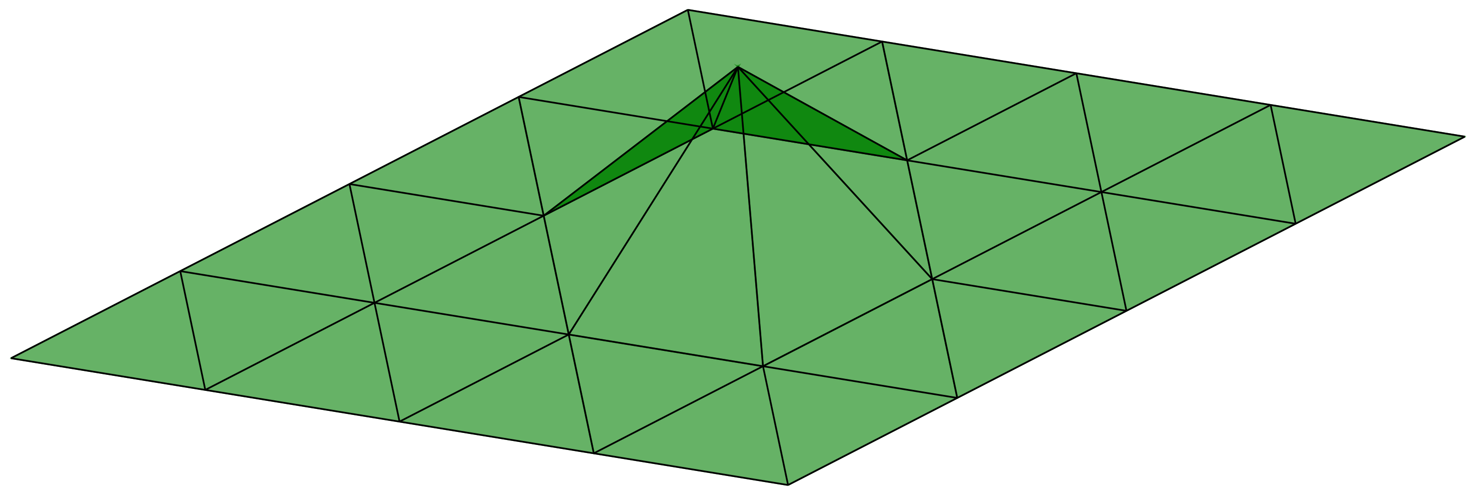
$g_h$ is only tt-continuous

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_k g_{ij})$$

$\mathfrak{R}(g_h)$ is nonlinear distribution!

# Angle defect in differential geometry and Regge calculus



Regge: General relativity without coordinates, Il Nuovo Cimento, 1961.

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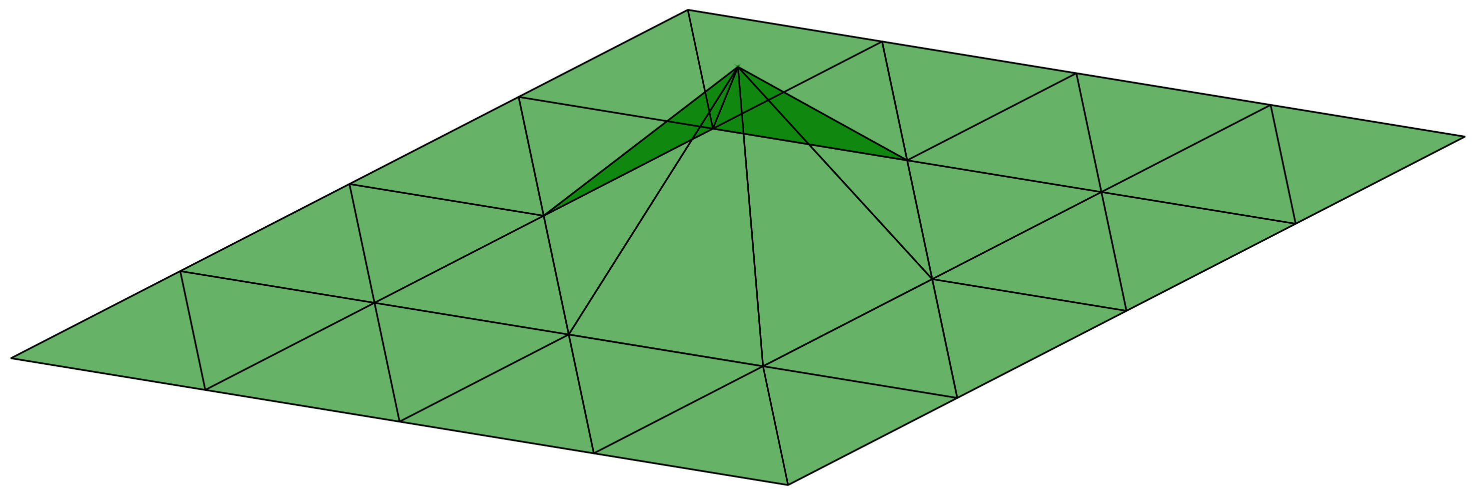

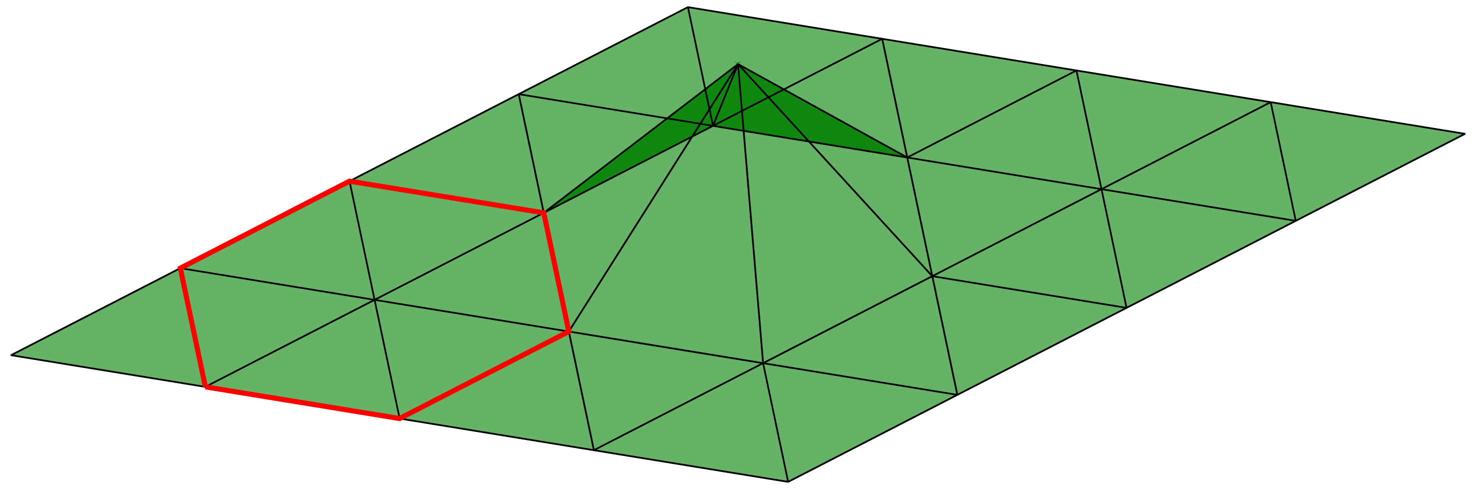

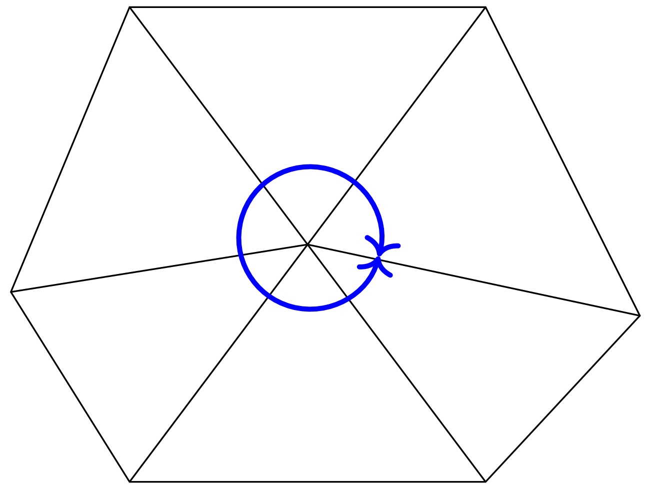

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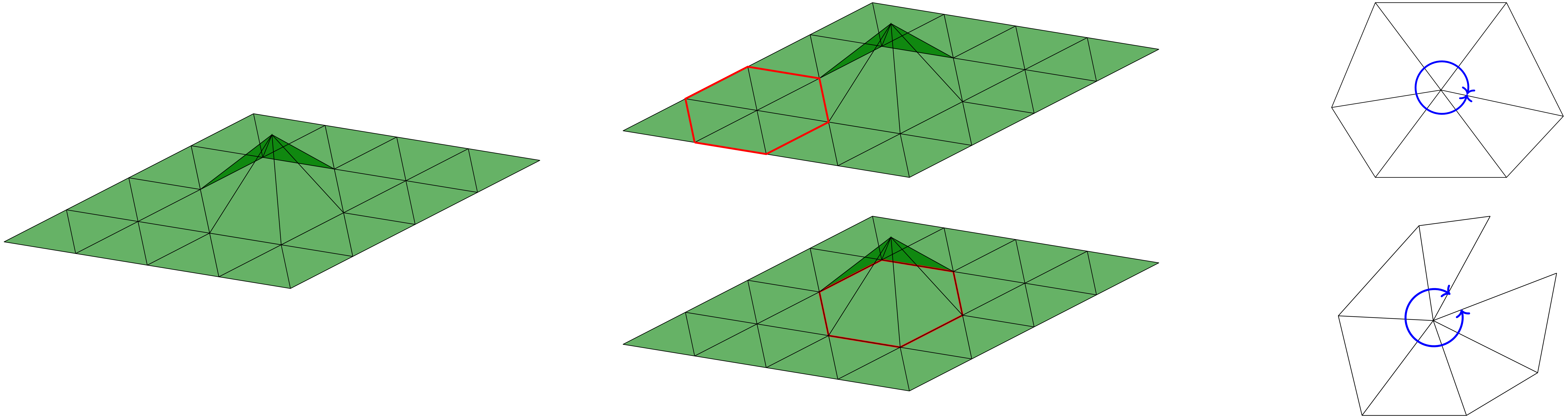

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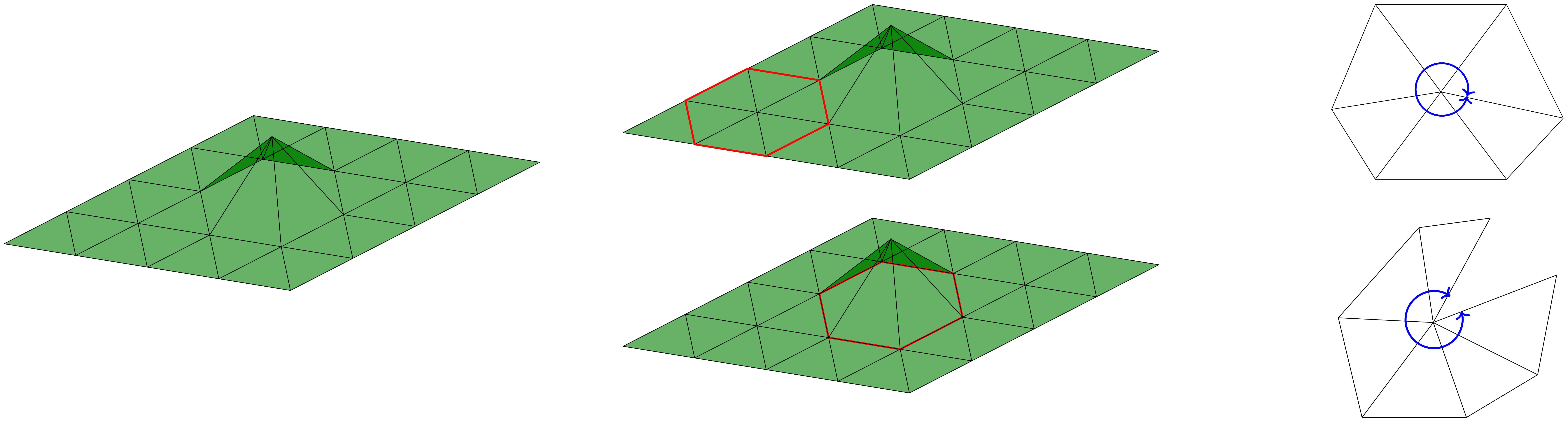

- Angle defect at vertices to measure curvature
- Discrete differential geometry and Regge calculus
- Proof of convergence in the sense of measure

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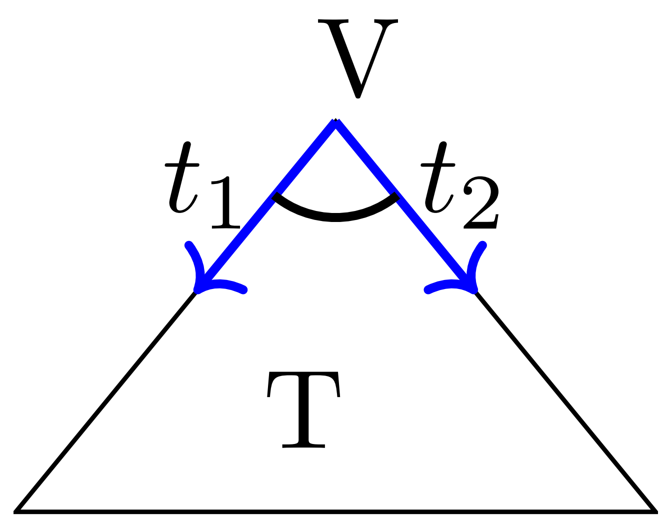
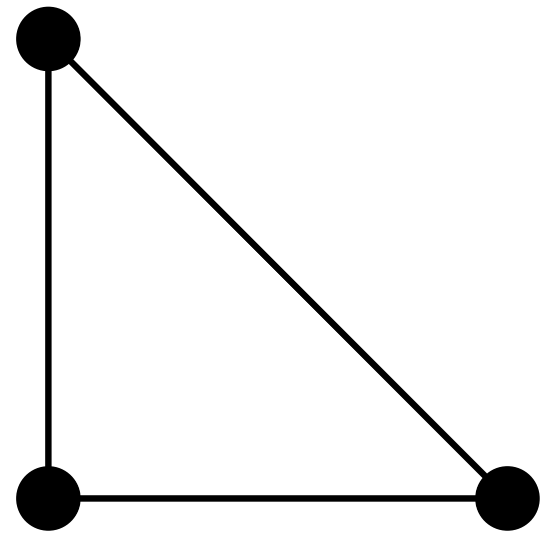
# Distributional Gauss curvature




- Angle defect acts on vertices → use as part of distribution

$$\sphericalangle_V(g) = 2\pi - \sum_{T \supset V} \arccos(g|_T(t_1, t_2))$$

$$\langle K(g_h), v_h \rangle := \sum_{V \in \mathscr{V}} \sphericalangle_V(g_h)\, v_h(V)$$



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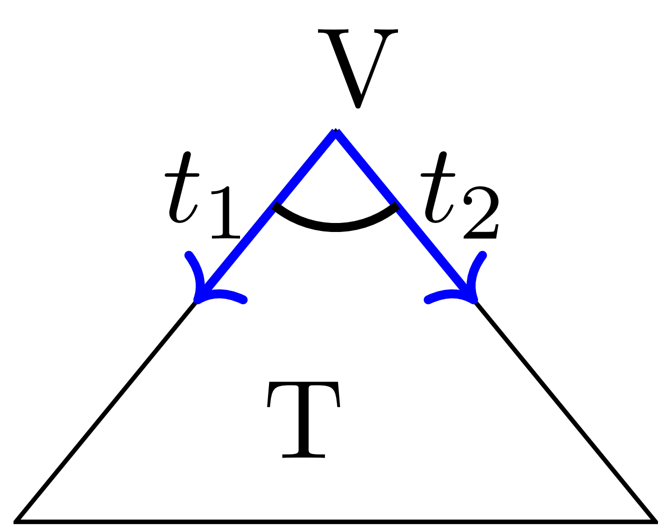


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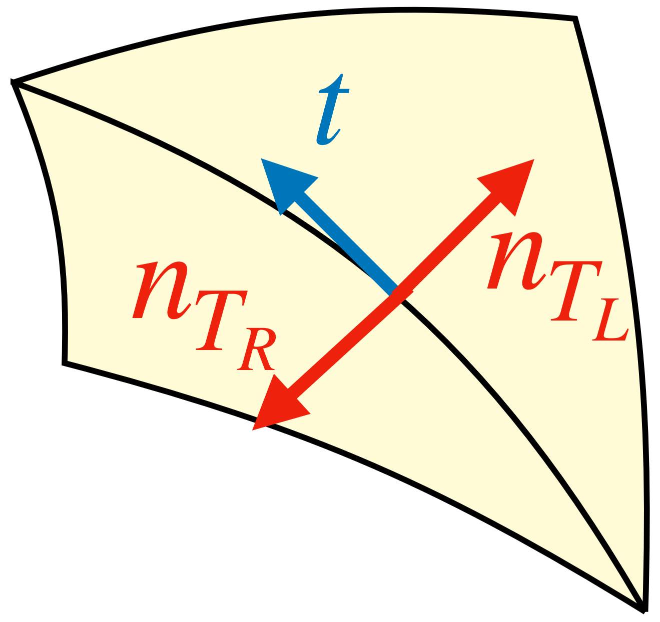
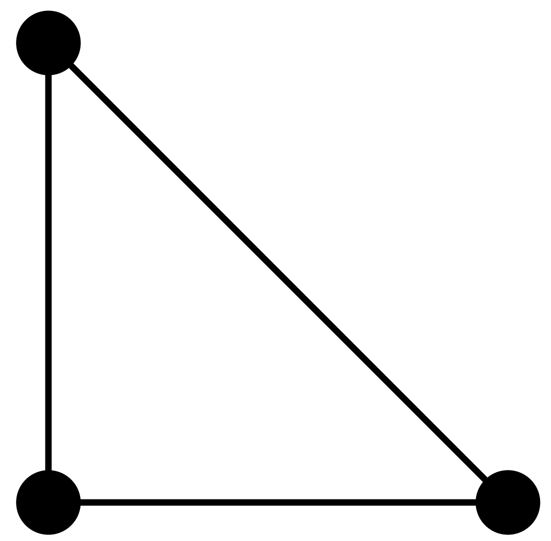



- Geodesic curvature different for non-$C^1$ interfaces:

$$[\![\kappa_g]\!] := \kappa_g|_{T_L} + \kappa_g|_{T_R} \neq 0$$

$$\kappa_g|_T = g(\nabla_t t, n)|_T$$



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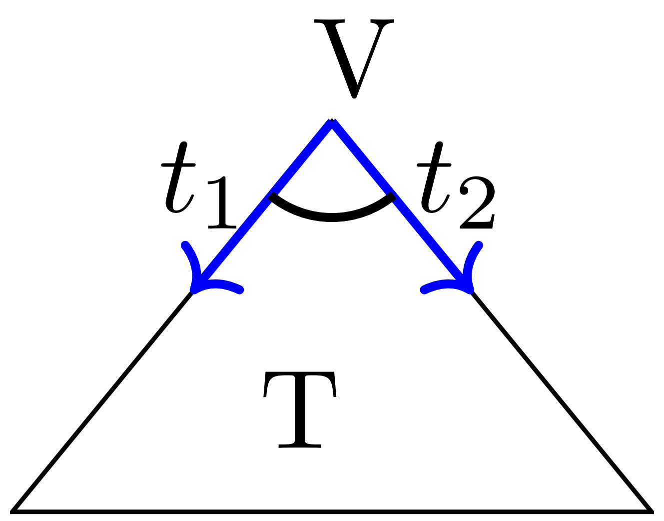


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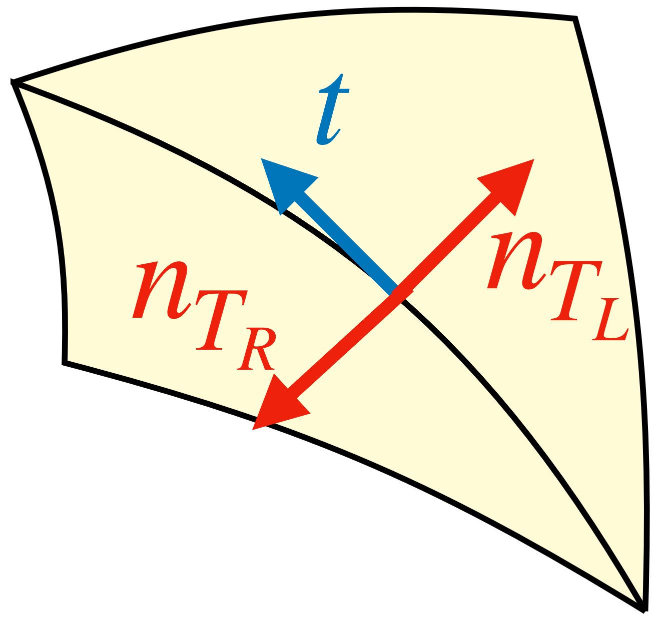


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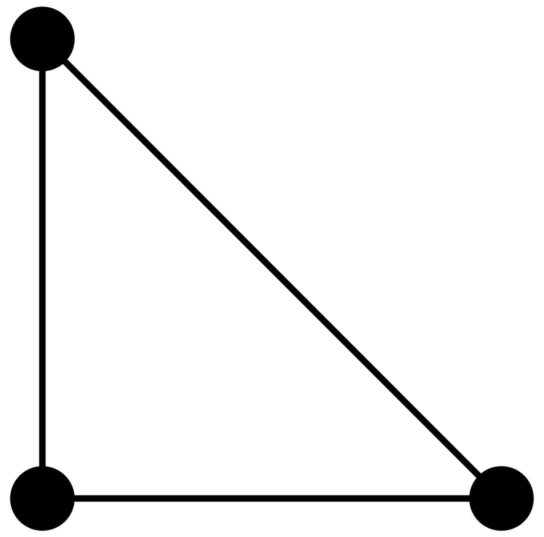

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# Distributional curvatures

- Gauss curvature: 2D, scalar, $K = \dfrac{\mathfrak{R}_{1221}}{\det g} = \dfrac{1}{2} S$
- Scalar curvature: nD, scalar, $S = \mathfrak{R}_{ijkl} g^{ik} g^{jl}$
- Einstein tensor: nD, matrix, $G_{ij} = \mathrm{Ric}_{ij} - \dfrac{1}{2} S\, g_{ij}$
- Riemann curvature tensor: nD, 4th order tensor, $\mathfrak{R}_{ijkl}$
- Ricci curvature tensor: nD, matrix, $\mathrm{Ric}_{ij} = \mathfrak{R}_{iajb} g^{ab}$


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# Integral representation

$$\langle K(g_h), u \rangle = \sum_{T \in \mathcal{T}} \int_T K(g_h)\,|_T u\, \omega_T + \sum_{E \in \mathcal{E}} \int_E [\![\kappa_g]\!]\, u\, \omega_E + \sum_{V \in \mathcal{V}} \measuredangle_V(g_h)\, u(V)$$

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$$\tilde{g}(t) = g + t\,\sigma, \qquad \frac{d}{dt}\big(K(\tilde{g}(t))\big)\,|_{t=0} = D_g K(g)[\sigma]$$

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- Hellan-Herrmann-Johnson method in covariant setting
- Integral representation of error

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Strategy applicable (with more work) for all curvature quantities!

# Convergence results

Let $g$ be a sufficiently smooth metric and $g_h \in \mathrm{Reg}^k$ be an approximation such that $\|g_h - g\|_{L^2} \lesssim h^{k+1}$. Then, for $k \geq 0$ for $N = 2$ and $k \geq 1$ for $N \geq 3$ there holds

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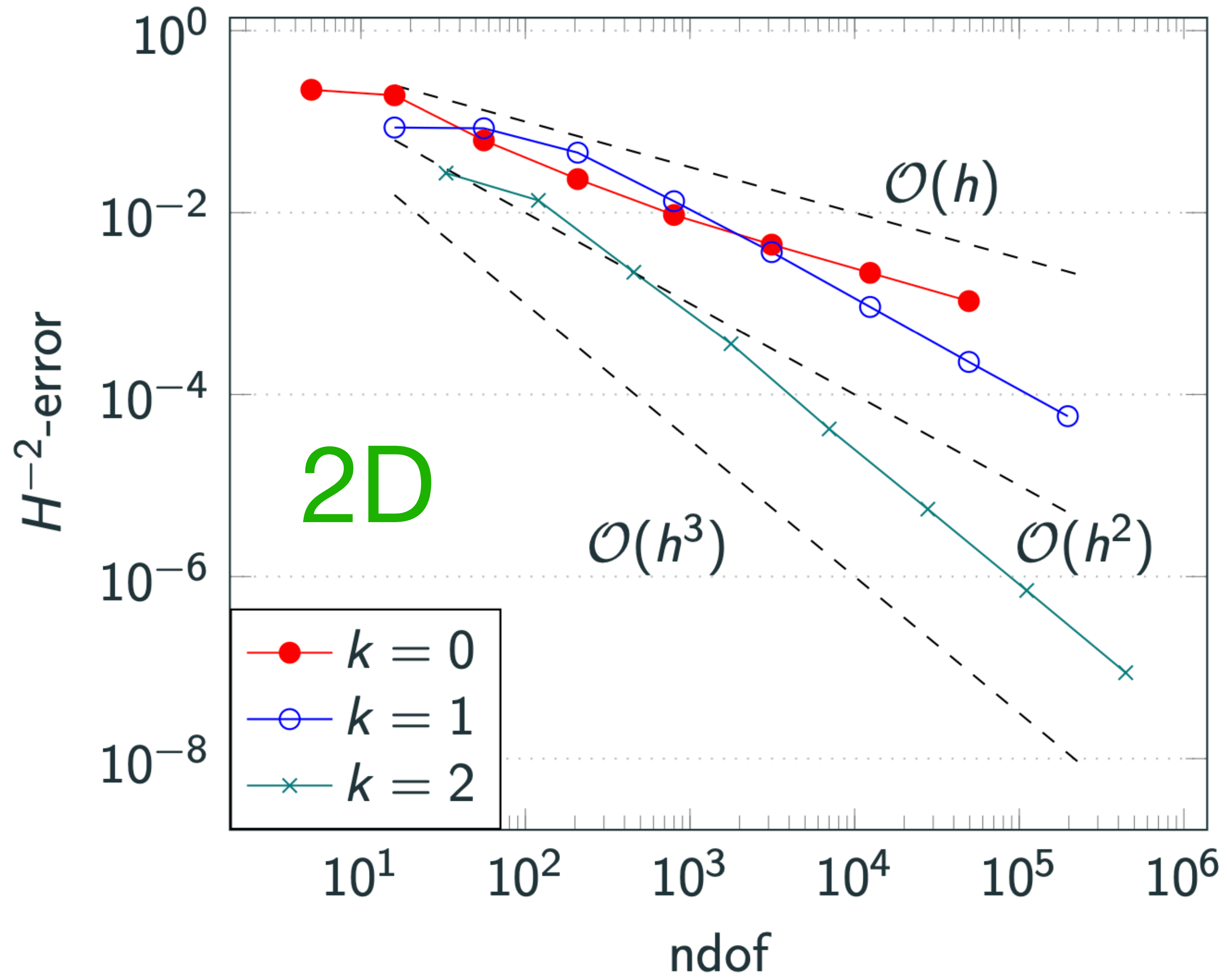
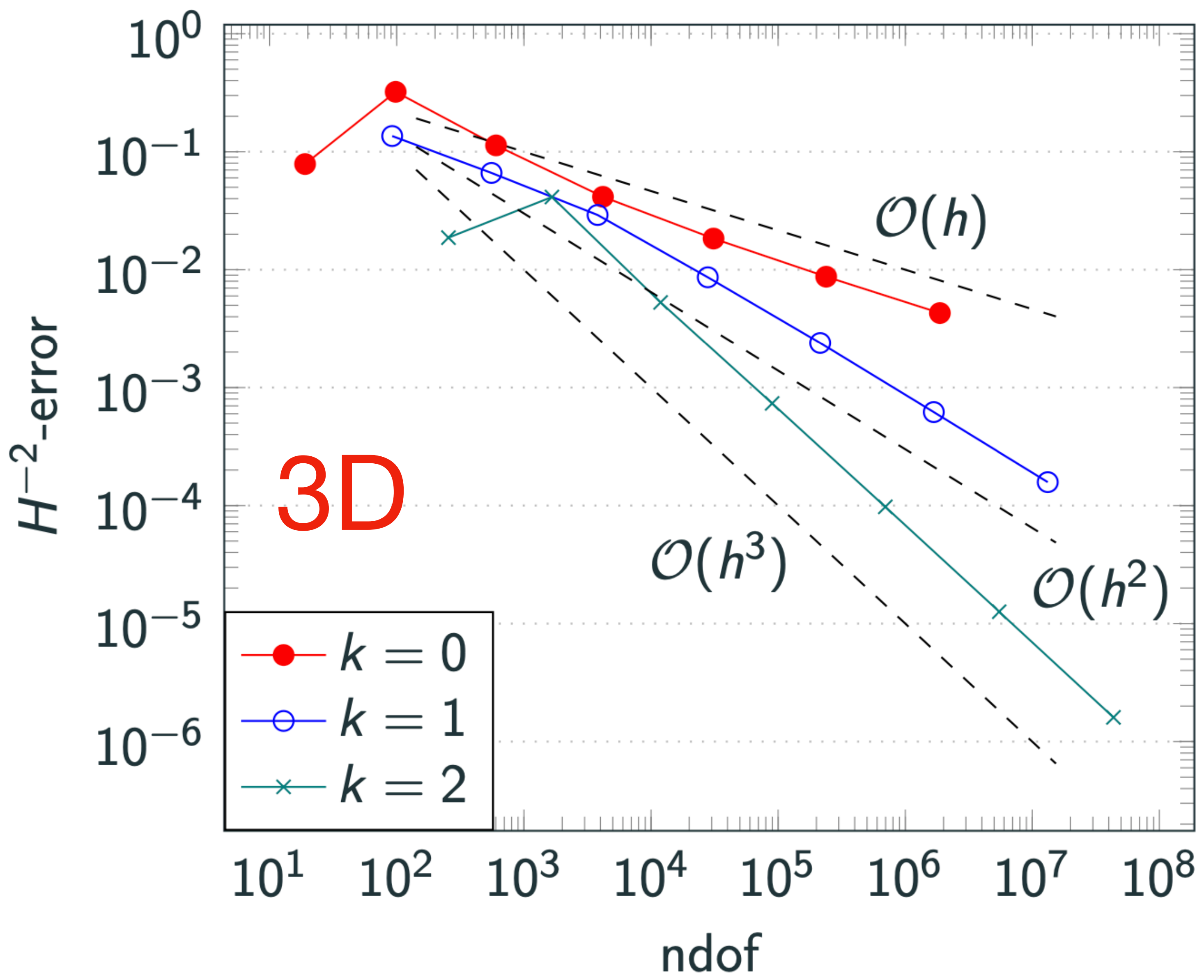
NGSolve





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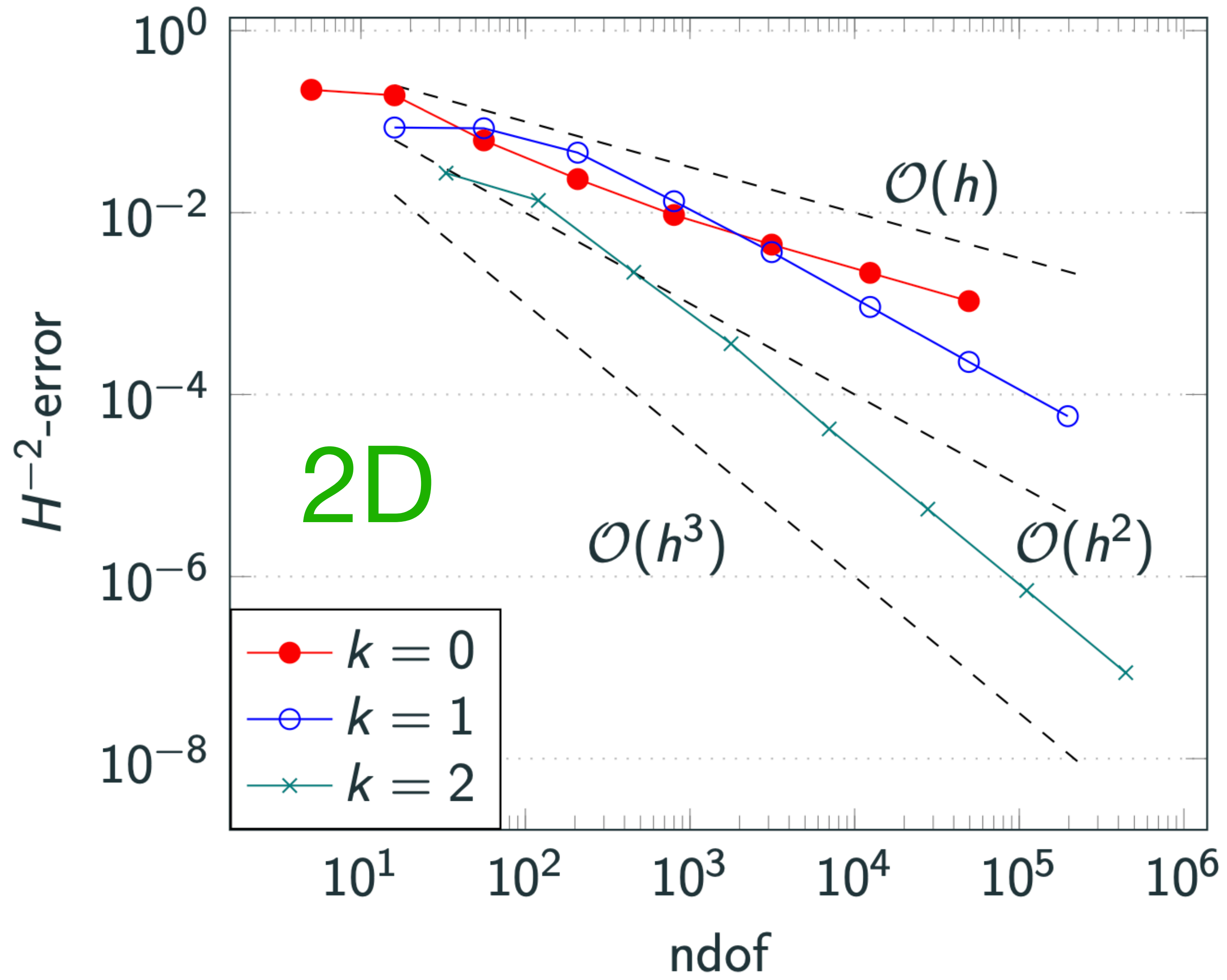


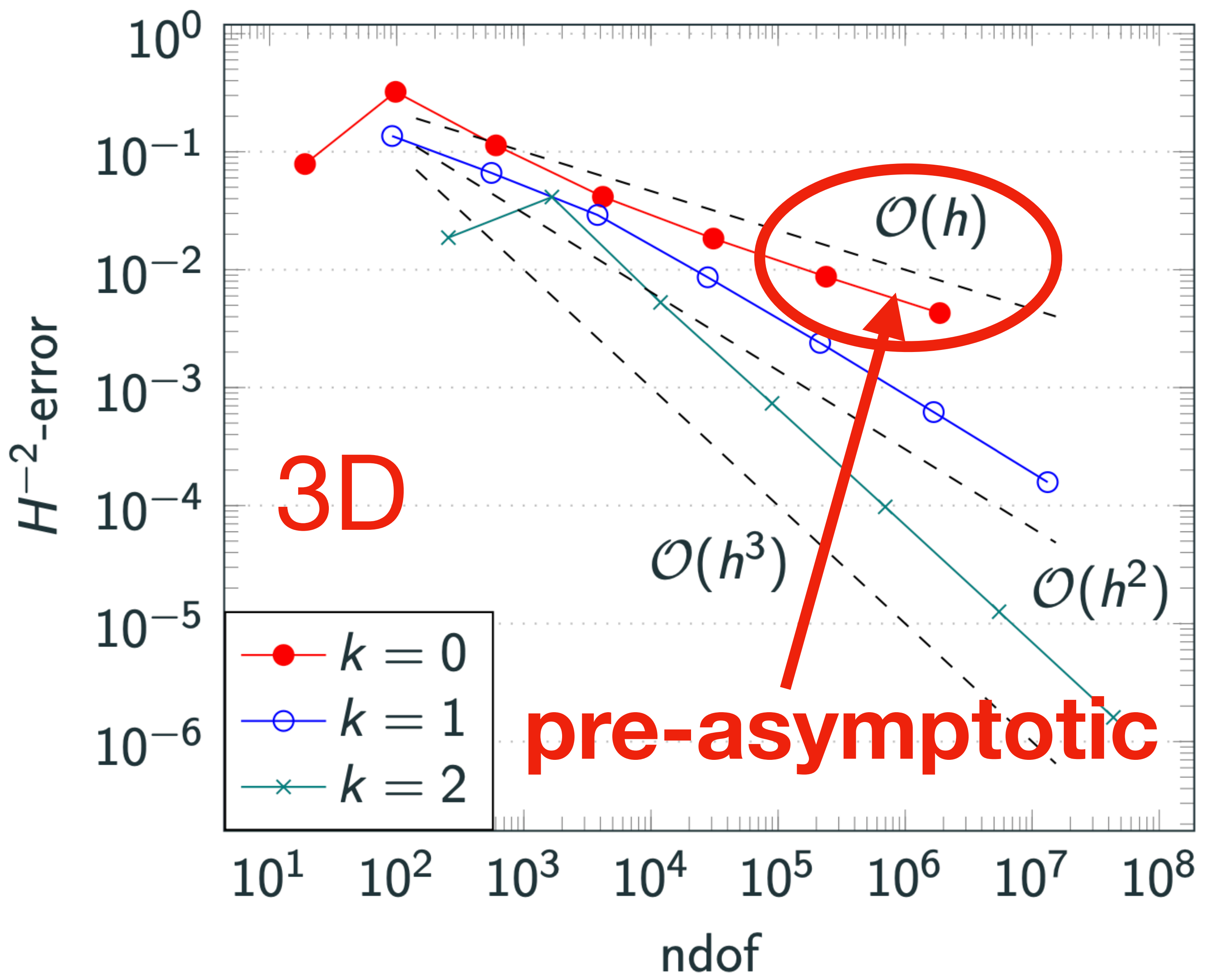



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# Discrete lifting of distributional curvatures

- need function instead of distribution sometimes
- Gaus curvature: solve with mass matrix for discrete Riesz representative $g_h \in \mathrm{Reg}^k$

Find $K_h \in \mathrm{Lag}^{k+1}$ such that $\int_\Omega K_h v_h \omega = \langle K(g_h), v_h \rangle \qquad \forall v_h \in \mathrm{Lag}^{k+1}$


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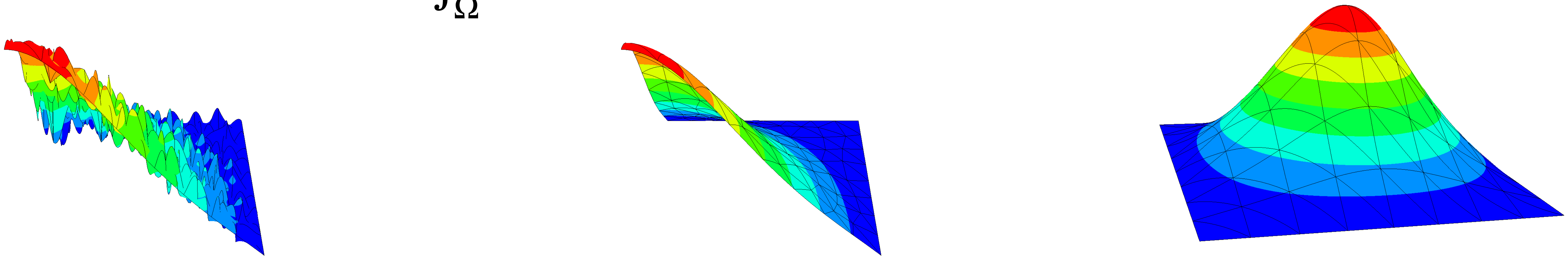


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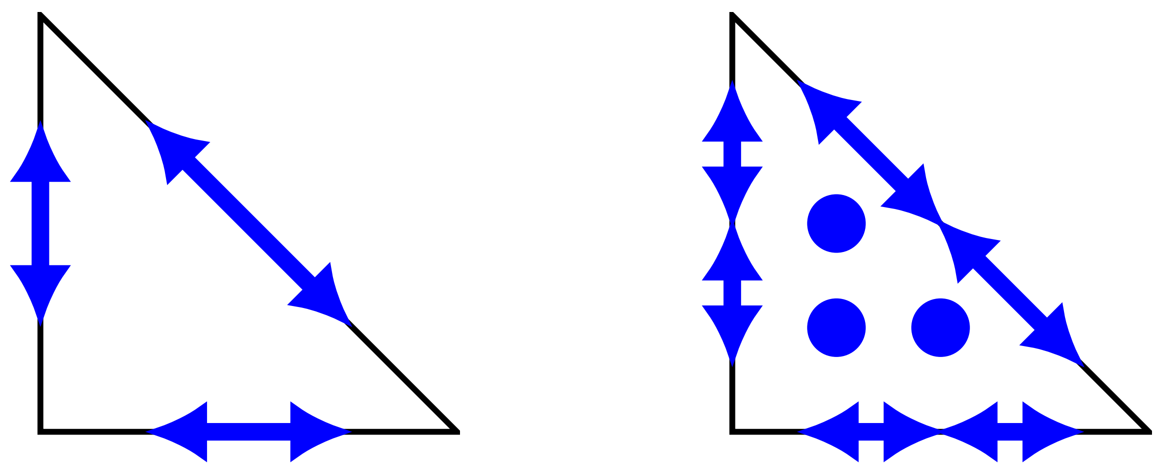
# Summary & Outlook



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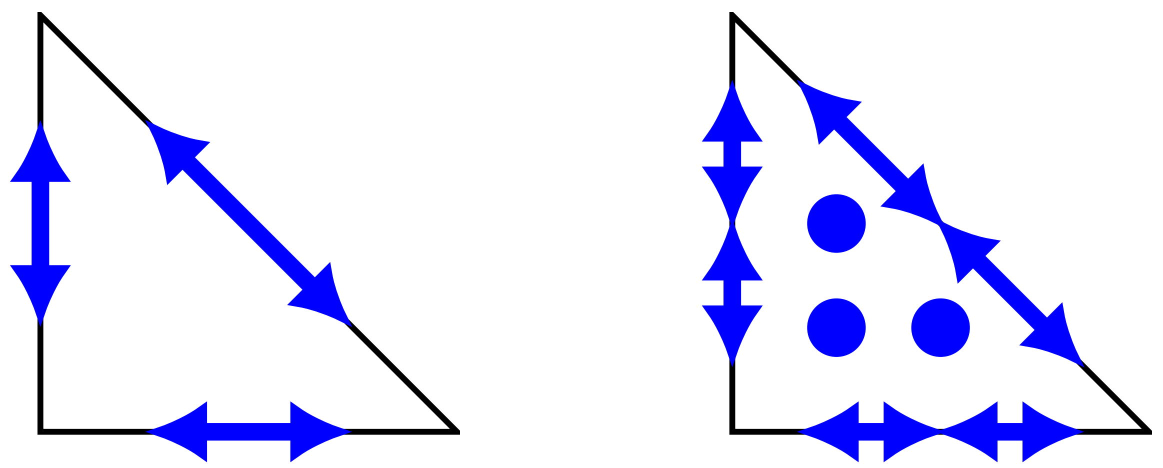

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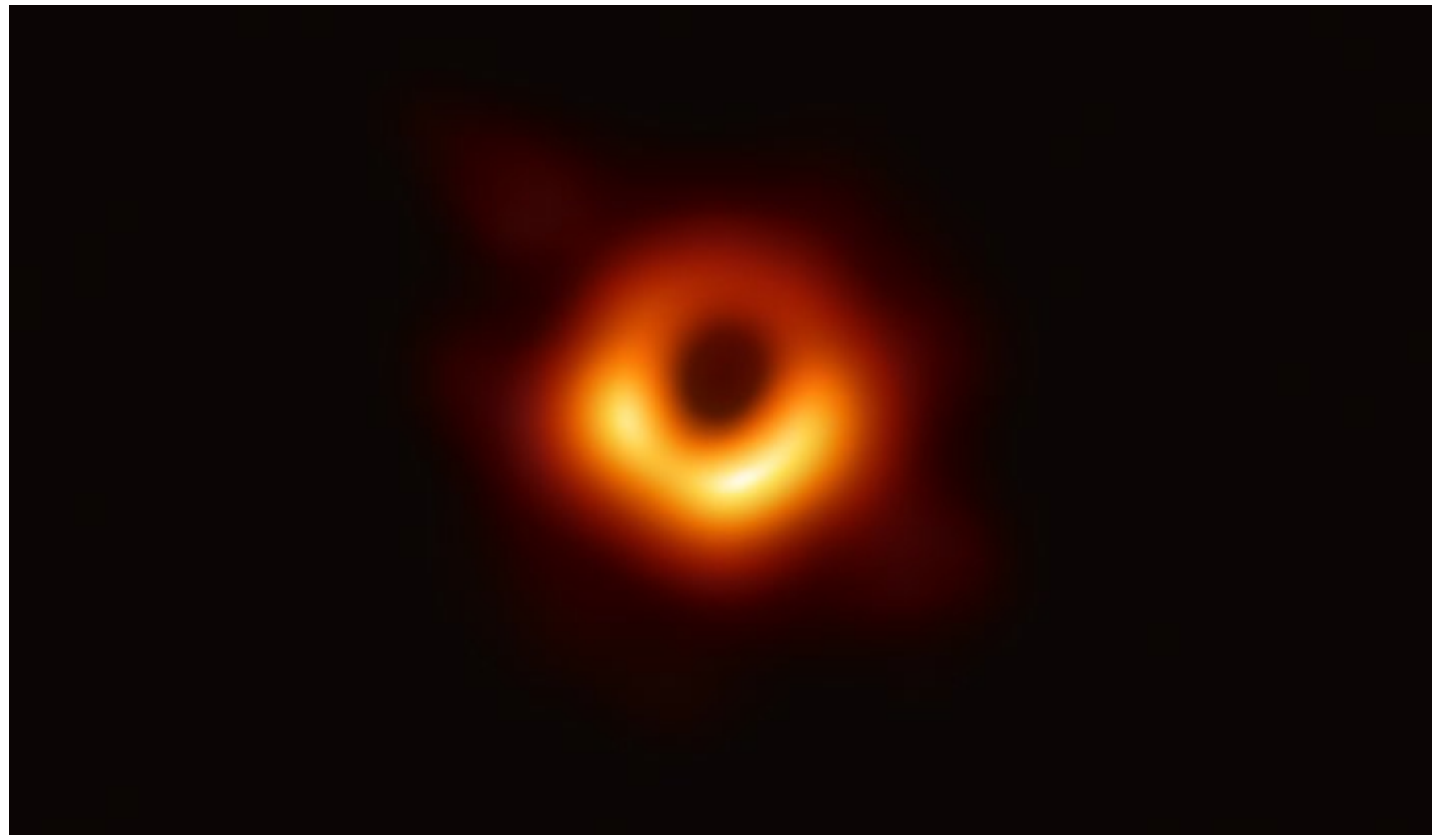
- Numerical error analysis via integral representation

- Finite elements for Riemann, Ricci, and Einstein tensor approximation
- Analysis of distributional covariant operators
- Theoretical & numerical framework solving PDEs on Riemannian manifolds
- Long-term goal: Application to geometric flows and numerical relativity



By Event Horizon Telescope (EHT)

# Literature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# Thank you for your attention!